15. MODELING OF IN-PLANE THIN-WALLED SHELLS UNDER IN-PLANE AND TRANSVERSE LOAD BY FINITE ELEMENT METHOD BASED SOFTWARE SYSTEMS

Szerző: Dr. András Szekrényes

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15.1 Plate elements subjected to bending

Flat plate elements are suitable to determine the internal forces, stress resultants and stress couples in plate shape structures. The plate element is the extension of beam elements so that bending, shear and torsion take place in two orthogonal planes involving some interactions. Similarly to the plane membrane elements, the triangle and quadrilateral shape elements are available for the modeling of shells. The application of general triangle shape elements is reasonable when the shape of the structure is irregular, triangular or similar to the triangle. In this section we overview primarily the plate elements subjected to transverse load. In that case when the plate is loaded in-plane and also transversely we can solve the problem by combining the plane membrane and plate bending elements. We have already seen by Eq.(14.3) that due to neglecting the transverse shear forces the rotations in an actual point of the plate are:

$$\beta = -w_{,x} \text{ and } \alpha = w_{,y} . \tag{15.1}$$

The curvatures related to the bending deformation are:

$$\underline{\kappa} = -\begin{bmatrix} w_{,xx} \\ w_{,yy} \\ 2w_{,xy} \end{bmatrix}, \text{ and } \underline{\varepsilon} = -z \cdot \underline{\kappa} . \tag{15.2}$$

For thin plates we assume plane stress state, i.e.:

$$\underline{\varepsilon}^T = \left[\varepsilon_x, \varepsilon_y, \gamma_{xy}\right]. \tag{15.3}$$

In the course of the introduction of Kirchhoff plate theory we have observed that the deflection surface is given by a two-variable $w(x,y)$ function, with that both the curvatures and strain components can be calculated. For plate bending problems this $w(x,y)$ function must be produced by interpolation polynomials, and then we can provide the element stiffness matrix and force vector. In the followings we give the details of few element types for plate bending.

15.2 Triangular plate bending element or Tocher triangle element

In the course of the finite element discretization of plate shape structures we approximate the transverse deflection by a third order polynomial in terms of the x and y coordinates [1,2]:

$$w(x, y) = a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 y^2 + a_6 x^3 + a_7 (x^2 y + xy^2) + a_8 y^3 . \tag{15.4}$$

This approximation was one of the first triangular finite elements, which was published by Tocher [1]. The element is shown in Fig.15.1. The deflection surface in vector form is:

$$w(x, y) = \underline{A}^T \underline{\lambda} , \tag{15.5}$$

where $\underline{A}$ is the vector unknown coefficients, $\underline{\lambda}$ is the vector of basis polynomials, respectively:

$$\underline{A}^T = [a_0 \quad a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5 \quad a_6 \quad a_7 \quad a_8], \qquad (15.6)$$
$$\underline{\lambda}^T = [1 \quad x \quad y \quad x^2 \quad xy \quad y^2 \quad x^3 \quad (x^2 y + xy^2) \quad y^3].$$

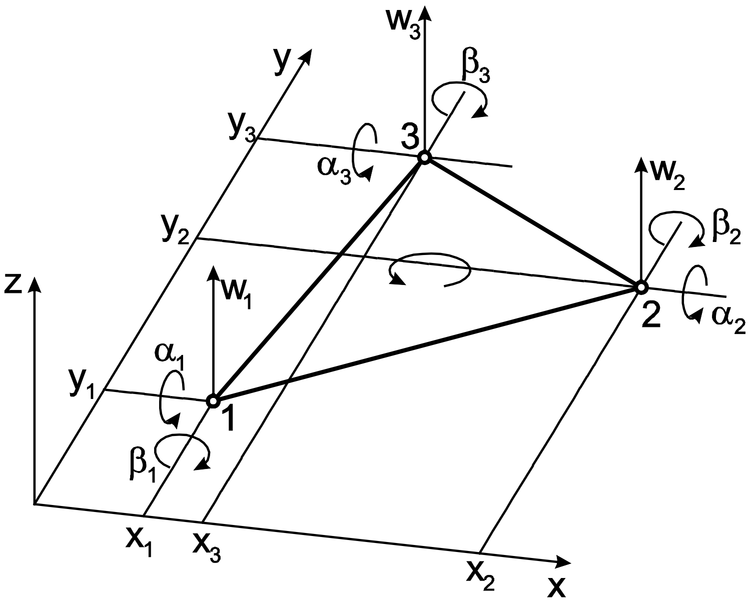

Fig.15.1. The nine degrees of freedom Tocher triangular plate element.

