17. MODELING OF CURVED AND DOUBLY-CURVED SHELLS BY FINITE ELEMENT METHOD BASED SOFTWARE SYSTEMS

Szerző: Dr. András Szekrényes

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17.1 Curved shell elements

Curved shell elements are suitable to model the midsurface geometry more accurately. In the case of certain surfaces – for example the cylindrical shell – it means the exact description of the original surface. For more complicated cases – similarly to the displacement field - the curvatures of the surface are approximated by interpolation functions. In this respect such elements belong to the parametric element types [1].

17.2 Thin-walled cylindrical shell element

The thin cylindrical shell element is presented in Fig.17.1. In accordance with the basic equations of the technical theory of thin shells the geometrical properties of the cylindrical shell are the followings [1,2]:

$$q_1 = x\,, H_1 = 1\,, R_1 = \infty\,, \qquad (17.1)$$
$$q_2 = \varphi\,, H_2 = R\,, R_2 = R\,.$$

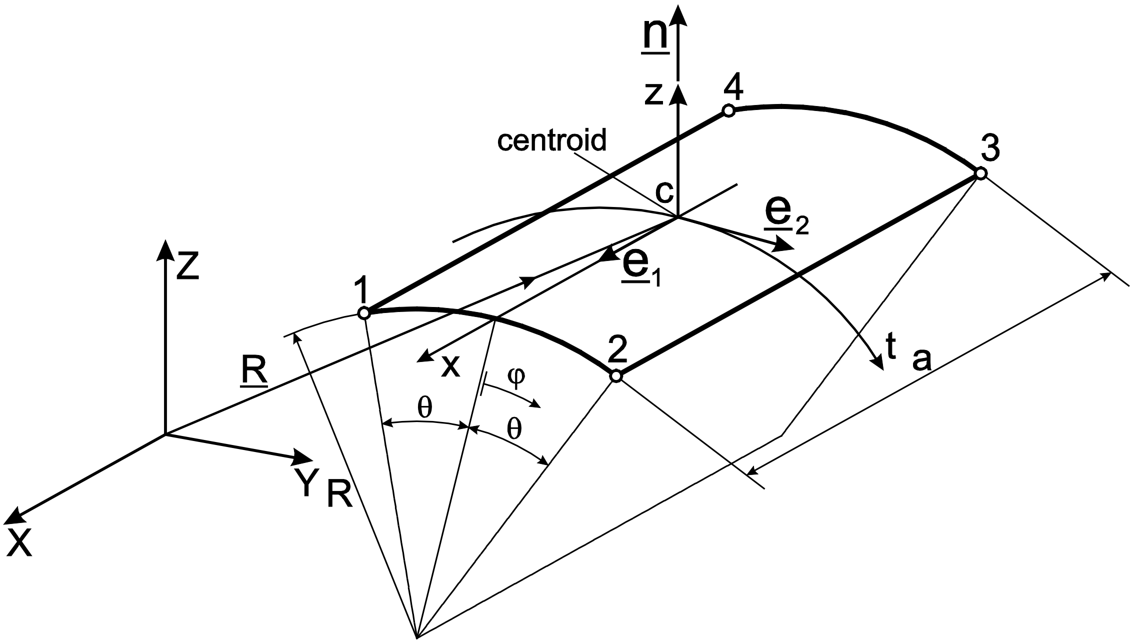

Fig.17.1. Parameters of the thin cylindrical shell element.

Apart from the displacement components *u*, *v* and *w* we can derive the angle of rotations by applying the basic equations of the technical theory of thin shells based on Eqs.(14.69) and (14.70):

$$\beta_1 = \beta_x = \frac{u}{R_1} - \frac{1}{H_1} w_{,1} = -w_{,x}\,, \qquad (17.2)$$
$$\beta_2 = \beta_\varphi = \frac{v}{R_2} - \frac{1}{H_2} w_{,2} = \frac{1}{R}(v - w_{,\varphi})\,,$$

$$\beta_3 = \frac{1}{2R}(Rv_{,x} - u_{,\varphi}).$$

Next, we calculate the strain components using the parameters of the cylindrical shell surface (see Eq.(14.67)):

$$\varepsilon_{11} = \varepsilon_x = \frac{1}{H_1}u_{,1} + \frac{H_{1,2}}{H_1 H_2}v + \frac{1}{R_1}w = u_{,x}, \qquad (17.3)$$

$$\varepsilon_{22} = \varepsilon_\varphi = \frac{1}{H_2}v_{,2} + \frac{H_{2,1}}{H_1 H_2}u + \frac{1}{R_2}w = \frac{1}{R}(u_{,\varphi} + w),$$

$$2\gamma_{12} = 2\gamma_{x\varphi} = \frac{H_1}{H_2}\left(\frac{u}{H_1}\right)_{,2} + \frac{H_2}{H_1}\left(\frac{v}{H_2}\right)_{,1} = \frac{1}{R}u_{,\varphi} + v_{,x},$$

$$\kappa_{11} = \kappa_x = \frac{1}{H_1}\beta_{1,1} + \frac{H_{1,2}}{H_1 H_2}\beta_2 = w_{,xx},$$

$$\kappa_{22} = \kappa_\varphi = \frac{1}{H_2}\beta_{2,2} + \frac{H_{2,1}}{H_1 H_2}\beta_1 = \frac{1}{R^2}(v_{,\varphi} - w_{,\varphi\varphi}),$$

$$2\kappa_{12} = 2\kappa_{x\varphi} = \frac{H_1}{H_2}\left(\frac{\beta_1}{H_2}\right)_{,2} + \frac{H_2}{H_1}\left(\frac{\beta_2}{H_1}\right)_{,1} + \left(\frac{1}{R_2} - \frac{1}{R_1}\right)\beta_3 = \frac{1}{R}(-2w_{,x\varphi} + v_{,x} + \beta_3).$$

The rigid body-like motion of the element involves six degrees of freedom, which are given by the displacement vector field given below [1]:

$$u_0 = a_1 + a_2 R(\cos\varphi - \cos\theta) - a_3 R\sin\varphi, \qquad (17.4)$$

$$v_0 = -a_2 x\sin\varphi + a_3 x\cos\varphi + a_4 R(\cos\varphi\cos\theta - 1) - a_5\sin\varphi + a_6\cos\varphi,$$

$$w_0 = a_2 x\cos\varphi + a_3 x\sin\varphi + a_4 R\sin\varphi\cos\varphi + a_5\cos\varphi + a_6\sin\varphi.$$

In matrix form:

$$\underline{u}_0 = \underline{\underline{\Phi}}_0 \underline{\alpha}_0, \qquad (17.5)$$

where:

$$\underline{u}_0^T = [u_0 \quad v_0 \quad w_0], \qquad (17.6)$$

$$\underline{\underline{\Phi}}_0 = \begin{bmatrix} 1 & R(\cos\varphi - \cos\theta) & -R\sin\varphi & 0 & 0 & 0 \\ 0 & -x\sin\varphi & x\cos\varphi & R(\cos\varphi\cos\theta - 1) & -\sin\varphi & \cos\varphi \\ 0 & x\cos\varphi & x\sin\varphi & R\sin\varphi\cos\varphi & \cos\varphi & \sin\varphi \end{bmatrix},$$

$$\underline{\alpha}_0^T = [a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_6].$$

where $\underline{\underline{\Phi}}_0$ is the matrix of interpolation functions, which capture the rigid body-like motions, $\underline{\alpha}_0$ is the vector of unknown coefficients. The displacement field of rigid body-like motion and that of the deformation together give the total displacement field, which is:

$$\underline{u} = \underline{u}_0 + \underline{u}_{01} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}, \qquad (17.7)$$

where:

$$\underline{u}_{01}^T = [u_{01} \quad v_{01} \quad w_{01}], \qquad (17.8)$$

$$u_{01} = a_7 x + a_8\varphi + a_9 x\varphi,$$

$$v_{01}=a_{10}\varphi+a_{11}x\varphi,$$

$$w_{01}=a_{12}x^2+a_{13}x\varphi+a_{14}\varphi^2+a_{15}x^3+a_{16}x^2\varphi+a_{17}x\varphi^2+a_{18}\varphi^3+$$
$$+a_{19}x^3\varphi+a_{20}x^2\varphi^2+a_{21}x\varphi^3+a_{22}x^3\varphi^2+a_{23}x^2\varphi^3+a_{24}x^3\varphi^3.$$

