16. MODELING OF SPATIAL THIN-WALLED SHELLS BY FINITE ELEMENT METHOD-BASED SOFTWARE SYSTEMS

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16.1 Simple flat shell elements

The stiffness matrix of flat shell elements are easily calculated using the stiffness matrices of the membrane and plate bending elements. Accordingly, it is possible to derive the different version of flat shell finite elements by combining the available triangle and rectangle shape elements [1,2]. The approximation of a curved surface by flat shell elements is shown by Fig.16.1. This kind of approximation is another source of error apart from the displacement field interpolation. By increasing the number of elements we can decrease the geometrical inaccuracies. The application of flat shell elements is justified, when the advantage of the higher order elements – namely the larger element size – can not be exploited. In the sequel we demonstrate the combination of the linear (membrane) triangle and the Tocher plate (bending) elements.

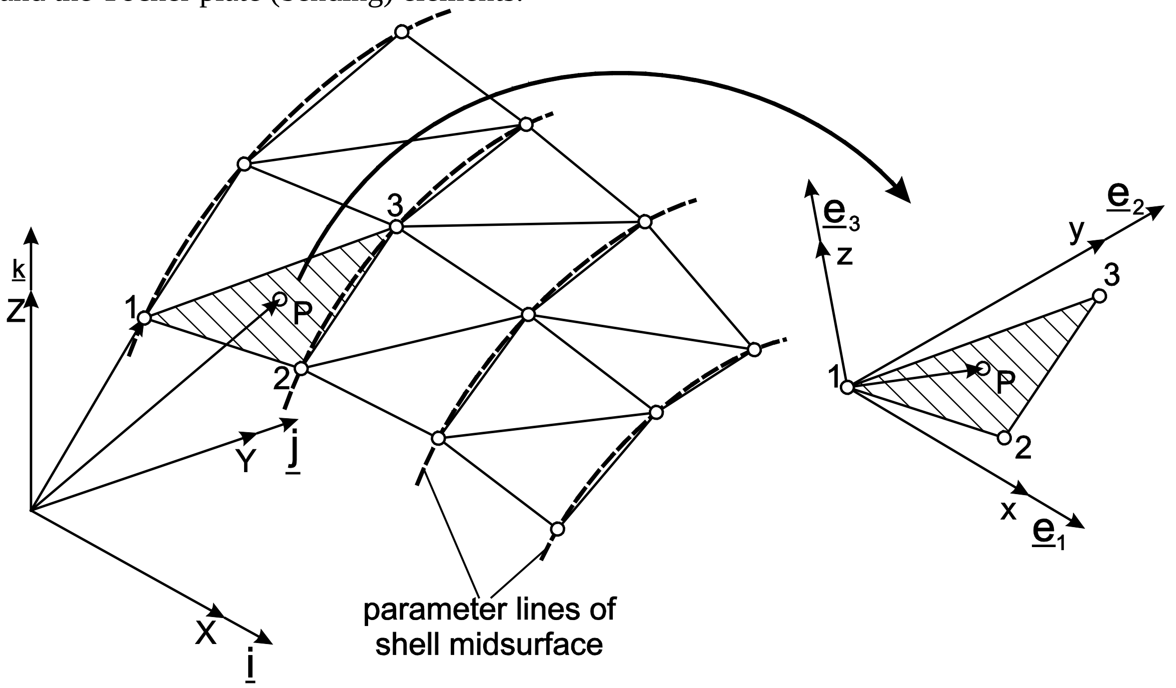

Fig.16.1. Triangular shape flat shell element in the global and local coordinate systems.

16.2 Superposition of the linear triangle and Tocher bending plate elements

The element mentioned above is not conform because of the discontinuity of displacements at the element boundaries [1,2]. However, due to its simplicity we use this combination to demonstrate the application of flat shell elements. The linear triangular membrane element (see Fig.12.2) has two degrees of freedom at each node, the stiffness matrix in the local element coordinate system is:

$$\underset{6\times 6}{\underline{\underline{\tilde{K}}}_e^m} = \begin{bmatrix} \left\{\begin{matrix}\tilde{k}_{11}^m \\ 2\times 2\end{matrix}\right\} & \left\{\begin{matrix}\tilde{k}_{12}^m \\ 2\times 2\end{matrix}\right\} & \left\{\begin{matrix}\tilde{k}_{13}^m \\ 2\times 2\end{matrix}\right\} \\ \left\{\begin{matrix}\tilde{k}_{21}^m \\ 2\times 2\end{matrix}\right\} & \left\{\begin{matrix}\tilde{k}_{22}^m \\ 2\times 2\end{matrix}\right\} & \left\{\begin{matrix}\tilde{k}_{23}^m \\ 2\times 2\end{matrix}\right\} \\ \left\{\begin{matrix}\tilde{k}_{31}^m \\ 2\times 2\end{matrix}\right\} & \left\{\begin{matrix}\tilde{k}_{32}^m \\ 2\times 2\end{matrix}\right\} & \left\{\begin{matrix}\tilde{k}_{33}^m \\ 2\times 2\end{matrix}\right\} \end{bmatrix}, \qquad (16.1)$$

where the submatrices ($\tilde{k}_{ij}^m$) correspond to the stiffness matrix components associated with nodes *i* and *j*. The tilde over the matrix indicates the local coordinate system; the superscript (*m*) refers to the membrane action. The finite element equation is:

$$\underset{6\times 6}{\underline{\underline{\tilde{K}}}_e^m}\, \underset{6\times 1}{\underline{\tilde{u}}_e^m} = \underset{6\times 1}{\underline{\tilde{F}}_e^m}, \qquad (16.2)$$

where the vector of nodal displacements and concentrated forces of the membrane element are:

$$\underset{6\times 1}{\underline{\tilde{u}}_e^{m\,T}} = \begin{bmatrix}\tilde{u}_1 & \tilde{v}_1 & \tilde{u}_2 & \tilde{v}_2 & \tilde{u}_3 & \tilde{v}_3\end{bmatrix}, \qquad (16.3)$$

$$\underset{6\times 1}{\underline{\tilde{F}}_{ec}^{m\,T}} = \begin{bmatrix}F_{x1} & F_{y1} & F_{x2} & F_{y1} & F_{x3} & F_{y3}\end{bmatrix},$$