The unknown coefficients can be calculated based on the nodal conditions. Namely, the displacement function must give back the actual nodal displacement if we substitute the nodal coordinates of the same node. Therefore, in Eq.(15.4) the number of terms is always equal to the number of degrees of freedom. For the Tocher triangle element the eighth term contains the sum of x^2y and xy^2. Actually, in vector $\underline{A}$ there is nine unknown coefficients. Following Fig.15.1 the vector of nodal degrees of freedom for a single element is:

$$\underline{u}_e^T = [w_1 \quad \alpha_1 \quad \beta_1 \quad w_2 \quad \alpha_2 \quad \beta_2 \quad w_3 \quad \alpha_3 \quad \beta_3], \qquad (15.7)$$

where w_i is the transverse displacement perpendicularly to the midplane, α_i and β_i are the rotations about axes x and y, respectively. Accordingly, the Tocher plate element has nine degrees of freedom. The nodal conditions for the calculation of the coefficients are [1,2]:

$$w(x_1, y_1) = w_1, \frac{\partial w}{\partial y}(x_1, y_1) = \alpha_1, -\frac{\partial w}{\partial x}(x_1, y_1) = \beta_1, \qquad (15.8)$$
$$w(x_2, y_2) = w_2, \frac{\partial w}{\partial y}(x_2, y_2) = \alpha_2, -\frac{\partial w}{\partial x}(x_2, y_2) = \beta_2,$$
$$w(x_3, y_3) = w_3, \frac{\partial w}{\partial y}(x_3, y_3) = \alpha_3, -\frac{\partial w}{\partial x}(x_3, y_3) = \beta_3.$$

We need the derivatives of the $w(x,y)$ function with respect to x and y to calculate both the coefficients and the strain components, i.e we can write using Eq. (15.4):

$$\frac{\partial w}{\partial x} = a_1 + 2a_3 x + a_4 y + 3a_6 x^2 + a_7(2xy + y^2), \qquad (15.9)$$
$$\frac{\partial w}{\partial y} = a_2 + a_4 x + 2a_5 y + a_7(x^2 + 2xy) + 3a_8 y^2,$$

$$\frac{\partial^2 w}{\partial x^2} = 2a_3 + 6a_6 x + 2a_7 y\,,$$

$$\frac{\partial^2 w}{\partial y^2} = 2a_5 + 2a_7 x + 6a_8 y\,,$$

$$\frac{\partial^2 w}{\partial x \partial y} = a_4 + 2a_7 (x + y)\,.$$

The substitution of the derivatives above into Eq.(15.7) leads to the following system of equation reduced to matrix form [1]:

$$\underline{u}_e = \underline{\underline{M}}\,\underline{A}\,, \qquad (15.10)$$

where:

$$\underline{\underline{M}} = \begin{bmatrix} 1 & x_1 & y_1 & x_1^2 & x_1 y_1 & y_1^2 & x_1^3 & (x_1^2 y_1 + x_1 y_1^2) & y_1^3 \\ 0 & 0 & 1 & 0 & x_1 & 2y_1 & 0 & (x_1^2 + 2x_1 y_1) & 3y_1^2 \\ 0 & -1 & 0 & -2x_1 & -y_1 & 0 & -3x_1^2 & -(2x_1 y_1 + y_1^2) & 0 \\ 1 & x_2 & y_2 & x_2^2 & x_2 y_2 & y_2^2 & x_2^3 & (x_2^2 y_2 + x_2 y_2^2) & y_2^3 \\ 0 & 0 & 1 & 0 & x_2 & 2y_2 & 0 & (x_2^2 + 2x_2 y_2) & 3y_2^2 \\ 0 & -1 & 0 & -2x_2 & -y_2 & 0 & -3x_2^2 & -(2x_2 y_2 + y_2^2) & 0 \\ 1 & x_3 & y_3 & x_3^2 & x_3 y_3 & y_3^2 & x_3^3 & (x_3^2 y_3 + x_3 y_3^2) & y_3^3 \\ 0 & 0 & 1 & 0 & x_3 & 2y_3 & 0 & (x_3^2 + 2x_3 y_3) & 3y_3^2 \\ 0 & -1 & 0 & -2x_3 & -y_2 & 0 & -3x_3^2 & -(2x_3 y_3 + y_3^2) & 0 \end{bmatrix}. \qquad (15.11)$$

The coefficients of the interpolation function are the solutions of the system of equation given by Eq.(15.10):

$$\underline{A} = \underline{\underline{M}}^{-1} \underline{u}_e\,. \qquad (15.12)$$

The expressions of the coefficients are extremely complicated; hence they are not detailed here. The vector of strain components based on Eqs.(14.5) and (15.5) are:

$$\underline{\varepsilon} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = -z \begin{bmatrix} w_{,xx} \\ w_{,yy} \\ 2 \cdot w_{,xy} \end{bmatrix} = \underline{\underline{R}}\,\underline{A}\,, \qquad (15.13)$$

where matrix $\underline{\underline{R}}$ is:

$$\underline{\underline{R}} = -z \cdot \begin{bmatrix} 0 & 0 & 0 & 2 & 0 & 0 & 6x & 2y & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2x & 6y \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 4(x+y) & 0 \end{bmatrix}. \qquad (15.14)$$