In matrix form we have:

$$\underline{u}=\underline{u}_0+\underline{u}_{01}=\underline{\underline{\Phi}}_0\underline{\alpha}_0+\underline{\underline{\Phi}}_1\underline{\alpha}_1, \qquad (17.9)$$

where $\underline{\underline{\Phi}}_1$ is the interpolation functions matrix related to the deformation displacement field:

$$\underline{\underline{\Phi}}_1=\begin{bmatrix} x & \varphi & x\varphi & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & \varphi & x\varphi & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & x^2 & x\varphi & \varphi^2 & x^3 & x^2\varphi & x\varphi^2 & \varphi^3 & x^3\varphi & x^2\varphi^2 & x\varphi^3 & x^3\varphi^2 & x^2\varphi^3 & x^3\varphi^3 \end{bmatrix}. \qquad (17.10)$$

$\underline{\alpha}_1$ is the vector of unknown coefficients:

$$\underline{\alpha}_1^T=[a_7\quad a_8\quad a_9\quad a_{10}\quad a_{11}\quad a_{12}\quad a_{13}\quad a_{14}\quad a_{15}\ldots\ldots\ldots$$
$$a_{16}\quad a_{17}\quad a_{18}\quad a_{19}\quad a_{20}\quad a_{21}\quad a_{22}\quad a_{23}\quad a_{24}]. \qquad (17.11)$$

In the expression above there are 24 unknown coefficients. To determine all of them 24 displacement parameters are required, let us choose the followings:

$$\underline{\tilde{u}}_e^T=[\tilde{u}_1\quad \tilde{v}_1\quad \tilde{w}_1\quad \tilde{\beta}_{x1}\quad \tilde{\beta}_{\varphi1}\quad \tilde{w}_{,x\varphi1}\quad \tilde{u}_2\quad \tilde{v}_2\quad \tilde{w}_2\quad \tilde{\beta}_{x2}\quad \tilde{\beta}_{\varphi2}\quad \tilde{w}_{,x\varphi2}\ldots\ldots$$
$$\ldots\ldots\ldots\tilde{u}_3\quad \tilde{v}_3\quad \tilde{w}_3\quad \tilde{\beta}_{x3}\quad \tilde{\beta}_{\varphi3}\quad \tilde{w}_{,x\varphi3}\quad \tilde{u}_4\quad \tilde{v}_4\quad \tilde{w}_4\quad \tilde{\beta}_{x4}\quad \tilde{\beta}_{\varphi4}\quad \tilde{w}_{,x\varphi4}], \qquad (17.12)$$

i.e. at each node there are displacements in the direction of the basis vectors, there are rotations about $\underline{e}_1$ and $\underline{e}_2$, the sixth degrees of freedom is chosen to be the mixed derivative $w_{,x\varphi}$. The degrees of freedom for the cylindrical shell element based on Eqs.(17.2) and (17.7) are:

$$u=u_0+u_{01},v=v_0+v_{01},w=w_0+w_{01}, \qquad (17.13)$$

$$\beta_x=-w_{,x}=-a_2\cos\varphi-a_3\sin\varphi-2a_{12}x-a_{13}\varphi-3a_{15}x^2-2a_{16}x\varphi-a_{17}\varphi^2+$$
$$-3a_{19}x^2\varphi-2a_{20}x\varphi^2-a_{21}\varphi^3-3a_{22}x^2\varphi^2-2a_{23}x\varphi^3-3a_{24}x^2\varphi^3,$$

$$\beta_\varphi=\frac{1}{R}(v-w_{,\varphi})=\frac{1}{R}(-a_4R+a_{10}\varphi+a_{11}x\varphi-a_{13}x-2a_{14}\varphi-a_{16}x^2-2a_{17}x\varphi-3a_{18}\varphi^2+$$
$$-a_{19}x^3-2a_{20}x^2\varphi-3a_{21}x\varphi^2-2a_{22}x^3\varphi-3a_{23}x^2\varphi^2-3a_{24}x^3\varphi^2),$$

$$w_{,x\varphi}=-a_2\sin\varphi+a_3\cos\varphi+a_{13}+2a_{16}x+2a_{17}\varphi+3a_{19}x^2+$$
$$+4a_{20}x\varphi+3a_{21}\varphi^2+6a_{22}x^2\varphi+6a_{23}x\varphi^2+9a_{24}x^2\varphi^2.$$

The conditions for the determination of the parameters a_i, i = 1...24 are:

$$u(L/2,-\theta)=\tilde{u}_1,u(L/2,\theta)=\tilde{u}_2, \qquad (17.14)$$
$$u(-L/2,\theta)=\tilde{u}_3,u(-L/2,-\theta)=\tilde{u}_4,$$
$$v(L/2,-\theta)=\tilde{v}_1,v(L/2,\theta)=\tilde{v}_2,$$
$$v(-L/2,\theta)=\tilde{v}_3,v(-L/2,-\theta)=\tilde{v}_4,$$
$$w(L/2,-\theta)=\tilde{w}_1,w(L/2,\theta)=\tilde{w}_2,$$
$$w(-L/2,\theta)=\tilde{w}_3,w(-L/2,-\theta)=\tilde{w}_4,$$
$$\beta_x(L/2,-\theta)=\tilde{\beta}_{x1},\beta_x(L/2,\theta)=\tilde{\beta}_{x2},$$
$$\beta_x(-L/2,\theta)=\tilde{\beta}_{x3},\ \beta_x(-L/2,-\theta)=\tilde{\beta}_{x4},$$

$$\beta_\varphi(L/2,-\theta)=\tilde{\beta}_{\varphi 1},\ \beta_\varphi(L/2,\theta)=\tilde{\beta}_{\varphi 2},$$
$$\beta_\varphi(-L/2,\theta)=\tilde{\beta}_{\varphi 3},\ \beta_\varphi(-L/2,-\theta)=\tilde{\beta}_{\varphi 4},$$
$$w_{,x\varphi}(L/2,-\theta)=\tilde{w}_{,x\varphi 1},\ w_{,x\varphi}(L/2,\theta)=\tilde{w}_{,x\varphi 2},$$
$$w_{,x\varphi}(-L/2,\theta)=\tilde{w}_{,x\varphi 3},\ w_{,x\varphi}(-L/2,-\theta)=\tilde{w}_{,x\varphi 4}.$$

The vector of nodal displacements is formulated similarly to the plate element presented in section 15, viz. [3]:

$$\underline{\tilde{u}}_e=\underline{\underline{M}}\,\underline{A}, \qquad (17.15)$$

where vector $\underline{A}$ contains the elements of $\underline{\alpha}_0$ and $\underline{\alpha}_1$ in order, i.e.:

$$\underline{A}^T=[a_1\quad a_2\quad a_3\quad a_4\quad a_5\quad a_6\quad a_7\quad a_8\quad a_9\quad a_{10}\quad a_{11}\quad a_{12}.........$$
$$..........a_{13}\quad a_{14}\quad a_{15}\quad a_{16}\quad a_{17}\quad a_{18}\quad a_{19}\quad a_{20}\quad a_{21}\quad a_{22}\quad a_{23}\quad a_{24}]. \qquad (17.16)$$

The coefficients of the interpolation polynomials are determined by the inversion of $\underline{\underline{M}}$:

$$\underline{A}=\underline{\underline{M}}^{-1}\underline{\tilde{u}}_e. \qquad (17.17)$$

Due to their large length, the coefficients are not detailed here. Assuming that the angle rotation, β_3 is approximately zero the strain components are formulated as follows:

$$\varepsilon_x=u_{,x},\ \varepsilon_\varphi=\frac{1}{R}(u_{,\varphi}+w),\ 2\gamma_{x\varphi}=\frac{1}{R}u_{,\varphi}+v_{,x}, \qquad (17.18)$$
$$\kappa_x=w_{,xx},\ \kappa_\varphi=\frac{1}{R^2}(v_{,\varphi}-w_{,\varphi\varphi}),\ 2\kappa_{x\varphi}=\frac{1}{R}(-2w_{,x\varphi}+v_{,x}).$$