where *u* is the displacement in the local *x*, *v* is the displacement in the local *y* direction. In the displacement vector we refer to the local parameters by using the tilde. In the force vector we identify the local parameters by lowercase *x*, *y* and *z* in the subscript of components. A distinction like that was not necessary until now, which can be explained by the fact that in all of the previous examples the local and global coordinate systems coincided.
At each node of the Tocher triangular plate element (see Fig.15.1) there are three degrees of freedom; therefore the stiffness matrix has nine rows and nine columns:

$$\underset{6\times 6}{\underline{\underline{\tilde{K}}}_e^b} = \begin{bmatrix} \left\{\begin{matrix}\tilde{k}_{11}^b \\ 3\times 3\end{matrix}\right\} & \left\{\begin{matrix}\tilde{k}_{12}^b \\ 3\times 3\end{matrix}\right\} & \left\{\begin{matrix}\tilde{k}_{13}^b \\ 3\times 3\end{matrix}\right\} \\ \left\{\begin{matrix}\tilde{k}_{21}^b \\ 3\times 3\end{matrix}\right\} & \left\{\begin{matrix}\tilde{k}_{22}^b \\ 3\times 3\end{matrix}\right\} & \left\{\begin{matrix}\tilde{k}_{23}^b \\ 3\times 3\end{matrix}\right\} \\ \left\{\begin{matrix}\tilde{k}_{31}^b \\ 3\times 3\end{matrix}\right\} & \left\{\begin{matrix}\tilde{k}_{32}^b \\ 3\times 3\end{matrix}\right\} & \left\{\begin{matrix}\tilde{k}_{33}^b \\ 3\times 3\end{matrix}\right\} \end{bmatrix}, \qquad (16.4)$$

where the superscript *b* indicates bending action. In the local coordinate system the displacement and concentrated force vectors of the Tocher triangular plate element are:

$$\underset{9\times 1}{\underline{\tilde{u}}_e^{h\,T}} = \begin{bmatrix}\tilde{w}_1 & \tilde{\alpha}_1 & \tilde{\beta}_1 & \tilde{w}_2 & \tilde{\alpha}_2 & \tilde{\beta}_2 & \tilde{w}_3 & \tilde{\alpha}_3 & \tilde{\beta}_3\end{bmatrix}, \qquad (16.5)$$

$$\underset{9\times 1}{\underline{\tilde{F}}_{ec}^{h\,T}} = \begin{bmatrix}F_{z1} & M_{x1} & M_{y1} & F_{z2} & M_{x2} & M_{y2} & F_{z3} & M_{x3} & M_{y3}\end{bmatrix}.$$

The degrees of freedom of the combined element are shown by Fig.16.2. The membrane and bending stiffness matrices have to be combined in accordance with the following observations [2]:

a. *for small displacement s the membrane and bending stiffnesses are uncoupled (independent),*
b. *the in-plane rotation ψ in the local x-y plane is not necessary for a single*

element, however, ψ and its conjugate moment M_z have to be considered in the analysis by including the appropriate number of zeros to obtain the element stiffness matrix for the purpose of assembling several elements or assembling the flat shell element with different type of elements.

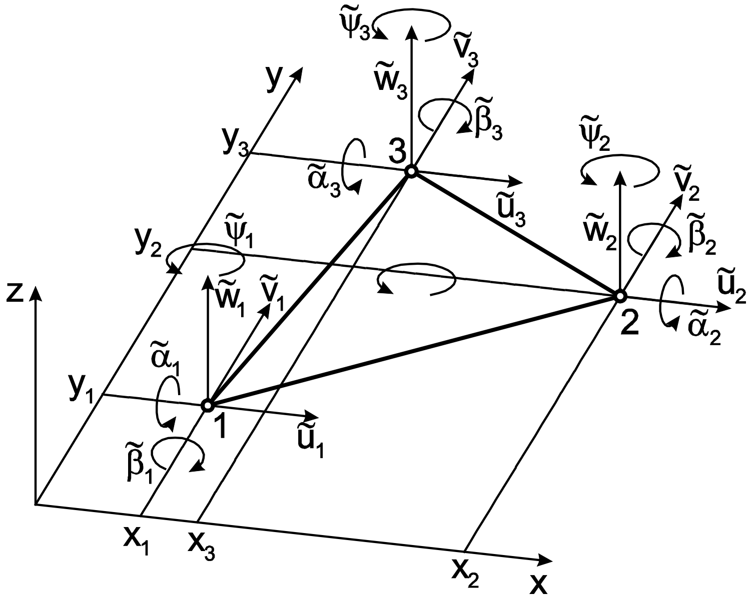

Fig.16.2. Combination of the linear membrane triangle element and the Tocher triangular plate bending element.

The nodal displacement vector of the combined element in the local coordinate system is:

$$\underset{18\times 1}{\underline{\tilde{u}}_e^{m+b\,T}} = \left[\tilde{u}_1 \quad \tilde{v}_1 \quad \tilde{w}_1 \quad \tilde{\alpha}_1 \quad \tilde{\beta}_1 \quad \tilde{\psi}_1 \quad \tilde{u}_2 \quad \tilde{v}_2 \quad \tilde{w}_2 \ldots\ldots\ldots\ldots\ldots\ldots \right.$$
$$\left.\ldots\ldots\ldots\ldots\ldots \tilde{\alpha}_2 \quad \tilde{\beta}_2 \quad \tilde{\psi}_2 \quad \tilde{u}_3 \quad \tilde{v}_3 \quad \tilde{w}_3 \quad \tilde{\alpha}_3 \quad \tilde{\beta}_3 \quad \tilde{\psi}_3 \right] \tag{16.6}$$

The vector of concentrated forces becomes:

$$\underset{18\times 1}{\underline{\tilde{F}}_{ec}^{m+b\,T}} = \left[F_{x1} \quad F_{y1} \quad F_{z1} \quad M_{x1} \quad M_{y1} \quad M_{z1} \quad F_{x2} \quad F_{y2} \quad F_{z2} \ldots\ldots\ldots\ldots\ldots\ldots \right.$$
$$\left.\ldots\ldots\ldots\ldots\ldots M_{x2} \quad M_{y2} \quad M_{y2} \quad F_{x3} \quad F_{y3} \quad F_{z3} \quad M_{x3} \quad M_{y3} \quad M_{z3} \right] \tag{16.7}$$