Taking back Eq.(15.12) into Eq.(15.13) the strain-displacement matrix can be derived:

$$\underline{\varepsilon} = \underline{\underline{R}}\,\underline{A} = \underline{\underline{R}}\,\underline{\underline{M}}^{-1} \underline{u}_e = \underline{\underline{B}}\,\underline{u}_e\,. \qquad (15.15)$$

For thin plates we assume plane stress state, consequently we can write:

$$\underline{\sigma} = \underline{\underline{C}}^{str} \underline{\varepsilon}\,,\ \ \underline{\sigma} = \underline{\underline{C}}^{str} \underline{\underline{B}}\,\underline{u}_e\,, \qquad (15.16)$$

where matrix $\underline{\underline{C}}^{str}$ refers to plane stress state. According to Eq.(14.43) the definition of the element stiffness matrix is:

$$\underline{\underline{K}}_e = \int_{V_e} \underline{\underline{B}}^T \underline{\underline{C}}^{str^T} \underline{\underline{B}} dV . \qquad (15.17)$$

Incorporating Eq.(15.15) we obtain:

$$\underline{\underline{K}}_e = (\underline{\underline{M}}^{-1})^T \int_{A_e} \left\{ \int_{-t/2}^{t/2} \underline{\underline{R}}^T \underline{\underline{C}}^{str^T} \underline{\underline{R}} dz \right\} dA (\underline{\underline{M}}^{-1}) . \qquad (15.18)$$

The middle term in the expression above is [1]:

$$\int_{A_e} \left\{ \int_{-t/2}^{t/2} \underline{\underline{R}}^T \underline{\underline{C}}^{sf^T} \underline{\underline{R}} dz \right\} dA =$$

$$= \int_{A_e} I_1 E_1 \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 4\nu & 12x & 4(\nu x + y) & 12\nu y \\ 0 & 0 & 0 & 0 & 2(1-\nu) & 0 & 0 & 4(1-\nu)(x+y) & 0 \\ 0 & 0 & 0 & 4\nu & 0 & 4 & 12\nu x & 4(x+\nu y) & 12y \\ 0 & 0 & 0 & 12x & 0 & 12\nu x & 36x^2 & 12x(\nu x + y) & 36\nu xy \\ 0 & 0 & 0 & 4(\nu x + y) & 4(1-\nu)(x+y) & 4(x+\nu y) & 12x(\nu x + y) & \begin{matrix}\{(12-8\nu)(x+y)^2 + \\ -8(1-\nu)xy\}\end{matrix} & 12y(x+\nu y) \\ 0 & 0 & 0 & 12\nu xy & 0 & 12y & 36\nu xy & 12y(x+\nu y) & 36y^2 \end{bmatrix} dA. \qquad (15.19)$$

where $I_1 = t^3/12$ and $E_1 = E/(1-\nu^2)$. To calculate the stiffness matrix the inverse of matrix $\underline{\underline{M}}$ is required. Since it is very complicated, it is not detailed here. In Eq.(15.19) it is possible to simplify the components by the surface integral transformations given below [1]:

$$\int_{A_e} dA = \int\int dxdy = A_e = \frac{1}{2}[(x_2 y_3 - x_3 y_2) + (x_3 y_1 - x_1 y_3) + (x_1 y_2 - x_2 y_1)], \qquad (15.20)$$

$$\int_{A_e} xdA = \int\int xdxdy = \frac{A_e}{3}(x_1 + x_2 + x_3),$$

$$\int_{A_e} ydA = \int\int ydxdy = \frac{A_e}{3}(y_1 + y_2 + y_3),$$

$$\int_{A_e} x^2 dA = \int\int x^2 dxdy = \frac{A_e}{6}(x_1^2 + x_2^3 + x_3^2 + x_1 x_2 + x_2 x_3 + x_3 x_1),$$

$$\int_{A_e} y^2 dA = \int\int y^2 dxdy = \frac{A_e}{6}(y_1^2 + y_2^3 + y_3^2 + y_1 y_2 + y_2 y_3 + y_3 y_1),$$

$$\int_{A_e} xydA = \int\int xydxdy = \frac{A_e}{12}(y_1(2x_1 + x_2 + x_3) + y_2(x_1 + 2x_2 + x_3) + y_3(x_1 + x_2 + 2x_3)),$$

$$\int_{A_e} x^2 ydA = \int\int x^2 ydxdy = \frac{A_e}{60}(y_1(3x_1^2 + x_2^2 + x_3^2 + 2x_1(x_2 + x_3) + x_2 x_3) +$$

$$+ y_2(x_1^2 + 3x_2^2 + x_3^2 + 2x_2(x_1 + x_3) + x_1 x_3) + y_3(x_1^2 + x_2^2 + 3x_3^2 + 2x_3(x_1 + x_2) + x_1 x_2)),$$

$$\int_{A_e} xy^2 dA = \int\int xy^2 dxdy = \frac{A_e}{60}(x_1(3y_1^2 + y_2^2 + y_3^2 + 2y_1(y_2 + y_3) + y_2 y_3) +$$