The strain components can be classified into two parts: strains, shear strains and the curvatures, respectively. We can write that:

$$\underline{\varepsilon}_0=\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ 2\gamma_{xy}\end{bmatrix}=\underline{\underline{R}}_0\underline{A},\ \underline{\kappa}=\begin{bmatrix}\kappa_x\\ \kappa_y\\ 2\kappa_{xy}\end{bmatrix}=\underline{\underline{R}}_1\underline{A}, \qquad (17.19)$$

where matrices $\underline{\underline{R}}_0$ and $\underline{\underline{R}}_1$ are calculated based on the derivatives of the displacement functions:

$$\underline{\underline{R}}_0=\left[\begin{matrix}0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \varphi & 0 & 0 & 0\\ 0 & \begin{Bmatrix}-\sin\varphi+\\ +\frac{x}{R}\cos\varphi\end{Bmatrix} & \begin{Bmatrix}-\cos\varphi+\\ +\frac{x}{R}\sin\varphi\end{Bmatrix} & \sin\varphi\cos\theta & \frac{\cos\varphi}{R} & \frac{\sin\varphi}{R} & 0 & \frac{1}{R} & \frac{x}{R} & 0 & 0 & \frac{x^2}{R}\\ 0 & -\frac{3}{2}\sin\varphi & \frac{1}{2}\cos\varphi & 0 & 0 & 0 & 0 & \frac{1}{2R} & \frac{x}{2R} & 0 & \varphi & 0\end{matrix}\right.....$$
$$....\left.\begin{matrix}0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ \frac{x\varphi}{R} & \frac{\varphi^2}{R} & \frac{x^3}{R} & \frac{x^2\varphi}{R} & \frac{x\varphi^2}{R} & \frac{\varphi^3}{R} & \frac{x^3\varphi}{R} & \frac{x^2\varphi^2}{R} & \frac{x\varphi^3}{R} & \frac{x^3\varphi^2}{R} & \frac{x^2\varphi^3}{R} & \frac{x^3\varphi^3}{R}\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\end{matrix}\right], \qquad (17.20)$$

$$\underline{\underline{R}}_1=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 6x & 2\varphi \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{R^2} & \frac{x}{R^2} & 0 & 0 & -\frac{2}{R^2} & 0 & 0 \\ 0 & \frac{\sin\varphi}{2R} & -\frac{\cos\varphi}{2R} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\varphi}{2R} & 0 & -\frac{1}{R} & 0 & 0 & -\frac{2x}{R} \end{bmatrix} ....$$

$$.... \begin{bmatrix} 0 & 0 & 6x\varphi & 2\varphi^2 & 0 & 6x\varphi^2 & 2\varphi^3 & 6x\varphi^3 \\ -\frac{2x}{R^2} & -\frac{6\varphi}{R^2} & 0 & -\frac{2x^2}{R^2} & -\frac{6x\varphi}{R^2} & -\frac{2x^3}{R^2} & -\frac{6x^2\varphi}{R^2} & -\frac{6x^3\varphi}{R^2} \\ -\frac{2\varphi}{R} & 0 & -\frac{3x^2}{R} & -\frac{4x\varphi}{R} & -\frac{3\varphi^2}{R} & -\frac{6x^2\varphi}{R} & -\frac{6x\varphi^2}{R} & -\frac{9x^2\varphi^2}{R} \end{bmatrix}. \tag{17.21}$$

For the calculation of the stiffness matrix and the force vector related to the distributed load we formulate the total potential energy of a single element. We note that the expression below contains only the strain energy and the work of the distributed load:

$$\Pi_e=\frac{1}{2}\int_{V_e}\underline{\sigma}^T\underline{\varepsilon}dV-\int_{A_{pe}}\underline{u}^T\underline{p}dA. \tag{17.22}$$

The constitutive law of the linear elastic material is:

$$\underline{\sigma}=\underline{\underline{C}}\underline{\varepsilon}, \tag{17.23}$$

where, similarly to the plates we assume plane stress state, i.e. $\underline{\underline{C}}=\underline{\underline{C}}^{str}$. Using Eqs.(17.17) and (17.19) we obtain:

$$\underline{\varepsilon}_0=\underline{\underline{R}}_0\underline{A}=\underline{\underline{R}}_0\underline{\underline{M}}^{-1}\underline{\tilde{u}}_e=\underline{\underline{B}}_0\underline{\tilde{u}}_e, \tag{17.24}$$

$$\underline{\underline{B}}_0=\underline{\underline{R}}_0\underline{\underline{M}}^{-1},\underline{\sigma}_0=\underline{\underline{C}}\underline{\varepsilon}_0=\underline{\underline{C}}\underline{\underline{B}}_0\underline{\tilde{u}}_e,$$

furthermore:

$$\underline{\kappa}=\underline{\underline{R}}_1\underline{A}=\underline{\underline{R}}_1\underline{\underline{M}}^{-1}\underline{u}_e=\underline{\underline{B}}_1\underline{\tilde{u}}_e, \tag{17.25}$$

$$\underline{\underline{B}}_1=\underline{\underline{R}}_1\underline{\underline{M}}^{-1},\underline{\sigma}_1=-z\underline{\underline{C}}\underline{\kappa}=-z\underline{\underline{C}}\underline{\underline{B}}_1\underline{\tilde{u}}_e$$

Based on Eq.(17.7) the displacement field becomes:

$$\underline{u}=\underline{\underline{\lambda}}\underline{A}=\underline{\underline{\lambda}}\underline{\underline{M}}^{-1}\underline{\tilde{u}}_e, \tag{17.26}$$

where:

$$\underline{\underline{\lambda}}=\left[\underline{\underline{\Phi}}_0 \quad \underline{\underline{\Phi}}_1\right]. \tag{17.27}$$

Eq.(17.24) is related to the stress resultants, while Eq.(17.25) is related to the stress couples. The total potential energy becomes:

$$\Pi_e=\frac{1}{2}\int_{V_e}\underline{\varepsilon}^T\underline{\underline{C}}^T\underline{\varepsilon}dV+\frac{1}{2}\int_{V_e}z^2\underline{\kappa}^T\underline{\underline{C}}^T\underline{\kappa}dV-\underline{\tilde{u}}_e^T\int_{A_{pe}}\underline{\underline{M}}^{-1T}\underline{\underline{\lambda}}^T\underline{p}dA, \tag{17.28}$$

which is written as:

$$\Pi_e = \frac{1}{2}\underline{\tilde{u}}_e^T \left\{ \int_{-L/2}^{L/2} \int_{-\theta}^{\theta} t \cdot \underline{\underline{B}}_0^T \underline{\underline{C}}^T \underline{\underline{B}}_0 R \cdot d\varphi \cdot dx \right\} \underline{\tilde{u}}_e + \frac{1}{2}\underline{\tilde{u}}_e^T \left\{ \int_{-L/2}^{L/2} \int_{-\theta}^{\theta} \frac{t^3}{12} \underline{\underline{B}}_1^T \underline{\underline{C}}^T \underline{\underline{B}}_1 R \cdot d\varphi \cdot dx \right\} \underline{\tilde{u}}_e +$$
$$- \underline{\tilde{u}}_e^T \int_{-L/2}^{L/2} \int_{-\theta}^{\theta} \underline{\underline{M}}^{-1^T} \underline{\underline{\lambda}}^T \underline{p} \cdot R \cdot d\varphi \cdot dx. \qquad (17.29)$$

It is important to note that in Eq.(17.29) the term related to the concentrated loads is excluded, consequently the vector of concentrated loads should be produced additionally. This is an easy task based on the nodal degrees of freedom:

$$\underline{\tilde{F}}_{ec}^T = [F_{x1} \quad F_{\varphi 1} \quad F_{z1} \quad M_{x1} \quad M_{\varphi 1} \quad M_{x\varphi 1} \quad F_{x2} \quad F_{\varphi 2} \quad F_{z2} \quad M_{x2} \quad M_{\varphi 2} \quad M_{,x\varphi 2} \ldots..$$
$$\ldots\ldots\ldots F_{x3} \quad F_{\varphi 3} \quad F_{z3} \quad M_{x3} \quad M_{\varphi 3} \quad M_{x\varphi 3} \quad F_{x4} \quad F_{\varphi 4} \quad F_{z4} \quad M_{x4} \quad M_{\varphi 4} \quad M_{x\varphi 4}], \qquad (17.30)$$

and the completed total potential energy becomes:

$$\Pi_e = \frac{1}{2}\underline{\tilde{u}}_e^T \underline{\underline{\tilde{K}}}_e \underline{\tilde{u}}_e - \underline{\tilde{u}}_e^T (\underline{\tilde{F}}_{ep} + \underline{\tilde{F}}_{ec}) . \qquad (17.31)$$

In Eq.(17.31) $\underline{\underline{\tilde{K}}}_e$ is the element stiffness matrix in the local coordinate system

$$\underline{\underline{\tilde{K}}}_e = \int_{-L/2}^{L/2} \int_{-\theta}^{\theta} \left(t \cdot \underline{\underline{B}}_0^T \underline{\underline{C}}^T \underline{\underline{B}}_0 + \frac{t^3}{12} \underline{\underline{B}}_1^T \underline{\underline{C}}^T \underline{\underline{B}}_1 \right) R \cdot d\varphi \cdot dx . \qquad (17.32)$$

The force vector from the distributed load is:

$$\underline{\tilde{F}}_{ep} = \int_{-L/2}^{L/2} \int_{-\theta}^{\theta} \underline{\underline{M}}^{-1^T} \underline{\underline{\lambda}}^T \underline{p} \cdot R \cdot d\varphi \cdot dx . \qquad (17.33)$$

Finally the well-known finite element equilibrium equation in the local system is:

$$\underline{\underline{\tilde{K}}}_e \underline{\tilde{u}}_e = \underline{\tilde{F}}_e , \qquad (17.34)$$

which is applicable only for a single element. The global equation of developed by a proper transformation. The structural equation is required when there are several elements connected to each other, which is mathematically the same as Eq. (14.86). The advantage of the thin cylindrical shell element is that the cylindrical surface is captured exactly; as a consequence it provides accurate result even if the number of elements is relatively low.