Accordingly, the stiffness matrix is shown below [1,2]:

$$\underset{18\times 18}{\underline{\underline{\tilde{K}}}_e^{m+b}} = \left[\begin{array}{c|c|c|c|c|c|c|c|c}
\left\{\begin{matrix}\tilde{k}_{11}^m \\ 2\times 2\end{matrix}\right\} & \begin{matrix}0&0&0\\0&0&0\end{matrix} & \begin{matrix}0\\0\end{matrix} & \left\{\begin{matrix}\tilde{k}_{12}^m \\ 2\times 2\end{matrix}\right\} & \begin{matrix}0&0&0\\0&0&0\end{matrix} & \begin{matrix}0\\0\end{matrix} & \left\{\begin{matrix}\tilde{k}_{13}^m \\ 2\times 2\end{matrix}\right\} & \begin{matrix}0&0&0\\0&0&0\end{matrix} & \begin{matrix}0\\0\end{matrix} \\ \hline
\begin{matrix}0&0\\0&0\\0&0\end{matrix} & \left\{\begin{matrix}\tilde{k}_{11}^b \\ 3\times 3\end{matrix}\right\} & \begin{matrix}0\\0\\0\end{matrix} & \begin{matrix}0&0\\0&0\\0&0\end{matrix} & \left\{\begin{matrix}\tilde{k}_{12}^b \\ 3\times 3\end{matrix}\right\} & \begin{matrix}0\\0\\0\end{matrix} & \begin{matrix}0&0\\0&0\\0&0\end{matrix} & \left\{\begin{matrix}\tilde{k}_{13}^b \\ 3\times 3\end{matrix}\right\} & \begin{matrix}0\\0\\0\end{matrix} \\ \hline
\begin{matrix}0&0\end{matrix} & \begin{matrix}0&0&0\end{matrix} & 0 & \begin{matrix}0&0\end{matrix} & \begin{matrix}0&0&0\end{matrix} & 0 & \begin{matrix}0&0\end{matrix} & \begin{matrix}0&0&0\end{matrix} & 0 \\ \hline
\left\{\begin{matrix}\tilde{k}_{21}^m \\ 2\times 2\end{matrix}\right\} & \begin{matrix}0&0&0\\0&0&0\end{matrix} & \begin{matrix}0\\0\end{matrix} & \left\{\begin{matrix}\tilde{k}_{22}^m \\ 2\times 2\end{matrix}\right\} & \begin{matrix}0&0&0\\0&0&0\end{matrix} & \begin{matrix}0\\0\end{matrix} & \left\{\begin{matrix}\tilde{k}_{23}^m \\ 2\times 2\end{matrix}\right\} & \begin{matrix}0&0&0\\0&0&0\end{matrix} & \begin{matrix}0\\0\end{matrix}
\end{array}\right.$$

$$
\begin{bmatrix}
\begin{matrix}0&0\\0&0\\0&0\end{matrix} & \left\{\begin{matrix}\tilde{k}^{b}_{21}\\ 3\times3\end{matrix}\right\} & \begin{matrix}0\\0\\0\end{matrix} & \begin{matrix}0&0\\0&0\\0&0\end{matrix} & \left\{\begin{matrix}\tilde{k}^{b}_{22}\\ 3\times3\end{matrix}\right\} & \begin{matrix}0\\0\\0\end{matrix} & \begin{matrix}0&0\\0&0\\0&0\end{matrix} & \left\{\begin{matrix}k^{b}_{23}\\ 3\times3\end{matrix}\right\} & \begin{matrix}0\\0\\0\end{matrix} \\
\begin{matrix}0&0\end{matrix} & \begin{matrix}0&0&0\end{matrix} & 0 & \begin{matrix}0&0\end{matrix} & \begin{matrix}0&0&0\end{matrix} & 0 & \begin{matrix}0&0\end{matrix} & \begin{matrix}0&0&0\end{matrix} & 0 \\
\left\{\begin{matrix}\tilde{k}^{m}_{31}\\ 2\times2\end{matrix}\right\} & \begin{matrix}0&0&0\\0&0&0\end{matrix} & \begin{matrix}0\\0\end{matrix} & \left\{\begin{matrix}\tilde{k}^{m}_{32}\\ 2\times2\end{matrix}\right\} & \begin{matrix}0&0&0\\0&0&0\end{matrix} & \begin{matrix}0\\0\end{matrix} & \left\{\begin{matrix}\tilde{k}^{m}_{33}\\ 2\times2\end{matrix}\right\} & \begin{matrix}0&0&0\\0&0&0\end{matrix} & \begin{matrix}0\\0\end{matrix} \\
\begin{matrix}0&0\\0&0\\0&0\end{matrix} & \left\{\begin{matrix}\tilde{k}^{b}_{31}\\ 3\times3\end{matrix}\right\} & \begin{matrix}0\\0\\0\end{matrix} & \begin{matrix}0&0\\0&0\\0&0\end{matrix} & \left\{\begin{matrix}\tilde{k}^{b}_{32}\\ 3\times3\end{matrix}\right\} & \begin{matrix}0\\0\\0\end{matrix} & \begin{matrix}0&0\\0&0\\0&0\end{matrix} & \left\{\begin{matrix}\tilde{k}^{b}_{33}\\ 3\times3\end{matrix}\right\} & \begin{matrix}0\\0\\0\end{matrix} \\
\begin{matrix}0&0\end{matrix} & \begin{matrix}0&0&0\end{matrix} & 0 & \begin{matrix}0&0\end{matrix} & \begin{matrix}0&0&0\end{matrix} & 0 & \begin{matrix}0&0\end{matrix} & \begin{matrix}0&0&0\end{matrix} & 0
\end{bmatrix}
\qquad (16.8)
$$

The stiffness matrix above is valid in the local coordinate system. We highlight again, that the tilde over the matrices and vectors refers to the local system. In the analysis of three-dimensional structures in which different finite elements have different orientations, it is necessary to transform the local stiffness matrices to a common set of global coordinates. In the quantities of global coordinate system there is no tilde indicated. The transformation of the element stiffness matrix is given by the expression below:

$$\underset{18\times18}{\underline{\underline{K}}_{e}^{m+b}} = \underline{\underline{\lambda}}^{T}\,\underset{18\times18}{\underline{\underline{\tilde{K}}}_{e}^{m+b}}\,\underline{\underline{\lambda}}\,, \qquad (16.9)$$

where $\underline{\underline{\lambda}}$ is the transformation matrix with dimension of 18 x 18:

$$\underline{\underline{\lambda}} = \begin{bmatrix} \underline{\underline{L}} & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{L}} & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{L}} & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{L}} & \underline{\underline{0}} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{L}} & \underline{\underline{0}} \\ \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{0}} & \underline{\underline{L}} \end{bmatrix}, \qquad (16.10)$$

and:

$$\underline{\underline{0}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \qquad (16.11)$$