$$+ x_2(y_1^2 + 3y_2^2 + y_3^2 + 2y_2(y_1 + y_3) + y_1 y_3) + x_3(y_1^2 + y_2^2 + 3y_3^2 + 2y_3(y_1 + y_2) + y_1 y_2)),$$

$$\int_{A_e} x^3 dA = \int\int x^3 dxdy = \frac{A_e}{20}(x_1^3 + x_2^3 + x_3^2 + x_1 x_2^2 + x_1^2 x_2 + x_2 x_3^2 + x_2^2 x_3 + x_1 x_3^2 + x_1^2 x_3 + x_1 x_2 x_3),$$

$$\int_{A_e} y^3 dA = \int\int x^3 dxdy = \frac{A_e}{20}(y_1^3 + y_2^3 + y_3^2 + y_1 y_2^2 + y_1^2 y_2 + y_2 y_3^2 + y_2^2 y_3 + y_1 y_3^2 + y_1^2 y_3 + y_1 y_2 y_3),$$

where A_e is the triangle area, x_i and y_i, i = 1, 2, 3 are the nodal coordinates, respectively. In most of the cases the force vector is composed by two terms. The force vector related to the distributed force can be derived by expressing the work of external force:

$$W_e = \int_{A_{pe}} pw(x,y)dA = \underline{u}_e^T \underline{F}_{ep}. \qquad (15.21)$$

The calculation of $\underline{F}_{ep}$ is difficult, we need the inverse of matrix $\underline{\underline{M}}$ and the simplification of surface integrals, respectively. The concentrated forces and moments are collected in a vector in accordance with the nodal degrees of freedom:

$$\underline{F}_{ec}^T = \left[F_{z1} \quad M_{x1} \quad M_{y1} \quad F_{z2} \quad M_{x2} \quad M_{y2} \quad F_{z3} \quad M_{x3} \quad M_{y3}\right], \qquad (15.22)$$

where F_{zi} is the concentrated force perpendicularly to the midplane of plate, M_{xi} and M_{yi} are the concentrated moments acting in the x and y directions. In the sequel we present a detailed example.

15.3 Example for the application of the Tocher triangle plate element

Determine the displacement and the reactions of the built-in plate depicted in Fig.15.2 [1]!

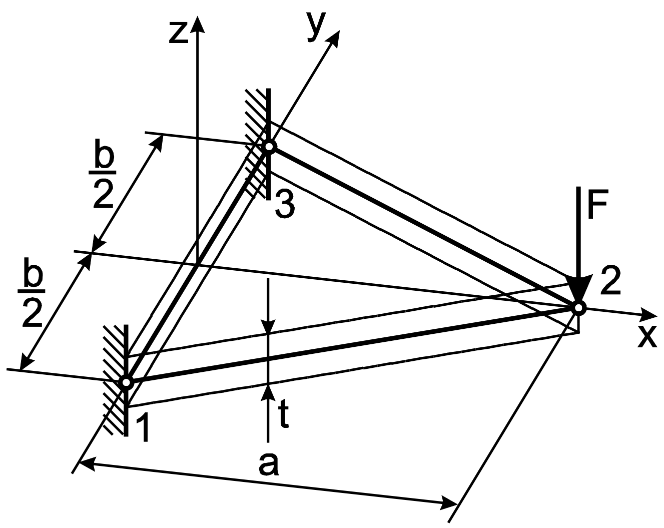

Fig.15.2. Triangle shape built-in plate loaded by concentrated force.

Given:
E = 200 GPa, ν = 0,3, t = 5 mm, F = 1 kN, a = 200 mm, b = 75 mm.

The nodal coordinates are:

node	***x***	***y***
1	*0*	-b/*2*
2	a	*0*
3	*0*	b/*2*

In the sequel the distances are calculated in [mm], the force is given in [N]. Because of the kinematic constraints (built-in nodes) the vector of nodal displacements becomes:

$$\underline{u}_e^T = \begin{bmatrix} 0 & 0 & 0 & w_2 & \alpha_2 & \beta_2 & 0 & 0 & 0 \end{bmatrix}. \qquad (15.23)$$

For the calculation of stiffness matrix we need the constitutive matrix, which is:

$$\underline{\underline{C}} = \underline{\underline{C}}^{str} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix} = \begin{bmatrix} 219,78 & 65,93 & 0 \\ 65,93 & 219,78 & 0 \\ 0 & 0 & 76,92 \end{bmatrix} \text{GPa} . \qquad (15.24)$$

Utilizing the nodal coordinates we calculate matrix $\underline{\underline{M}}$ based on Eq.(15.11):

$$\underline{\underline{M}} = \begin{bmatrix} 1 & 0 & -75 & 0 & 0 & 5625 & 0 & 0 & -421875 \\ 0 & 0 & 1 & 0 & 0 & -150 & 0 & 0 & 16875 \\ 0 & -1 & 0 & 0 & 75 & 0 & 0 & -5625 & 0 \\ 1 & 200 & 0 & 40000 & 0 & 0 & 8000000 & 0 & 0 \\ 0 & 0 & 1 & 0 & 200 & 0 & 0 & 40000 & 0 \\ 0 & -1 & 0 & -400 & 0 & 0 & -120000 & 0 & 0 \\ 1 & 0 & 75 & 0 & 0 & 5625 & 0 & 0 & 421875 \\ 0 & 0 & 1 & 0 & 0 & 150 & 0 & 0 & 16875 \\ 0 & -1 & 0 & 0 & -75 & 0 & 0 & -5625 & 0 \end{bmatrix}. \qquad (15.25)$$