17.3 Axisymmetric shell problems – conical shell element

The midsurface of axisymmetric shells is produced by the rotation of the meridian curve about a straight axis [1]. An example is shown by Fig.17.2.

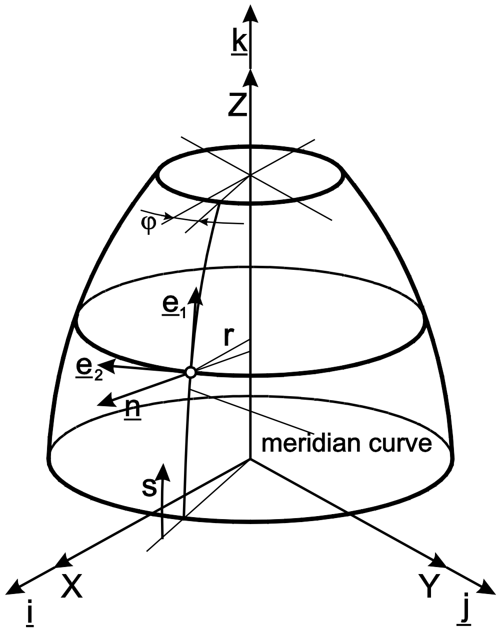

Fig.17.2. Axisymmetric shell.

The meridian curves and the circular curves perpendicularly to the meridian curves are principal curvature lines of the surface. If the load of the structure is axisymmetric, then in this kind of problem the displacement field is the function of arc length along the meridian curve only.
The meridian curve can be modeled by straight lines, and so we approximate the original shell structure by conical shell elements. Referring to the basic equations of the technical theory of thin shells, the parameters of the conical shell element shown in Fig.17.3 are:

$$q_1 = s\,, H_1 = 1\,, R_1 = \infty\,, \tag{17.35}$$

$$q_2 = \varphi\,, H_2 = r\,, R_2 = \frac{r}{\cos\theta}\,,$$

where s is the arc length, φ is the angle coordinate, r is the radius for a point P, θ is the half angle of inclination. To calculate the strain components we need to determine the $r(s)$ relationship, based on Fig.17.3 we have:

$$r(s) = s\sin\theta + r_1 \text{ and } \frac{dr}{ds} = \sin\theta\,. \tag{17.36}$$

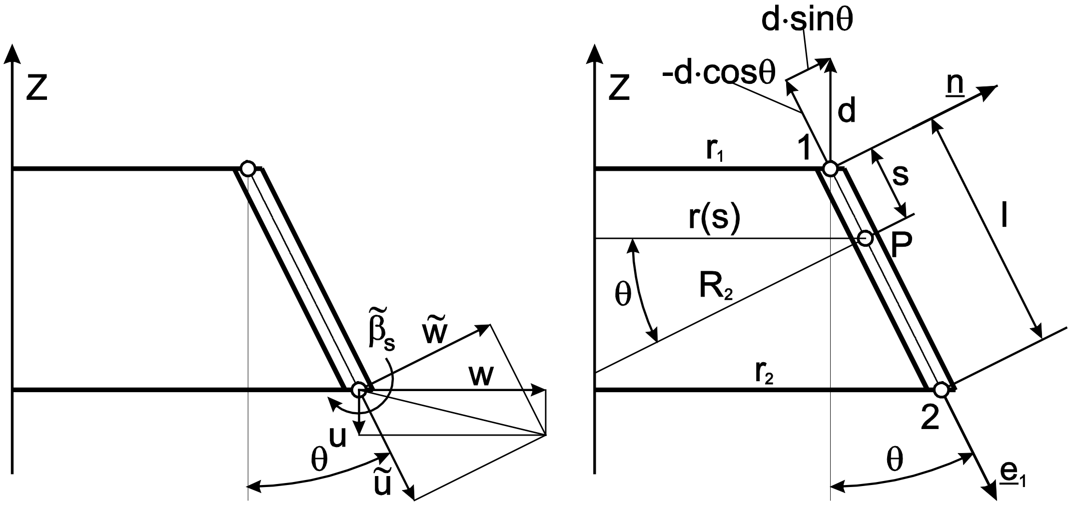

Fig.17.3. Axisymmetric conical shell element and its nodal parameters.

Using Eqs.(14.67), (14.69) and (14.70) of the technical theory of thin shells we can calculate the strain components as:

$$\beta_1 = \beta_s = -w_{,s}\,, \beta_2 = 0\,, \beta_3 = 0\,, \qquad (17.37)$$

$$\varepsilon_{11} = \varepsilon_s = u_{,s}\,, \varepsilon_{22} = \varepsilon_\varphi = \frac{1}{r}(u\sin\theta + w\cos\theta)\,, \gamma_{12} = 0\,,$$

$$\kappa_{11} = \kappa_s = -w_{,ss}\,,\ \kappa_{22} = \kappa_\varphi = -\frac{\sin\theta}{r} w_{,s}\,, \kappa_{12} = 0\,.$$

The displacement in the tangential direction at point P is $v = 0$ due to the axisymmetry. The admissible rigid body-like motion of the element is a displacement given by d in direction Z, for which the displacement components are $u = -d\cdot\cos\theta$ and $w = d\cdot\sin\theta$ (see Fig.17.3). We consider three degrees of freedom at each node, these are: u (displacement along the meridian direction), w (displacement perpendicularly to the meridian curve) and β_s (angle of rotation about the axis perpendicularly to the meridian curve in accordance with Fig.17.3), therefore the element has six degrees of freedom. The displacement in the meridian direction is interpolated by a linear function of the arc length. On the other hand we apply third order interpolation with respect to the displacement in the normal direction:

$$\begin{bmatrix} u \\ w \end{bmatrix} = \begin{bmatrix} 1 & s & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & s & s^2 & s^3 \end{bmatrix} \underline{\alpha} = \underline{\underline{\Phi}}\,\underline{\alpha}\,, \qquad (17.38)$$

where $\underline{\alpha}$ is the vector of unknown coefficients:

$$\underline{\alpha}^T = [a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_6]. \qquad (17.39)$$

The vector of nodal displacements is:

$$\underline{\tilde{u}}_e^T = [\tilde{u}_1 \quad \tilde{w}_1 \quad \tilde{\beta}_{s1} \quad \tilde{u}_2 \quad \tilde{w}_2 \quad \tilde{\beta}_{s2}]. \qquad (17.40)$$

The conditions required for the determination of the coefficients are:

$$u(s_1) = a_1 + a_2 s_1 = \tilde{u}_1\,, \qquad (17.41)$$

$$w(s_1) = a_3 + a_4 s_1 + a_5 s_1^2 + a_6 s_1^3 = \tilde{w}_1\,,$$

$$\beta_s(s_1) = a_4 + 2a_5 s_1 + 3a_6 s_1^2 = \tilde{\beta}_{s1}\,,$$

$$u(s_2) = a_1 + a_2 s_2 = \tilde{u}_2\,,$$

$$w(s_2) = a_3 + a_4 s_2 + a_5 s_2^2 + a_6 s_2^3 = \tilde{w}_2\,,$$

$$\beta_s(s_2)=a_4+2a_5s_2+3a_6s_2^2=\tilde{\beta}_{s2}.$$

The solutions for the coefficients are moderately complicated, therefore they are not included here. The displacement functions can be formulated also in the way presented below:

$$u(s)=N_1\tilde{u}_1+N_2\tilde{u}_2, \tag{17.42}$$

$$w(s)=N_3\tilde{w}_1+N_4\tilde{\beta}_{s1}+N_5\tilde{w}_2+N_6\tilde{\beta}_{s2},$$

where N_i, i = 1...6 are the interpolation functions:

$$N_1=\frac{s_2-s}{s_2-s_1},\; N_2=-\frac{s_1-s}{s_2-s_1}, \tag{17.43}$$

$$N_3=\frac{(s-s_2)^2(3s_1-s_2-2s)}{(s_2-s_1)^3},\; N_4=\frac{(s-s_2)^2(s_1-s)}{(s_2-s_1)^2},$$

$$N_5=\frac{(s-s_1)^2(3s_2-s_1-2s)}{(s_2-s_1)^3},\; N_6=-\frac{(s-s_1)^2(s_2-s)}{(s_2-s_1)^2},$$

and:

$$\underline{u}=\begin{bmatrix}u\\w\end{bmatrix}=\begin{bmatrix}N_1 & 0 & 0 & N_2 & 0 & 0\\0 & N_3 & N_4 & 0 & N_5 & N_6\end{bmatrix}\underline{\tilde{u}}_e=\underline{\underline{N}}\underline{\tilde{u}}_e. \tag{17.44}$$