Matrix $\underline{\underline{L}}$ contains the unit basis vectors of the local coordinate system, $\underline{e}_1$, $\underline{e}_2$ and $\underline{e}_3$, (see Fig.16.1) in the form of column vectors formulated in the global coordinate system:

$$\underline{\underline{L}} = \begin{bmatrix} e_{1X} & e_{2X} & e_{3X} \\ e_{1Y} & e_{2Y} & e_{3Y} \\ e_{1Z} & e_{3Z} & e_{3Z} \end{bmatrix}. \qquad (16.12)$$

Eventually matrix $\underline{\underline{L}}$ contains the direction cosines of the angles between the local and global axes. The definition of the direction cosines for an optional $\underline{A}$ vector based on Fig.16.3 is [4]:

$$l = \cos\alpha = \frac{A_x}{\sqrt{A_x^2 + A_y^2 + A_z^2}}, \qquad (16.13)$$

$$m = \cos\beta = \frac{A_y}{\sqrt{A_x^2 + A_y^2 + A_z^2}},$$

$$n = \cos\gamma = \frac{A_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}}.$$

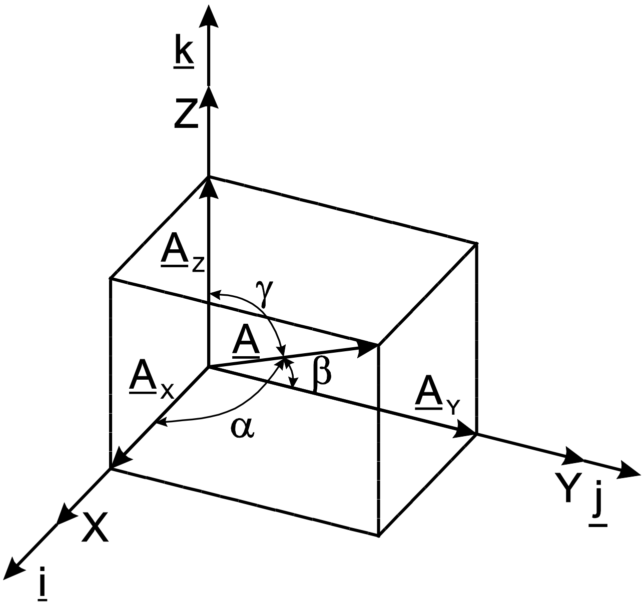

Fig.16.3. Direction cosines of vector A.

Since the basis vectors $\underline{e}_1$, $\underline{e}_2$ and $\underline{e}_3$ are unit vectors it is not easy to see, that their components are eventually the direction cosines. To construct the structural stiffness matrix the local quantities have to be transformed into the global system. The vector of nodal displacement and vector of forces in the global system are:

$$\underline{u}_e^{m+b} = \underline{\underline{\lambda}}^T \underline{\tilde{u}}_e^{m+b}, \qquad (16.14)$$

$$\underline{F}_e^{m+b} = \underline{\underline{\lambda}}^T \underline{\tilde{F}}_e^{m+b}.$$

For shell structures the most common load type is the constant pressure perpendicularly to the shell surface, i.e. in direction of the local z axis. It is a reasonable assumption, that there is membrane stress state, under these assumptions the force vector is: [2]:

$$\underline{\tilde{F}}_{ep}^{m+b} = p\frac{A_e}{3}[0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0], \qquad (16.15)$$

where A_e is the triangular area. If the pressure on the shell surface is not constant, but its change over the element area is insignificant, then we can still use the vector above, but we use the pressure averaged by the nodal loads instead of p:

$$p = \frac{1}{3}(p_1 + p_2 + p_3). \qquad (16.16)$$

Finally we summarize the finite element equations. In the local x, y, z system the quantities indicated by the tilde are used, i.e.:

$$\underline{\underline{\tilde{K}}}_e^{m+b} \underline{\tilde{u}}_e^{m+b} = \underline{\tilde{F}}_e^{m+b}. \qquad (16.17)$$

Transforming Eq.(16.17) into the global *X,Y,Z* system by using the transformation matrix, we have:

$$\underline{\underline{K}}_e^{m+b}\underline{u}_e^{m+b}=\underline{F}_e^{m+b}, \qquad (16.18)$$

where the quantities in the global coordinate system are calculated based on Eqs.(16.9) and (16.14). For the whole structure the finite element equation is:

$$\underline{\underline{K}}^{m+b}\underline{U}^{m+b}=\underline{F}^{m+b}, \qquad (16.19)$$

of which solutions are the components of vector $\underline{U}^{m+b}$, which are the nodal displacements in the global coordinate system. From that we can calculate the global element displacement vectors, $\underline{u}_e^{m+b}$, and then we can transform them into the local system by the following expression:

$$\underline{\tilde{u}}_e^{m+b}=\underline{\underline{\lambda}}^{T^{-1}}\underline{u}_e^{m+b}. \qquad (16.20)$$

The transformation matrix is orthogonal, therefore we can write that: $\underline{\underline{\lambda}}^{-1}=\underline{\underline{\lambda}}^{T}$ and $\underline{\underline{\lambda}}^{T^{-1}}\underline{\underline{\lambda}}^{T}=\underline{\underline{E}}$, viz.:

$$\underline{\tilde{u}}_e^{m+b}=\underline{\underline{\lambda}}\underline{u}_e^{m+b}. \qquad (16.21)$$

Using the local displacements in the nodes we can calculate the membrane and bending stresses.

A significant advantage of the flat shell elements is a novel software can be easily constructed by combining the softwares of the existing membrane and plate elements, which can be used for engineering calculations [2,3]. This computation requires only the knowledge of matrix $\underline{\underline{L}}$. The accuracy of the results depends on the element size. Higher mesh resolution is necessary, where the curvature of the surface is larger, or the change in stresses is expected to be more significant. The expected error of the calculation is higher in the vicinity of the sides, notches and the connection of different surfaces. Let us solve an example to understand the application of the method!