The determinant of matrix $\underline{\underline{M}}$ is $-4,86 \cdot 10^{12}$, i.e. the matrix is not singular, its inverse exists. The stiffness matrix is obtained by calculating matrix $\underline{\underline{R}}$ (see Eq.(15.14)) and computing the surface integrals:

$$\underline{\underline{K}}_e = (\underline{\underline{M}}^{-1})^T \int_{A_e} \left\{ \int_{-t/2}^{t/2} \underline{\underline{R}}^T \underline{\underline{C}}^{str\,T} \underline{\underline{R}} dz \right\} dA (\underline{\underline{M}}^{-1}) =$$

$$= \begin{bmatrix} 629,1 & 30559,9 & -15383,6 & -80,5 & 16472,0 & -7887,6 & -548,7 & 41301,7 & 7174,0 \\ . & 3731523,2 & 236741,0 & -7055,9 & -392240,8 & -304505,1 & -23503,9 & 715499,7 & -1343422,4 \\ . & . & 2433109,2 & 10731,5 & -772851,1 & 765510,5 & 4652,2 & -966573,2 & -1052328,6 \\ . & . & . & 257,6 & 4829,2 & 17170,3 & -177,1 & 9470,5 & 23609,2 \\ . & . & . & . & 1716949,1 & 69754,5 & -21301,1 & 1508270,3 & 1668927,7 \\ . & . & . & . & . & 1717032,9 & -9282,7 & 339382,2 & 951522,4 \\ . & . & . & . & . & . & 725,71 & -50772,2 & -30783,3 \\ . & . & . & . & . & . & . & 4681778,8 & 2521292,8 \\ . & . & . & . & . & . & . & . & 4822646,8 \end{bmatrix} \frac{\text{N}}{\text{mm}}, \qquad (15.26)$$

In Eq.(15.26) only the independent components are indicated, the reason for that is the stiffness matrix is always symmetric. The force vector based on the concentrated loads is:

$$\underline{F}_{ec}^T = \begin{bmatrix} F_{z1} & M_{x1} & M_{y1} & -F & 0 & 0 & F_{z3} & M_{x3} & M_{y3} \end{bmatrix}. \qquad (15.27)$$

The condensed stiffness matrix and the resulting matrix equation for the calculation of nodal displacements is the following:

$$\begin{bmatrix} 257{,}6 & 4869{,}2 & 17170{,}3 \\ 4829{,}2 & 1716949{,}1 & 69754{,}5 \\ 17170{,}3 & 69754{,}5 & 1717033{,}0 \end{bmatrix} \begin{bmatrix} w_2 \\ \alpha_2 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} -1000 \\ 0 \\ 0 \end{bmatrix}. \qquad (15.28)$$

The nodal solutions are:

$$w_2 = -13{,}1764 \text{ mm}, \alpha_2 = 0{,}03176 \text{ rad}, \beta_2 = 0{,}130477 \text{ rad}. \quad (15.29)$$

It is seen that although the problem is symmetric with respect to axis x for both the geometry and load, the deformation of the triangle element is not symmetric. Taking the nodal displacements back to original equation we can determine the reactions:

$$F_{z1} = 554{,}5 \text{ N}, \; M_{x1} = 40784{,}5 \text{ Nmm}, \; M_{y1} = -66068{,}6 \text{ Nmm}. \qquad (15.30)$$

Using Eq.(15.15) the vectors of strain and stress components are:

$$\underline{\varepsilon} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \underline{\underline{R}}\,\underline{\underline{M}}^{-1}\underline{u}_e = \begin{bmatrix} 5{,}824 \cdot 10^{-4} z + 4{,}764 \cdot 10^{-7} zx - 1{,}5879 \cdot 10^{-6} zy \\ -1{,}5879 \cdot 10^{-6} zx \\ -7{,}9399 \cdot 10^{-7} z(4x + 4y) \end{bmatrix}, \qquad (15.31)$$

$$\underline{\sigma} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \underline{\underline{C}}^{str}\underline{\varepsilon} = \begin{bmatrix} 128{,}0z + 2{,}0 \cdot 10^{-10} zx - 0{,}349 zy \\ 38{,}4z - 0{,}3176 zx - 0{,}1047 zy \\ -0{,}06107 z(4x + 4y) \end{bmatrix}.$$

The strain and stress components can be obtained at any point of the triangle element by taking back the coordinates in [mm]. The example above was verified by a finite element code developed in Matlab [3] and we obtained the same results. In general the accuracy of the Tocher plate element is not satisfactory and even the convergence of the results is bad. To reduce the deficiencies of the Tocher triangle the so-called reduced triangle element was developed, where area coordinates are introduced [4]. Apart from the Tocher triangular plate element there are several more element types, e.g.: Adini or Cowper triangle element, Adini-Clough-Melosh, Bogner-Fox-Scmit rectangle element, etc [1]. In the sequel we present some rectangle shape plate elements.