The strain components in matrix form are:

$$\underline{\varepsilon}=\begin{bmatrix}\varepsilon_s\\ \varepsilon_\varphi\end{bmatrix}=\underline{\underline{B}}\underline{u}_e, \tag{17.45}$$

where the strain-displacement matrix is:

$$\underline{\underline{B}}=\begin{bmatrix}\frac{\partial N_1}{\partial s} & 0 & 0 & \frac{\partial N_2}{\partial s} & 0 & 0\\ \frac{N_1\sin\theta}{r} & \frac{N_3\cos\theta}{r} & \frac{N_4\cos\theta}{r} & \frac{N_2\sin\theta}{r} & \frac{N_5\cos\theta}{r} & \frac{N_6\cos\theta}{r}\end{bmatrix}. \tag{17.46}$$

We collect also the curvatures in matrix form:

$$\underline{\kappa}=\begin{bmatrix}\kappa_s\\ \kappa_\varphi\end{bmatrix}=\underline{\underline{H}}\underline{\tilde{u}}_e, \tag{17.47}$$

where:

$$\underline{\underline{H}}=\begin{bmatrix}0 & -\frac{\partial^2 N_3}{\partial s^2} & -\frac{\partial^2 N_4}{\partial s^2} & 0 & -\frac{\partial^2 N_5}{\partial s^2} & -\frac{\partial^2 N_6}{\partial s^2}\\ 0 & -\frac{\sin\theta}{r}\frac{\partial N_3}{\partial s} & -\frac{\sin\theta}{r}\frac{\partial N_4}{\partial s} & 0 & -\frac{\sin\theta}{r}\frac{\partial N_5}{\partial s} & -\frac{\sin\theta}{r}\frac{\partial N_6}{\partial s}\end{bmatrix}. \tag{17.48}$$

Based on the constitutive law the vector of stress components is:

$$\underline{\sigma}_0=\underline{\underline{C}}\underline{\varepsilon}=\underline{\underline{C}}\underline{\underline{B}}\underline{\tilde{u}}_e, \tag{17.49}$$

$$\underline{\sigma}_1=-z\underline{\underline{C}}\underline{\kappa}=-z\underline{\underline{C}}\underline{\underline{H}}\underline{\tilde{u}}_e.$$

The vectors of strain components and curvatures contain only two elements, therefore the constitutive matrix reduces to:

$$\underline{\underline{C}}=\frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu\\ \nu & 1\end{bmatrix}. \tag{17.50}$$

Taking the former back into the total potential energy (similarly to the cylindrical shell element) we can calculate the element stiffness matrix in the local coordinate system:

$$\underline{\underline{\tilde{K}}}_e = \int_{s_1}^{s_2} (t \cdot \underline{\underline{B}}^T \underline{\underline{C}}^T \underline{\underline{B}} + \frac{t^3}{12} \underline{\underline{H}}^T \underline{\underline{C}}^T \underline{\underline{H}}) \cdot 2\pi r ds = \int_0^l (t \cdot \underline{\underline{B}}^T \underline{\underline{C}}^T \underline{\underline{B}} + \frac{t^3}{12} \underline{\underline{H}}^T \underline{\underline{C}}^T \underline{\underline{H}}) \cdot 2\pi (s \sin\theta + r_1) ds . \quad (17.51)$$

In the above expression it was considered that $s_1 = 0$ and $s_2 = l$ and so $r = s \cdot \sin\theta$. The exact computation of the stiffness matrix is quite complicated, and consequently the finite element codes implement numerical methods, e.g. the Gauss rule presented in section 12 is suitable to calculate the matrix components. The force vector is composed by two terms. The vector of concentrated forces can be constructed based on the nodal degrees of freedom:

$$\underline{\tilde{F}}_{ec}^T = \begin{bmatrix} F_{s1} & F_{n1} & M_1 & F_{s2} & F_{n2} & M_2 \end{bmatrix}, \quad (17.52)$$

where F refers to the concentrated force, M is a concentrated moment about the same direction tan that of β_s. The force vector from the distributed load is calculated based on the work of the load:

$$W_e = \int_0^l (p_s u + p_n w) 2\pi r ds = \underline{\tilde{u}}_e^T \int_0^l \underline{\underline{N}}^T \begin{bmatrix} p_s \\ p_n \end{bmatrix} \cdot 2\pi r ds = \underline{\tilde{u}}_e^T \underline{\tilde{F}}_{ep} , \quad (17.53)$$

accordingly:

$$\underline{\tilde{F}}_{ep} = \int_0^l \underline{\underline{N}}^T \begin{bmatrix} p_s \\ p_n \end{bmatrix} \cdot 2\pi (s \sin\theta + r_1) ds . \quad (17.54)$$

Considering that $l \cdot \sin\theta = r_2 - r_1$ and assuming that both p_s and p_n are constants, we obtain:

$$\underline{\tilde{F}}_{ep} = \begin{bmatrix} \frac{1}{3} p_s \pi l (2r_1 + r_2) \\ \frac{1}{10} p_n \pi l (7r_1 + 3r_2) \\ \frac{1}{30} p_n \pi l^3 (3r_1 + 2r_2) \\ \frac{1}{3} p_s \pi l (r_1 + 2r_2) \\ \frac{1}{10} p_n \pi l (3r_1 + 7r_2) \\ -\frac{1}{30} p_n \pi l^3 (2r_1 + 3r_2) \end{bmatrix} . \quad (17.55)$$

In the local coordinate system the nodal displacement and reactions are calculated form the usual:

$$\underline{\underline{\tilde{K}}}_e \underline{\tilde{u}}_e = \underline{\tilde{F}}_e \quad (17.56)$$

equation, where:

$$\underline{\tilde{F}}_e = \underline{\tilde{F}}_{ec} + \underline{\tilde{F}}_{ep} . \quad (17.57)$$

For a finite element structure we need the structural equation given by Eq.(14.86). Since the elements are connected under a given angle, the local displacement coordinates should be transformed into the global cylindrical coordinate system with longitudinal axis given by Z. The transformation can be performed based on Fig.17.3:

$$\begin{bmatrix} u \\ w \\ \beta_s \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \tilde{u} \\ \tilde{w} \\ \tilde{\beta}_s \end{bmatrix} = \underline{\underline{L}}^T \underline{\tilde{u}} \,. \qquad (17.58)$$

Based on the former the transformation of the stiffness matrix becomes:

$$\underline{\underline{K}}_e = \underline{\underline{\lambda}}^T \underline{\underline{\tilde{K}}}_e \underline{\underline{\lambda}} \,, \qquad (17.59)$$

where:

$$\underline{\underline{T}} = \begin{bmatrix} \underline{\underline{L}}^T & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{L}}^T \end{bmatrix}, \qquad (17.60)$$

is an orthogonal transformation matrix. The transformed force vector is:

$$\underline{F}_e = \underline{\underline{T}} \underline{\tilde{F}}_e \,. \qquad (17.61)$$

For a single element the finite element equation in the global system is:

$$\underline{\underline{K}}_e \underline{u}_e = \underline{F}_e \,, \qquad (17.62)$$

Moreover, for the whole structure we have:

$$\underline{\underline{K}} \underline{U} = \underline{F} \,. \qquad (17.63)$$

In the finite element literature there are more element types, e.g. curved axisymmetric shell element [4,5,6], which operates similarly to the conical shell element.