16.3 Example for the combination of the linear triangle and Tocher triangle elements

Solve the shell problem given in Fig.16.4! Calculate the nodal displacements, reactions in the local coordinate system, and transform the results into the global coordinate system!

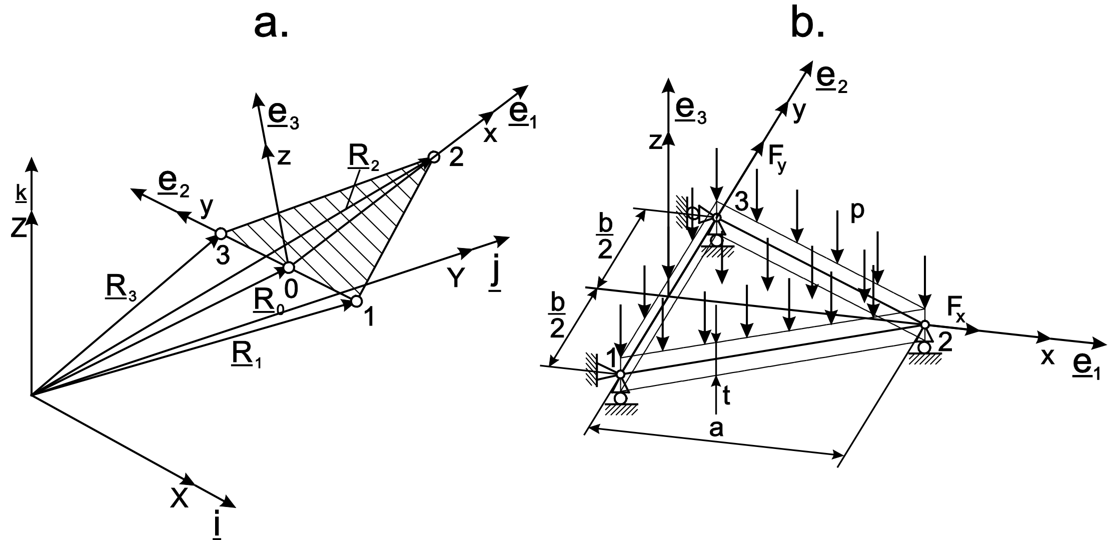

Fig.16.4. Flat triangular shell element in the local and global coordinate systems (a), application example for the flat shell element (b).

Given:

a = 0,8 m, b = 0,5 m, t = 3 mm, E = 200 GPa, ν = 0,3, F_x = 6000 kN, F_y = 8000 kN, p = 1200 N/m^2

The distances are substituted in [m], the force is interpreted in [N]. The nodal coordinates in the local coordinate system are:

node	*x*	*y*	*z*
1	0	-b/2	0
2	a	0	0
3	0	b/2	0

We give the nodal coordinates also in the global coordinate system. We note that the global coordinates depend on how the element is built-in the actual structure.

node	*X* [m]	*Y* [m]	*Z* [m]
1	0,6795	0,57	0,2
2	0,4	1,225	0,5
3	0,5004	0,6	0,4

The vectors of nodal displacements and concentrated forces in the local coordinate system are:

$$\underset{18\times1}{\underline{\tilde{u}}_e^{m+b\,T}} = \left[0 \quad 0 \quad 0 \quad \tilde{\alpha}_1 \quad \tilde{\beta}_2 \quad \tilde{\psi}_1 \quad \tilde{u}_2 \quad \tilde{v}_2 \quad 0 \quad \tilde{\alpha}_2 \quad \tilde{\beta}_2 \quad \tilde{\psi}_2 \quad 0 \quad \tilde{v}_3 \quad 0 \quad \tilde{\alpha}_3 \quad \tilde{\beta}_3 \quad \tilde{\psi}_3\right], \tag{16.22}$$

$$\underset{18\times1}{\underline{\tilde{F}}_{ec}^{m+b\,T}} = \left[F_{x1} \quad F_{y1} \quad F_{z1} \quad 0 \quad 0 \quad 0 \quad F_x \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad F_{x3} \quad F_y \quad F_{z3} \quad 0 \quad 0 \quad 0\right].$$

The terms in. the force vector related to the distributed load can be calculated based on the integral transformation formulae presented in section 15. We can construct the force vector of the Tocher triangle from distributed load by formulating the work of the distributed load:

$$W_e = \int_{A_{pe}} p w(x,y) dA = \underline{\tilde{u}}_e^T \underline{\tilde{F}}_{ep}. \tag{16.23}$$

Based on Eq.(16.23) and the integral transformation expressions given by Eq.(15.20) we have:

$$\underset{9\times 1}{\underline{\tilde{F}}_{ep}^{b}} = p\begin{bmatrix} 7/40ba - 1/80b^2 \\ 1/120ab^2 - 1/40ba^2 \\ 1/30ab^3 - 1/480b^3 \\ 3/20ba \\ 1/30ba^2 \\ -1/120b^3 \\ 7/40ba + 1/80b^2 \\ -1/120ab^2 - 1/40ba^2 \\ -1/30ab^2 - 1/480b^3 \end{bmatrix} = p\begin{bmatrix} 0{,}0669 \\ -0{,}0063 \\ 0{,}0064 \\ 0{,}060 \\ 0{,}0107 \\ -0{,}0010 \\ 0{,}0731 \\ -0{,}0097 \\ -0{,}0069 \end{bmatrix}. \qquad (16.24)$$