15.4 Incompatible rectangular shape plate element

Fig.15.3 presents one of the first rectangle shape elements in a global, local and natural coordinate system.

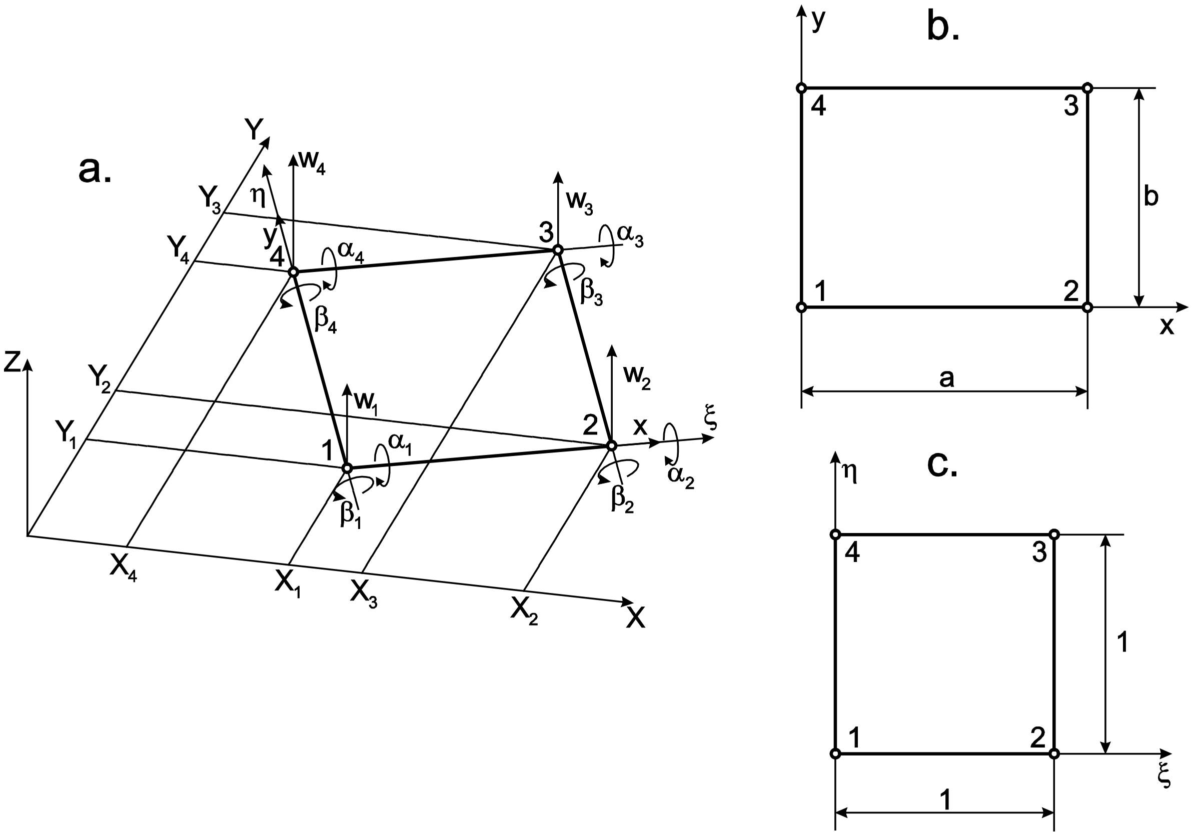

Fig.15.3. Incompatible rectangle shape plate element in a global (a), local (b) and natural (c) co-ordinate system.

The dimensionless local ξ and η coordinates are:

$$\xi = \frac{x}{a},\ \eta = \frac{y}{b}, \text{ and: } d\xi = \frac{1}{a}dx,\ d\eta = \frac{1}{b}dy\,. \qquad (15.32)$$

The following differential quotients are also required:

$$\frac{d}{dx} = \frac{1}{a}\frac{d}{d\xi}, \frac{d}{dy} = \frac{1}{b}\frac{d}{d\eta}\,. \qquad (15.33)$$