17.4 Thick-walled shell elements

For the solution of three-dimensional problems we can apply the spatial (SOLID type) elements. Fig.17.4 shows a 20 node isoparametric element. Isoparametric representation means that the geometry and the displacement field is described by the same set of interpolation functions [1,4,5]:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \sum_{i=1}^{20} N_i(\xi,\eta,\zeta) \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix}, \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \sum_{i=1}^{20} N_i(\xi,\eta,\zeta) \begin{bmatrix} u_i \\ v_i \\ w_i \end{bmatrix}. \qquad (17.64)$$

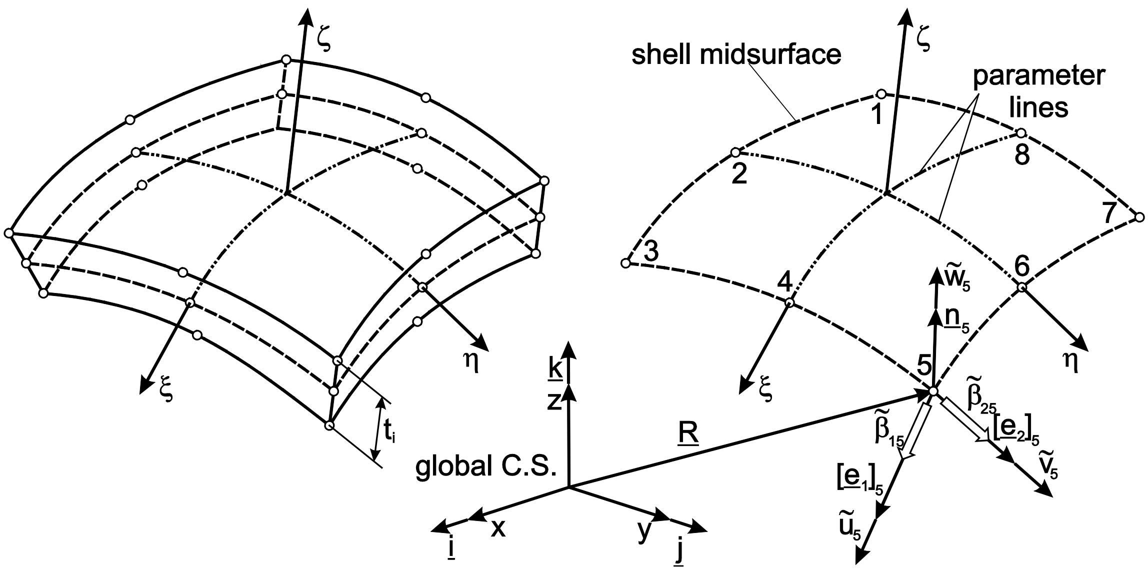

Fig.17.4. Quadratic two and three dimensional elements.

The thick-walled shell elements are constructed in accordance with isoparametric formulation, in this respect we point out that the sides perpendicularly to the shell midsurface are straight, i.e. the interpolation in the thickness direction is linear. The element is determined by the 8 nodes of the $\zeta = 0$ midsurface. As it can be seen in Fig.17.4 the direction of the unit basis vectors changes from point to point, therefore the nodal number is indicated by subscript „*i*". The coordinates of the points on the midsurface of the thick-walled shell element are given by:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \sum_{i=1}^{8} N_i(\xi,\eta) \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} + \sum_{i=1}^{8} N_i(\xi,\eta) \frac{\zeta}{2} t_i \underline{n}_i \,, \qquad (17.65)$$

where $\underline{n}_i$ are the column vector of normal vectors at the midsurface nodes, t_i is the thickness in the actual node, N_i are the interpolation functions, respectively. The interpolation functions are the same as those of the quadratic isoparametric plane membrane element (see section 12). The compact form of the interpolation function is:

$$N_i = \frac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i)(\xi\xi_i + \eta\eta_i - 1)\,,\ i = 1, 3, 5, 7, \qquad (17.66)$$

$$N_i = \frac{1}{2}\xi_i^2(1+\xi\xi_i)(1-\eta^2) + \frac{1}{2}\eta_i^2(1+\eta\eta_i)(1-\xi^2)\,,\ i = 2, 4, 6, 8,$$

where ξ_i and η_i are the local nodal coordinates. On the midsurface the ξ and η coordinate lines are orthogonal, therefore the basis vectors are calculated as:

$$\underline{n}_i = \frac{\underline{R}_{i,\xi} \times \underline{R}_{i,\eta}}{\left|\underline{R}_{i,\xi} \times \underline{R}_{i,\eta}\right|}\,, \underline{e}_{2i} = \frac{\underline{R}_{i,\eta}}{\left|\underline{R}_{i,\eta}\right|}\,, \underline{e}_{1i} = \underline{e}_{2i} \times \underline{n}_i \,. \qquad (17.67)$$

The nodal displacement parameters are the u_i, v_i, w_i displacements and the β_{1i} and β_{2i} angle of rotations. In the case of eight nodes it means that the element has 40 degrees of freedom. Vector $\underline{n}_i$ can be formulated by using the rotations and the basis vectors $\underline{e}_{1i}$, $\underline{e}_{2i}$:

$$\underline{n}_i = \tilde{\beta}_2 \underline{e}_{1i} - \tilde{\beta}_1 \underline{e}_{2i} \,, \qquad (17.68)$$

which is the term capturing the transverse shear deformation, it causes an increment in u and v. According to the isoparametric representation the displacement field becomes:

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \sum_{i=1}^{8} N_i(\xi,\eta) \begin{bmatrix} \tilde{u}_i \\ \tilde{v}_i \\ \tilde{w}_i \end{bmatrix} + \sum_{i=1}^{8} N_i(\xi,\eta) \frac{\zeta}{2} t_i (\tilde{\beta}_{2i} \underline{e}_{1i} - \tilde{\beta}_{1i} \underline{e}_{2i})\,. \qquad (17.69)$$

To calculate the stiffness matrix we have to establish the strain-displacement relationship. The derivatives of the displacement parameters with respect to the local coordinates are:

$$\begin{bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \\ \dfrac{\partial u}{\partial \zeta} \end{bmatrix} = \sum_{i=1}^{8} \begin{bmatrix} \dfrac{\partial N_i}{\partial \xi}[1 & -\dfrac{1}{2} t_i \zeta e_{2i}^x & \dfrac{1}{2} t_i \zeta e_{1i}^x] \\ \dfrac{\partial N_i}{\partial \eta}[1 & -\dfrac{1}{2} t_i \zeta e_{2i}^x & \dfrac{1}{2} t_i \zeta e_{1i}^x] \\ N_i[0 & -\dfrac{1}{2} t_i e_{2i}^x & \dfrac{1}{2} t_i e_{1i}^x] \end{bmatrix} \begin{bmatrix} \tilde{u}_i \\ \tilde{\beta}_{1i} \\ \tilde{\beta}_{2i} \end{bmatrix}. \qquad (17.70)$$

For the other two components we obtain similar equations. The further computations require the Jacobi matrix and determinant [1,4,5]:

$$\begin{bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial \zeta} \end{bmatrix} = \underline{\underline{J}} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix}, \text{ and: } \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} = \underline{\underline{J}}^{-1} \begin{bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \\ \frac{\partial}{\partial \zeta} \end{bmatrix}. \qquad (17.71)$$

The elements of the Jacobi matrix can be obtained using Eq.(17.65). Also, the derivatives of the displacement components can be determined in the global coordinate system. For example, the derivatives of the component u in matrix form are:

$$\begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \\ \frac{\partial u}{\partial z} \end{bmatrix} = \begin{bmatrix} J_{11}^{(-1)} & J_{12}^{(-1)} & J_{13}^{(-1)} \\ J_{21}^{(-1)} & J_{22}^{(-1)} & J_{23}^{(-1)} \\ J_{31}^{(-1)} & J_{32}^{(-1)} & J_{33}^{(-1)} \end{bmatrix} \begin{bmatrix} \frac{\partial u}{\partial \xi} \\ \frac{\partial u}{\partial \eta} \\ \frac{\partial u}{\partial \zeta} \end{bmatrix}, \qquad (17.72)$$

where $J_{ij}^{(-1)}$ are the elements of the inverse Jacobi matrix. Based on Eq.(17.70) we obtain the following:

$$\frac{\partial u}{\partial x} = \sum_{i=1}^{8} (J_{11}^{(-1)} \frac{\partial N_i}{\partial \xi} + J_{12}^{(-1)} \frac{\partial N_i}{\partial \eta}) \tilde{u}_i - \frac{1}{2} t_i e_{2i}^x \left\{ \zeta (J_{11}^{(-1)} \frac{\partial N_i}{\partial \xi} + J_{12}^{(-1)} \frac{\partial N_i}{\partial \eta}) + J_{13}^{(-1)} N_i \right\} \tilde{\beta}_{1i} +$$
$$+ \frac{1}{2} t_i e_{1i}^x \left\{ \zeta (J_{11}^{(-1)} \frac{\partial N_i}{\partial \xi} + J_{12}^{(-1)} \frac{\partial N_i}{\partial \eta}) + J_{13}^{(-1)} N_i \right\} \tilde{\beta}_{2i} = \sum_{i=1}^{8} \frac{\partial N_i}{\partial x} \tilde{u}_i + G_i^x (g_{1i}^x \tilde{\beta}_{1i} + g_{2i}^x \tilde{\beta}_{2i}), \qquad (17.73)$$

where:

$$G_i^x = \zeta (J_{11}^{(-1)} \frac{\partial N_i}{\partial \xi} + J_{12}^{(-1)} \frac{\partial N_i}{\partial \eta}) + J_{13}^{(-1)} N_i, \qquad (17.74)$$

and:

$$\underline{g}_1^i = -\frac{1}{2} t_i \underline{e}_{2i}, \underline{g}_2^i = \frac{1}{2} t_i \underline{e}_{1i}. \qquad (17.75)$$