We note that in the Tocher triangle the distributed load is divided into three parts and put into the nodes; however the division is made in unequal degree, as it is seen in the vector of forces. On the other hand, by summing the forces in direction z and the moments about x and y we obtain:

$$\left[\underline{\tilde{F}}_{ep}^{b}\right]_1 + \left[\underline{\tilde{F}}_{ep}^{b}\right]_4 + \left[\underline{\tilde{F}}_{ep}^{b}\right]_7 = pA_e = \frac{1}{2}pab\,, \qquad (16.25)$$

$$\left[\underline{\tilde{F}}_{ep}^{b}\right]_2 + \left[\underline{\tilde{F}}_{ep}^{b}\right]_5 + \left[\underline{\tilde{F}}_{ep}^{b}\right]_8 = -\frac{pa}{30}A_e = -\frac{pa^2b}{15}\,,$$

$$\left[\underline{\tilde{F}}_{ep}^{b}\right]_2 + \left[\underline{\tilde{F}}_{ep}^{b}\right]_5 + \left[\underline{\tilde{F}}_{ep}^{b}\right]_8 = -\frac{pb^2}{40a}A_e = -\frac{pab^2}{20}\,,$$

which are the resultant forces in direction z and the resultant moments about axes x and y. The force vector of the 18 degrees of freedom flat shell element in the local coordinate system is:

$$\underset{18\times 1}{\tilde{\underline{F}}_{e}^{m+b\,T}}=\underset{18\times 1}{\tilde{\underline{F}}_{ec}^{m+b\,T}}+\underset{18\times 1}{\tilde{\underline{F}}_{ep}^{m+b\,T}}=\begin{bmatrix}F_{x1}\\F_{y1}\\F_{z1}\\0\\0\\0\\F_{x}\\0\\F_{z2}\\0\\0\\0\\F_{x3}\\F_{y}\\F_{z3}\\0\\0\\0\end{bmatrix}-p\begin{bmatrix}0\\0\\0.0069\\-0.0063\\0.0064\\0\\0\\0\\0.0600\\0.0107\\-0.0010\\0\\0\\0\\0.0731\\-0.0097\\-0.0069\\0\end{bmatrix}=\begin{bmatrix}F_{x1}\\F_{y1}\\F_{z1}-80{,}25\\7{,}6\\-7{,}6875\\0\\6000000\\0\\F_{z2}-72{,}0\\-12{,}8\\1{,}25\\0\\F_{x3}\\8000000\\F_{z3}-87{,}75\\11{,}6\\8{,}3125\\0\end{bmatrix}. \tag{16.26}$$

The stiffness matrix of the linear (membrane) triangle element based on the calculations of section 15 is:

$$\underset{6\times 6}{\tilde{\underline{\underline{K}}}_{e}^{m}}=\begin{bmatrix}2{,}48 & 1{,}44 & -1{,}26 & -1{,}15 & -1{,}22 & -0{,}29\\ . & 6{,}64 & -1{,}73 & -0{,}36 & 0{,}29 & -6{,}28\\ . & . & 2{,}52 & 0 & -1{,}26 & 1{,}73\\ . & . & . & 0{,}72 & 1{,}15 & -0{,}36\\ . & . & . & . & 2{,}48 & -1{,}44\\ . & . & . & . & . & 6{,}64\end{bmatrix}\cdot 10^{8}\,\frac{\mathrm{N}}{\mathrm{m}}, \tag{16.27}$$

where due to symmetry of the matrix only the independent components are indicated. The stiffness matrix of the Tocher triangular element in the local coordinate system is:

$$\underset{\underset{9\times 9}{=}}{\tilde{K}}{}_e^{b}=\begin{bmatrix}12{,}90 & 2{,}28 & -0{,}91 & -0{,}99 & 0{,}88 & -3{,}88 & -11{,}90 & 3{,}05 & 0{,}51\\ . & 0{,}96 & 0{,}10 & -0{,}33 & -0{,}09 & -0{,}05 & -1{,}95 & 0{,}19 & -0{,}32\\ . & . & 0{,}59 & 0{,}53 & -0{,}17 & 0{,}16 & 0{,}38 & -0{,}26 & -0{,}32\\ . & . & . & 2{,}90 & 0{,}15 & 0{,}77 & -1{,}90 & 0{,}40 & 1{,}01\\ . & . & . & . & 0{,}29 & -0{,}0007 & -1{,}03 & 0{,}28 & 0{,}29\\ . & . & . & . & . & 0{,}31 & -0{,}38 & 0{,}05 & 0{,}15\\ . & . & . & . & . & . & 13{,}80 & -3{,}45 & -1{,}52\\ . & . & . & . & . & . & . & 1{,}15 & 0{,}53\\ . & . & . & . & . & . & . & . & 0{,}97\end{bmatrix}\cdot 10^3. \tag{16.28}$$

The combination of the membrane and bending stiffness matrices based on Eq.(16.8) lead s to:

$$\underset{\underset{18\times 18}{=}}{\tilde{K}}{}_e^{m+b}=$$

$$\left[\begin{array}{c|c|c|c|c|c|c|c|c}
\begin{matrix}2{,}48 & 1{,}44\\ 1{,}44 & 6{,}64\end{matrix}\cdot 10^8 & \begin{matrix}0&0&0\\0&0&0\end{matrix} & \begin{matrix}0\\0\end{matrix} & \begin{matrix}-1{,}26 & -1{,}15\\ -1{,}73 & -0{,}36\end{matrix}\cdot 10^8 & \begin{matrix}0&0&0\\0&0&0\end{matrix} & \begin{matrix}0\\0\end{matrix} & \begin{matrix}-1{,}22 & -0{,}29\\ 0{,}29 & -6{,}28\end{matrix}\cdot 10^8 & \begin{matrix}0&0&0\\0&0&0\end{matrix} & \begin{matrix}0\\0\end{matrix}\\ \hline
\begin{matrix}0&0\\0&0\\0&0\end{matrix} & \begin{matrix}12{,}9 & 2{,}28 & -0{,}91\\ 2{,}28 & 0{,}96 & 0{,}10\\ -0{,}91 & 0{,}10 & 0{,}59\end{matrix}\cdot 10^3 & \begin{matrix}0\\0\\0\end{matrix} & \begin{matrix}0&0\\0&0\\0&0\end{matrix} & \begin{matrix}-0{,}99 & 0{,}88 & -3{,}88\\ -0{,}33 & -0{,}09 & -0{,}05\\ 0{,}52 & -0{,}17 & 0{,}16\end{matrix}\cdot 10^3 & \begin{matrix}0\\0\\0\end{matrix} & \begin{matrix}0&0\\0&0\\0&0\end{matrix} & \begin{matrix}-11{,}9 & 3{,}05 & 0{,}51\\ -1{,}95 & 0{,}19 & -0{,}32\\ 0{,}38 & -0{,}26 & -0{,}32\end{matrix}\cdot 10^3 & \begin{matrix}0\\0\\0\end{matrix}\\ \hline
0\quad 0 & 0\quad 0\quad 0 & 0 & 0\quad 0 & 0\quad 0\quad 0 & 0 & 0\quad 0 & 0\quad 0\quad 0 & 0\\ \hline
\begin{matrix}-1{,}26 & -1{,}73\\ -1{,}15 & -0{,}36\end{matrix}\cdot 10^8 & \begin{matrix}0&0&0\\0&0&0\end{matrix} & \begin{matrix}0\\0\end{matrix} & \begin{matrix}2{,}52 & 0\\ 0 & 0{,}72\end{matrix}\cdot 10^8 & \begin{matrix}0&0&0\\0&0&0\end{matrix} & \begin{matrix}0\\0\end{matrix} & \begin{matrix}-1{,}26 & 1{,}73\\ 1{,}15 & -0{,}36\end{matrix}\cdot 10^8 & \begin{matrix}0&0&0\\0&0&0\end{matrix} & \begin{matrix}0\\0\end{matrix}\\ \hline
\begin{matrix}0&0\\0&0\\0&0\end{matrix} & \begin{matrix}-0{,}99 & -0{,}33 & 0{,}52\\ 0{,}88 & -0{,}09 & -0{,}17\\ -3{,}88 & -0{,}05 & 0{,}16\end{matrix}\cdot 10^3 & \begin{matrix}0\\0\\0\end{matrix} & \begin{matrix}0&0\\0&0\\0&0\end{matrix} & \begin{matrix}2{,}90 & 0{,}15 & 0{,}77\\ 0{,}15 & 0{,}29 & -0{,}0007\\ 0{,}77 & -0{,}0007 & 0{,}31\end{matrix}\cdot 10^3 & \begin{matrix}0\\0\\0\end{matrix} & \begin{matrix}0&0\\0&0\\0&0\end{matrix} & \begin{matrix}-1{,}90 & 0{,}40 & 1{,}01\\ -1{,}03 & 0{,}28 & 0{,}29\\ -0{,}38 & 0{,}05 & 0{,}15\end{matrix}\cdot 10^3 & \begin{matrix}0\\0\\0\end{matrix}\\ \hline
0\quad 0 & 0\quad 0\quad 0 & 0 & 0\quad 0 & 0\quad 0\quad 0 & 0 & 0\quad 0 & 0\quad 0\quad 0 & 0\\ \hline
\begin{matrix}-1{,}22 & 0{,}29\\ -0{,}29 & -6{,}28\end{matrix}\cdot 10^8 & \begin{matrix}0&0&0\\0&0&0\end{matrix} & \begin{matrix}0\\0\end{matrix} & \begin{matrix}-1{,}26 & 1{,}15\\ 1{,}73 & -0{,}36\end{matrix}\cdot 10^8 & \begin{matrix}0&0&0\\0&0&0\end{matrix} & \begin{matrix}0\\0\end{matrix} & \begin{matrix}2{,}48 & -1{,}44\\ -1{,}44 & 6{,}64\end{matrix}\cdot 10^8 & \begin{matrix}0&0&0\\0&0&0\end{matrix} & \begin{matrix}0\\0\end{matrix}\\ \hline
\begin{matrix}0&0\\0&0\\0&0\end{matrix} & \begin{matrix}-11{,}9 & -1{,}95 & 0{,}38\\ 3{,}05 & 0{,}19 & -0{,}26\\ 0{,}51 & -0{,}32 & -0{,}32\end{matrix}\cdot 10^3 & \begin{matrix}0\\0\\0\end{matrix} & \begin{matrix}0&0\\0&0\\0&0\end{matrix} & \begin{matrix}-1{,}90 & -1{,}03 & 0{,}38\\ 0{,}40 & 0{,}28 & 0{,}05\\ 1{,}01 & 0{,}29 & 0{,}15\end{matrix}\cdot 10^3 & \begin{matrix}0\\0\\0\end{matrix} & \begin{matrix}0&0\\0&0\\0&0\end{matrix} & \begin{matrix}13{,}8 & -3{,}45 & -1{,}52\\ -3{,}45 & 1{,}15 & 0{,}53\\ -1{,}52 & 0{,}53 & 0{,}97\end{matrix}\cdot 10^3 & \begin{matrix}0\\0\\0\end{matrix}\\ \hline
0\quad 0 & 0\quad 0\quad 0 & 0 & 0\quad 0 & 0\quad 0\quad 0 & 0 & 0\quad 0 & 0\quad 0\quad 0 & 0
\end{array}\right] \tag{16.29}$$

The next step is the construction of the finite element equation using Eq.(16.17). The nodal displacements are calculated from the 4th, 5th, 7th, 8th, 10th, 11th, 14th, 16th and 17th equations of the system of equations. The solutions are:

$$\tilde{\alpha}_1 = 0{,}00406 \text{ rad}, \tilde{\beta}_1 = -0{,}02278 \text{ rad}, \tag{16.30}$$

$$\tilde{u}_2 = 0{,}0187 \text{ m}, \tilde{v}_2 = 0{,}00368 \text{ m}, \tilde{\alpha}_2 = -0{,}0948 \text{ rad}, \tilde{\beta}_2 = 0{,}00319 \text{ rad},$$

$$\tilde{v}_3 = 0{,}00737 \text{ m}, \tilde{\alpha}_3 = 0{,}01833 \text{ rad}, \tilde{\beta}_3 = 0{,}01994 \text{ rad}.$$

In the knowledge of the displacements we can determine the reactions utilizing the 1st, 2nd, 3rd, 9th, 13th, and 15th equations:

$$F_{x1} = -3000000\ \mathrm{N}\,, F_{y1} = -8000000\ \mathrm{N}\,, F_{z1} = 91{,}88\ \mathrm{N}\,, \qquad (16.31)$$

$$F_{z2} = 74.34\ \mathrm{N}\,,\ F_{x3} = -3000000\ \mathrm{N}\,, F_{z3} = 73{,}78\ \mathrm{N}\,.$$

We note that the results by Eq.(16.31) are the components of the vector of concentrated forces, which is the first term in Eq.(16.26). Moreover the 6th, 12th and 18th component equations were not utilized here, which is explained by the fact that these equations are associated to the local rotations about axis *z*, and their values are zero in the local system. The total force vector by using Eq.(16.31) and Eq.(16.26) becomes:

$$\underset{18\times1}{\underline{\tilde{F}}_e^{m+b\;T}} = \begin{bmatrix} -3000000 \\ 8000000 \\ 11.629 \\ 7,6 \\ -7,6876 \\ 0 \\ 6000000 \\ 0 \\ 2,344 \\ -12,8 \\ 1,25 \\ 0 \\ -3000000 \\ 8000000 \\ -13,973 \\ 11,6 \\ 8,3125 \\ 0 \end{bmatrix}. \qquad (16.32)$$