According to Fig.15.3b we consider three degrees of freedom in the local coordinate system at each node, which are the displacement *w* perpendicularly to the midplane of plate and the rotations about the *x* and *y* axes, respectively. The vector of nodal displacements for a single element is:

$$\underline{u}_e^T = \begin{bmatrix} w_1 & \alpha_1 & \beta_1 & w_2 & \alpha_2 & \beta_2 & w_3 & \alpha_3 & \beta_3 & w_4 & \alpha_4 & \beta_4 \end{bmatrix}, \qquad (15.34)$$

i.e. totally the element has 12 degrees of freedom, where the rotations can be determined by means of Eq.(15.1) and the Kirchhoff-Love hypothesis. The displacement in direction *z* and the rotations are not independent of each other and there can only be to a maximum of 12 unknown parameters in the interpolation function. Along the element edges the expression of *w* should be a third order function, and accordingly the derivative in the normal direction should vary linearly [2]. A complete third order function contains 10 terms, but in accordance with the number of nodal parameters we need two additional terms in the interpolated function. We can choose from the three possibilities below:

$$\xi^3\eta \text{ and } \xi\eta^3, \text{ or: } \xi^3\eta^2 \text{ and } \xi^2\eta^3, \text{ or: } \xi^2\eta^2 \text{ and } \xi^3\eta^3. \qquad (15.35)$$

Any of the above possibilities is chosen, we obtain a cubic change in the derivatives in the normal direction instead of the expected linear one [1,2]. Therefore this element is not compatible, in other words it is incompatible. Choosing the first alternative we have:

$$w(\xi,\eta) = a_0 + a_1\xi + a_2\eta + a_3\xi^2 + a_4\xi\eta + a_5\eta^2 + a_6\xi^3 + + a_7\xi^2\eta + a_8\xi\eta^2 + a_9\eta^3 + a_{10}\xi^3\eta + a_{11}\xi\eta^3. \quad (15.36)$$

The nodal conditions for the determination of the unknown coefficients are:

$$w(0,0) = w_1,\ \frac{1}{b}\frac{\partial w}{\partial \eta}(0,0) = \alpha_1,\ -\frac{1}{a}\frac{\partial w}{\partial \xi}(0,0) = \beta_1, \quad (15.37)$$

$$w(1,0) = w_2,\ \frac{1}{b}\frac{\partial w}{\partial \eta}(1,0) = \alpha_2,\ -\frac{1}{a}\frac{\partial w}{\partial \xi}(1,0) = \beta_2,$$

$$w(1,1) = w_3,\ \frac{1}{b}\frac{\partial w}{\partial \eta}(1,1) = \alpha_3,\ -\frac{1}{a}\frac{\partial w}{\partial \xi}(1,1) = \beta_3,$$

$$w(0,1) = w_4,\ \frac{1}{b}\frac{\partial w}{\partial \eta}(0,1) = \alpha_4,\ -\frac{1}{a}\frac{\partial w}{\partial \xi}(0,1) = \beta_4.$$

Taking back the coefficients into the function given by Eq.(15.36), moreover by utilizing the fact that the displacement function can be formulated as the product of interpolation functions and nodal parameters it is possible to obtain:

$$w(\xi,\eta) = N_1 w_1 + N_2\alpha_1 + N_3\beta_1 + N_4 w_2 + N_5\alpha_2 + N_6\beta_2 + + N_7 w_3 + N_8\alpha_3 + N_9\beta_3 + N_{10} w_4 + N_{11}\alpha_4 + N_{12}\beta_4, \quad (15.38)$$

from which we obtain the mathematical form of the interpolation functions:

$$N_1 = 2(\eta-1)(\xi-1)\left(\frac{1}{2}(1+\xi+\eta) - \xi^2 - \eta^2\right), \quad (15.39)$$

$$N_2 = -b\eta(\eta-1)^2(\xi-1),$$

$$N_3 = a\xi(\eta-1)(\xi-1)^2,$$

$$N_4 = 2(\eta-1)\xi\left(\xi^2 + \eta^2 - \frac{3}{2}\xi - \frac{1}{2}\eta\right),$$

$$N_5 = b\xi\eta(\eta-1)^2,$$

$$N_6 = a\xi^2(\eta-1)(\xi-1),$$

$$N_7 = 2\eta\xi\left(-\xi^2 - \eta^2 - \frac{1}{2} + \frac{3}{2}(\xi+\eta)\right),$$

$$N_8 = b\xi\eta^2(\eta-1),$$

$$N_9 = -a\xi^2\eta(\xi-1),$$

$$N_{10} = 2\eta(\xi-1)\left(\xi^2 + \eta^2 - \frac{1}{2}\xi - \frac{3}{2}\eta\right),$$

$$N_{11} = -b\eta^2(\eta-1)(\xi-1),$$

$$N_{12} = -a\xi\eta(\xi-1)^2,$$

and:

$$w(\xi,\eta) = \underline{N}^T\underline{u}_e, \quad (15.40)$$

where:

$$\underline{N}^T = [N_1 \quad N_2 \quad N_3 \quad N_4 \quad N_5 \quad N_6 \quad N_7 \quad N_8 \quad N_9 \quad N_{10} \quad N_{11} \quad N_{12}], \tag{15.41}$$

is the vector of interpolation polynomials. As a next step we express the vector of strain components using Eq.(14.5):

$$\underline{\varepsilon} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = -z \begin{bmatrix} w_{,xx} \\ w_{,yy} \\ 2 \cdot w_{,xy} \end{bmatrix} = -z\underline{\kappa}, \tag{15.42}$$