The derivatives with respect to the other two coordinates are:

$$\frac{\partial u}{\partial y} = \sum_{i=1}^{8} (J_{21}^{(-1)} \frac{\partial N_i}{\partial \xi} + J_{22}^{(-1)} \frac{\partial N_i}{\partial \eta}) \tilde{u}_i - \frac{1}{2} t_i e_{2i}^x \left\{ \zeta (J_{21}^{(-1)} \frac{\partial N_i}{\partial \xi} + J_{22}^{(-1)} \frac{\partial N_i}{\partial \eta}) + J_{23}^{(-1)} N_i \right\} \tilde{\beta}_{1i} +$$
$$+ \frac{1}{2} t_i e_{1i}^x \left\{ \zeta (J_{21}^{(-1)} \frac{\partial N_i}{\partial \xi} + J_{22}^{(-1)} \frac{\partial N_i}{\partial \eta}) + J_{23}^{(-1)} N_i \right\} \tilde{\beta}_{2i} = \sum_{i=1}^{8} \frac{\partial N_i}{\partial y} \tilde{u}_i + G_i^y (g_{1i}^x \tilde{\beta}_{1i} + g_{2i}^x \tilde{\beta}_{2i}), \qquad (17.76)$$

$$\frac{\partial u}{\partial z} = \sum_{i=1}^{8} (J_{31}^{(-1)} \frac{\partial N_i}{\partial \xi} + J_{32}^{(-1)} \frac{\partial N_i}{\partial \eta}) \tilde{u}_i - \frac{1}{2} t_i e_{2i}^x \left\{ \zeta (J_{31}^{(-1)} \frac{\partial N_i}{\partial \xi} + J_{32}^{(-1)} \frac{\partial N_i}{\partial \eta}) + J_{33}^{(-1)} N_i \right\} \tilde{\beta}_{1i} +$$
$$+ \frac{1}{2} t_i e_{1i}^x \left\{ \zeta (J_{31}^{(-1)} \frac{\partial N_i}{\partial \xi} + J_{32}^{(-1)} \frac{\partial N_i}{\partial \eta}) + J_{33}^{(-1)} N_i \right\} \tilde{\beta}_{2i} = \sum_{i=1}^{8} \frac{\partial N_i}{\partial z} \tilde{u}_i + G_i^z (g_{1i}^x \tilde{\beta}_{1i} + g_{2i}^x \tilde{\beta}_{2i}),$$

where:

$$G_i^y = \zeta (J_{21}^{(-1)} \frac{\partial N_i}{\partial \xi} + J_{22}^{(-1)} \frac{\partial N_i}{\partial \eta}) + J_{23}^{(-1)} N_i, \qquad (17.77)$$

$$G_i^z = \zeta\left(J_{31}^{(-1)}\frac{\partial N_i}{\partial \xi} + J_{32}^{(-1)}\frac{\partial N_i}{\partial \eta}\right) + J_{33}^{(-1)} N_i .$$

Written in matrix form we have:

$$\begin{bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \\ \dfrac{\partial u}{\partial z} \end{bmatrix} = \sum_{i=1}^{8} \begin{bmatrix} \dfrac{\partial N_i}{\partial x} & G_i^x g_{1i}^x & G_i^x g_{2i}^x \\ \dfrac{\partial N_i}{\partial y} & G_i^y g_{1i}^x & G_i^y g_{2i}^x \\ \dfrac{\partial N_i}{\partial z} & G_i^z g_{1i}^x & G_i^z g_{2i}^x \end{bmatrix} \begin{bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \\ \dfrac{\partial u}{\partial \zeta} \end{bmatrix}. \qquad (17.78)$$

The derivatives of the other two components can be provided similarly. Using the derivatives we can calculate matrix $\underline{\underline{B}}$, which is the relationship between the strain components and the nodal displacement parameters:

$$\underline{\tilde{\varepsilon}} = \underline{\underline{B}}\,\underline{\tilde{u}}_e , \qquad (17.79)$$

where $\underline{\tilde{u}}_e$ is the vector of nodal parameters in the local coordinate system. The vectors of strain and stress components in the global system are:

$$\underline{\varepsilon}^T = \begin{bmatrix} \varepsilon_x & \varepsilon_y & \varepsilon_z & \gamma_{xy} & \gamma_{yz} & \gamma_{xz} \end{bmatrix}, \qquad (17.80)$$
$$\underline{\sigma}^T = \begin{bmatrix} \sigma_x & \sigma_y & \sigma_z & \tau_{xy} & \tau_{yz} & \tau_{xz} \end{bmatrix}.$$

Hooke’s law in the local system can be written as:

$$\underline{\tilde{\sigma}} = \underline{\underline{C}}\,\underline{\tilde{\varepsilon}} , \qquad (17.81)$$

where $\underline{\underline{C}}$ is the constitutive matrix:

$$\underline{\underline{C}} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 & 0 & 0 & 0 \\ \nu & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \dfrac{1-\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & k\dfrac{1-\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & k\dfrac{1-\nu}{2} \end{bmatrix}. \qquad (17.82)$$

The matrix above differs from the general three dimensional case in accordance with the followings. The stress normal to the shell surface is zero (3rd row, 3rd column). Since the element is thick-walled it considers also the effect of transverse shear deformation, but only in the form of an average stress. The constant in the elements of the 5th row, 5th column, and the 6th row, 6th column is a shear correction factor, k = 5/6 [1,4,5]. The reason for that is the real distribution of the shear stresses is assumed to be parabolic over the thickness, and it is not constant as considered in the shell model. The correction factor k is the ratio of the strain energies from the two different distributions. Based on the transformation of local stress and strain components we can write the followings:

$$\underline{\tilde{\sigma}} = \underline{\underline{T}}\,\underline{\sigma} ,\ \underline{\sigma} = \underline{\underline{T}}^T\,\underline{\tilde{\sigma}} , \qquad (17.83)$$
$$\underline{\tilde{\varepsilon}} = \underline{\underline{T}}\,\underline{\varepsilon} ,\ \underline{\varepsilon} = \underline{\underline{T}}^T\,\underline{\tilde{\varepsilon}} ,$$

where $\underline{\underline{T}}$ is the transformation matrix for general spatial stress and strain states. The calculation of $\underline{\underline{T}}$ is possible using the definitions given by Eq.(11.62). Taking back Eq.(17.83) into Hooke's law we have:

$$\underline{\underline{T}}\,\underline{\sigma} = \underline{\underline{C}}\,\underline{\underline{T}}\,\underline{\varepsilon}\,. \tag{17.84}$$

The premultiplication with $\underline{\underline{T}}^{-1}$ leads to:

$$\underline{\underline{T}}^{-1}\underline{\underline{T}}\,\underline{\sigma} = \underline{\underline{T}}^{-1}\underline{\underline{C}}\,\underline{\underline{T}}\,\underline{\varepsilon}\,. \tag{17.85}$$

Since $\underline{\underline{T}}$ is an orthogonal matrix we can write that: $\underline{\underline{T}}^{-1}\underline{\underline{T}} = \underline{\underline{E}}$, and $\underline{\underline{T}}^{T} = \underline{\underline{T}}^{-1}$, viz.:

$$\underline{\sigma} = \underline{\underline{T}}^{T}\underline{\underline{C}}\,\underline{\underline{T}}\,\underline{\varepsilon}\,. \tag{17.86}$$