In the sequel, we transform the results into the global *X*,*Y*,*Z* coordinate system. The global position vectors of the nodes based on the global coordinates are:

$$\underline{R}_1 = \begin{bmatrix} 0,6795 \\ 0,4 \\ 0,5004 \end{bmatrix}\mathrm{m}\,, \underline{R}_2 = \begin{bmatrix} 0,57 \\ 1,225 \\ 0,6 \end{bmatrix}\mathrm{m}\,, \underline{R}_3 = \begin{bmatrix} 0,2 \\ 0,5 \\ 0,4 \end{bmatrix}\mathrm{m}\,. \qquad (16.33)$$

The position vector of the origin of local coordinate system can be given in the global system as:

$$\underline{R}_0 = \frac{1}{2}\left(R_1 + R_3\right) = \begin{bmatrix} 0,43975 \\ 0,45 \\ 0,4502 \end{bmatrix}\mathrm{m}\,. \qquad (16.34)$$

We determine the unit basis vectors based on the position vectors in the global system using Fig.16.4:

$$\underline{e}_1 = \frac{\underline{R}_2 - \underline{R}_0}{|\underline{R}_2 - \underline{R}_0|} = \begin{bmatrix} 0{,}162825 \\ 0{,}96875 \\ 0{,}18725 \end{bmatrix}. \quad (16.35)$$

Similarly, the unit vectors $\underline{e}_2$ and $\underline{e}_3$ are

$$\underline{e}_2 = \frac{\underline{R}_3 - \underline{R}_0}{|\underline{R}_3 - \underline{R}_0|} = \begin{bmatrix} -0{,}959 \\ 0{,}2 \\ -0{,}2008 \end{bmatrix}, \quad (16.36)$$

$$\underline{e}_3 = \frac{\underline{R}_3 - \underline{R}_0}{|\underline{R}_3 - \underline{R}_0|} = \begin{bmatrix} -0{,}23197 \\ -0{,}14688 \\ 0{,}96159 \end{bmatrix}.$$

Based on the unit basis vectors and Eq.(16.12) matrix $\underline{\underline{L}}$ becomes:

$$\underline{\underline{L}} = [\underline{e}_1 \quad \underline{e}_2 \quad \underline{e}_3] = \begin{bmatrix} 0{,}162825 & -0{,}959 & -0{,}23197 \\ 0{,}96875 & 0{,}2 & -0{,}14688 \\ 0{,}18725 & -0{,}2008 & 0{,}96159 \end{bmatrix}. \quad (16.37)$$

With that it is possible to construct matrix $\underline{\underline{\lambda}}$ with dimension of 18x18. Using Eqs.(16.10), (16.22) and (16.30) the nodal displacements in the global coordinate system are:

$$u_1 = 0\,, v_1 = 0\,, w_1 = 0\,, \quad (16.38)$$

$$\alpha_1 = -0{,}02141\,\text{rad}\,, \beta_1 = -0{,}00845\,\text{rad}\,, \psi_1 = 0{,}002405\,\text{rad}\,,$$

$$u_2 = 0{,}00662\,\text{m}\,, v_2 = -0{,}017215\,\text{m}\,, w_2 = -0{,}004883\,\text{m}\,,$$

$$\alpha_2 = -0{,}012352\,\text{rad}\,, \beta_2 = 0{,}091582\,\text{rad}\,, \psi_2 = 0{,}02153\,\text{rad}\,,$$

$$u_3 = 0{,}0071365\,\text{m}\,, v_3 = 0{,}0014733\,\text{m}\,, w_3 = -0{,}001082\,\text{m}\,,$$

$$\alpha_3 = 0{,}0223\,\text{rad}\,, \beta_3 = -0{,}013587\,\text{rad}\,, \psi_3 = -0{,}0071797\,\text{rad}\,.$$

It is seen that although in the local coordinate system the rotations about *z* are zero, in the global system as a result of the transformation even rotations about *Z* exist. The nodal forces are the followings:

$$F_{X1} = -8238600\text{ N}\,, F_{Y1} = 1277000\text{ N}\,, F_{Z1} = 1871000\text{ N}\,, \quad (16.39)$$

$$M_{X1} = -6{,}21\text{ Nm}\,, M_{Y1} = -8{,}8259\text{ Nm}\,, M_{Z1} = -0{,}6338\text{ Nm}\,,$$

$$F_{X2} = 976920\text{ N}\,, F_{Y2} = -5754000\text{ N}\,, F_{Z3} = -1391800\text{ N}\,,$$

$$M_{X2} = -0{,}8732\text{ Nm}\,, M_{Y2} = 12{,}525\text{ Nm}\,, M_{Z2} = 2{,}7856\text{ Nm}\,,$$

$$F_{X3} = 7261500\text{ N}\,, F_{Y3} = 4477000\text{ N}\,, F_{Z3} = -479100\text{ N}\,,$$

$$M_{X3} = 9{,}9414\text{ Nm}\,, M_{Y3} = -9{,}4615\text{ Nm}\,, M_{Z3} = -3{,}9118\text{ Nm}\,.$$

Accordingly, in the global system there are bending moments about axis *Z*, which are in fact the projections of the moments about local *x* and *y* axes with respect to *Z*. The solution method is applicable also for rectangle shape elements.

16.4 Bibliography

[1] Singiresu S. Rao, *The finite element method in engineering – fourth edition*. Elsevier Science & Technology Books, 2004.

[2] Imre Bojtár, Gábor Vörös, *Application of the finite element method to plate and shell structures.* Technical Book Publisher, 1986, Budapest (in Hungarian).

[3] Klaus-Jürgen Bathe, *Finite element procedures*. Prentice Hall, Upper Saddle River, 1996, New Jersey 17458.

[4] Pei Chi Chou, Nicholas J. Pagano, *Elasticity – Tensor, dyadic and engineering approaches*. D. Van Nostrand Company, Inc., 1967, Princeton, New Jersey, Toronto, London.