The transformation matrix is [4]:

$$\underline{\underline{T}} = \begin{bmatrix} l_{1i}^2 & m_{1i}^2 & n_{1i}^2 & l_{1i}m_{1i} & m_{1i}n_{1i} & n_{1i}l_{1i} \\ l_{2i}^2 & m_{2i}^2 & n_{2i}^2 & l_{2i}m_{2i} & m_{2i}n_{2i} & n_{2i}l_{2i} \\ l_{3i}^2 & m_{3i}^2 & n_{3i}^2 & l_{3i}m_{3i} & m_{3i}n_{3i} & n_{3i}l_{3i} \\ 2l_{1i}l_{2i} & 2m_{1i}m_{2i} & 2n_{1i}n_{2i} & l_{1i}m_{2i}+l_{2i}m_{1i} & m_{1i}n_{2i}+m_{2i}n_{1i} & n_{1i}l_{2i}+n_{2i}l_{1i} \\ 2l_{2i}l_{3i} & 2m_{2i}m_{3i} & 2n_{2i}n_{3i} & l_{2i}m_{3i}+l_{3i}m_{2i} & m_{2i}n_{3i}+m_{3i}n_{2i} & n_{2i}l_{3i}+n_{3i}l_{2i} \\ 2l_{3i}l_{1i} & 2m_{3i}m_{1i} & 2n_{3i}n_{1i} & l_{3i}m_{1i}+l_{1i}m_{3i} & m_{3i}n_{1i}+m_{1i}n_{3i} & n_{3i}l_{1i}+n_{1i}l_{3i} \end{bmatrix}, \tag{17.87}$$

where l_i, m_i and n_i are the direction cosines of the unit basis vectors at the actual point [4,7]:

$$l_{1i} = \cos(\underline{i},\underline{e}_{1i}),\; m_{1i} = \cos(\underline{j},\underline{e}_{1i}),\; n_{1i} = \cos(\underline{k},\underline{e}_{1i}), \tag{17.88}$$

$$l_{2i} = \cos(\underline{i},\underline{e}_{2i}),\; m_{2i} = \cos(\underline{j},\underline{e}_{2i}),\; n_{2i} = \cos(\underline{k},\underline{e}_{2i}),$$

$$l_{3i} = \cos(\underline{i},\underline{e}_{3i}),\; m_{3i} = \cos(\underline{j},\underline{e}_{3i}),\; n_{3i} = \cos(\underline{k},\underline{e}_{3i}).$$

The transformation matrix should be evaluated in the nodes, moreover due to the numerical integration even in the integration points. The stiffness matrix in the global coordinate system can be calculated using Eq.(15.17):

$$\underline{\underline{K}}_e = \int_{V_e}\underline{\underline{B}}^T\underline{\underline{T}}\,\underline{\underline{C}}^T\underline{\underline{T}}^T\underline{\underline{B}}\,dV = \int_{V_e}\underline{\underline{B}}^T\underline{\underline{T}}\,\underline{\underline{C}}^T\underline{\underline{T}}^T\underline{\underline{B}}\,Jd\xi d\eta d\zeta\,, \tag{17.89}$$

where J is the Jacobi determinant, which can be calculated using Eqs.(17.65) and (17.71). For the determination of the force vector we recall the displacement vector field in the usual form:

$$\underline{u}(\xi,\eta,\zeta) = \underline{\underline{N}}\,\underline{\tilde{u}}_e\,, \tag{17.90}$$

where $\underline{\underline{N}}$ is the matrix of interpolation polynomials. As a result, the vectors of body, surface and line forces in the global coordinate system are:

$$\underline{\tilde{F}}_{eb} = \int_{V_e}\underline{\underline{N}}^T\underline{p}_b\,dV = \int_{V_e}\underline{\underline{N}}^T\underline{p}_b\,Jd\xi d\eta d\zeta\,, \tag{17.91}$$

$$\underline{\tilde{F}}_{ep} = \int_{A_e}\underline{\underline{N}}^T\underline{p}_p\,dA = \int_{A_e}\underline{\underline{N}}^T\underline{p}_p\,Jd\xi d\eta\,,$$

$$\underline{\tilde{F}}_{el} = \int_{S}\underline{\underline{N}}^T\underline{p}_l\,dS\,,$$

which can be determined by transformation into the global system in a similar way to that presented in section 16. In the nodes concentrated forces may act, the relevant vector can be obtained in the same way as that shown in plate elements. Because of he high number of nodes it is not detailed here. The finite element equilibrium equation is formed in the usual way, for a single element it is:

$$\underline{\underline{K}}_e \underline{u}_e = \underline{F}_e , \qquad (17.92)$$

where $\underline{F}_e$ is the sum of the vectors of body, surface, line and concentrated forces. Finally, the structural equation is:

$$\underline{\underline{K}}\underline{U} = \underline{F} . \qquad (17.93)$$

17.5 A shell-solid transition element

In complex structures sometimes there is the necessity of the simultaneous application of solid and thick-walled shell elements. These elements can not be connected directly, because the nodal degrees of freedom are not identical. In these cases it is reasonable to use a transition element between the solid and shell elements [1,4,5]. A quadratic transition element is shown in Fig.17.5, where the nodes 1-8 are located in the solid side, nodes 10-12 care located in the shell side of the element.

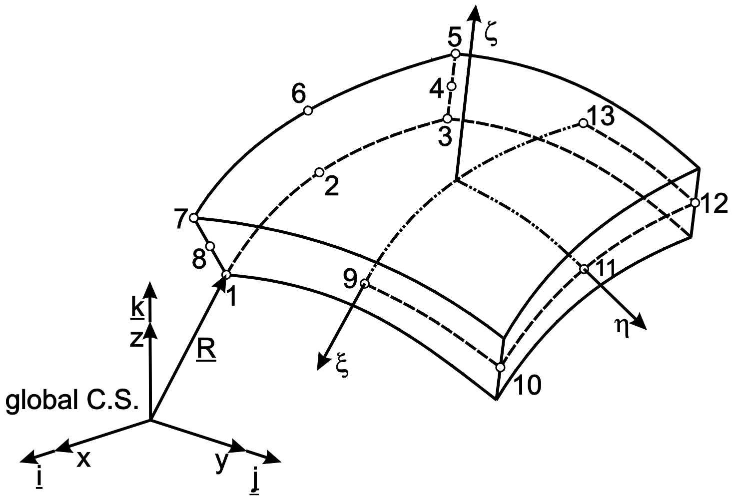

Fig.17.5. A shell-solid transition element.

The geometry of the transition element is captured by the function below:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \sum_{i=1}^{8} N_i(\xi,\eta,\zeta) \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} + \sum_{i=9}^{13} N_i(\xi,\eta) \begin{bmatrix} x_i \\ y_i \\ z_i \end{bmatrix} + \sum_{i=9}^{13} N_i(\xi,\eta) \frac{\zeta}{2} t_i \underline{n}_i . \qquad (17.94)$$

The indices i = 1...8 refer to the interpolation function of the solid element given by Eq.(17.64), if i = 9...13 then the actual interpolation functions of the thick-walled shell elements are referred to in accordance with Eq.(17.65). The composed system of functions satisfies the following conditions [1]:

$$\sum_{i=1}^{8} N_i(\xi,\eta,\zeta) + \sum_{i=9}^{13} N_i(\xi,\eta) = 1 , \qquad (17.95)$$

$$N_i(\xi_j,\eta_j,\zeta_j) = \begin{cases} 0, & \text{if} \quad i \neq j \\ 1, & \text{if} \quad i = j \end{cases} ,$$

$$N_i(\xi_j,\eta_j)=\begin{cases}0, & \text{if} \quad i\neq j\\ 1, & \text{if} \quad i=j\end{cases},$$

where ξ_j, η_j and ζ_j are the nodal coordinates in the local coordinate system. Similarly to the thick-walled shell elements the displacement field is expressed by:

$$\begin{bmatrix}u\\ v\\ w\end{bmatrix}=\sum_{i=1}^{8}N_i(\xi,\eta,\zeta)\begin{bmatrix}\tilde{u}_i\\ \tilde{v}_i\\ \tilde{w}_i\end{bmatrix}+\sum_{i=8}^{13}N_i(\xi,\eta)\begin{bmatrix}\tilde{u}_i\\ \tilde{v}_i\\ \tilde{w}_i\end{bmatrix}+\sum_{i=8}^{13}N_i(\xi,\eta)\frac{\zeta}{2}t_i(\tilde{\beta}_{2i}\underline{e}_{1i}-\tilde{\beta}_{1i}\underline{e}_{2i})\,. \tag{17.96}$$

The degrees of freedom in nodes 1-8 are equal to three, in nodes 9-13 there are five degrees of freedom. Consequently the transition element has 49 degrees of freedom. The further calculations can be performed in similar fashion to that presented in the thick-walled shell element.

17.6 Bibliography

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