where:

$$\underline{\kappa} = \begin{bmatrix} \underline{N}^T_{,xx} \\ \underline{N}^T_{,yy} \\ 2 \cdot \underline{N}^T_{,xy} \end{bmatrix} \underline{u}_e = \underline{\underline{\kappa}} \underline{u}_e . \tag{15.43}$$

where $\underline{N}_{,xx}$, $\underline{N}_{,yy}$ and $\underline{N}_{,xy}$ are vectors containing the second order derivatives of the interpolation functions by Eq.(15.40) with respect to the corresponding subscript. Hence, the vector of strain components and the vector of stress components become:

$$\underline{\varepsilon} = -z\underline{\underline{\kappa}}\underline{u}_e , \tag{15.44}$$

$$\underline{\sigma} = \underline{\underline{C}}^{str} \underline{\varepsilon} = -z\underline{\underline{C}}^{str} \underline{\underline{\kappa}}\underline{u}_e .$$

The vector of bending and twisting moments can be given in vector form; they are calculated based on Eqs.(14.7) and (15.43):

$$\underline{M} = \begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = -I_1 E_1 \begin{bmatrix} \underline{N}^T_{,xx} + \nu \underline{N}^T_{,yy} \\ \underline{N}^T_{,yy} + \nu \underline{N}^T_{,xx} \\ (1-\nu)\underline{N}^T_{,xy} \end{bmatrix} \underline{u}_e . \tag{15.45}$$

Taking the previously calculated vectors back into the total potential energy we obtain:

$$\Pi_e = \frac{1}{2}\int_{V_e} \underline{\sigma}^T \underline{\varepsilon} dV - \int_{A_{pe}} \underline{u}^T \underline{p} dA = \frac{1}{2}\underline{u}_e^T \int_{V_e} z^2 \underline{\underline{\kappa}}^T \underline{\underline{C}}^{str\,T} \underline{\underline{\kappa}} dV \underline{u}_e - \int_{A_{pe}} pw(\xi,\eta) dA . \tag{15.46}$$

We transform the volume integral over the element by integration with respect to the parameters *x*, *y* and *z*. Moreover, we assume that in the second term the intensity of the distributed load is constant. Consequently we can write:

$$\Pi_e = \frac{1}{2}\underline{u}_e^T \left\{ \int_0^1\int_0^1 \frac{t^3}{12} ab \cdot \underline{\underline{\kappa}}^T \underline{\underline{C}}^{str\,T} \underline{\underline{\kappa}} d\eta d\xi \right\} \underline{u}_e - \underline{u}_e^T \int_0^1\int_0^1 p \cdot ab \underline{N} d\eta d\xi = \frac{1}{2}\underline{u}_e^T \underline{\underline{K}}_e \underline{u}_e - \underline{u}_e^T \underline{F}_{ep} , \tag{15.47}$$

where the element stiffness matrix is:

$$\underline{\underline{K}}_e = \int_0^1\int_0^1 \frac{t^3}{12} ab \cdot \underline{\underline{\kappa}}^T \underline{\underline{C}}^{str\,T} \underline{\underline{\kappa}} d\eta d\xi , \tag{15.48}$$

and the force vector from the uniformly distributed load is:

$$\underline{F}_{ep} = \int_0^1\int_0^1 p \cdot ab \underline{N} d\eta d\xi =$$

$$= \frac{pab}{4}\begin{bmatrix} 1 & \frac{b}{6} & -\frac{a}{6} & 1 & \frac{b}{6} & \frac{a}{6} & 1 & -\frac{b}{6} & \frac{a}{6} & 1 & -\frac{b}{6} & -\frac{a}{6} \end{bmatrix}^T ,$$

(15.49)

that is, similarly to the beam element subjected to bending the distributed load is represented by concentrated forces and moments at the nodes referring to the discretization procedure. It is also necessary to consider that there can be concentrated loads in the nodes, viz.:

$$\underline{F}_{ec}^{T} = \left[F_{z1} \quad M_{x1} \quad M_{y1} \quad F_{z2} \quad M_{x2} \quad M_{y2} \quad F_{z3} \quad M_{x3} \quad M_{y3} \quad F_{z4} \quad M_{x4} \quad M_{y4}\right], \tag{15.50}$$

and:

$$\underline{F}_{e} = \underline{F}_{ec} + \underline{F}_{ep} \,. \tag{15.51}$$

The application of the minimum principle yields the element equilibrium equation:

$$\underline{\underline{K}}_{e}\underline{u}_{e} = \underline{F}_{e} \,, \tag{15.52}$$

which can be used only if the structure consists of a single element. For multi-element structures we obtain the structural equation by summing the potential energies of the elements:

$$\underline{\underline{K}}\underline{U} = \underline{F} \,. \tag{15.53}$$

Let us solve an example for the incompatible rectangle shape element!

15.5 Example for the application of the incompatible rectangle shape element

Calculate the nodal displacements and the reactions of the built-in plate shown in Fig.15.4!

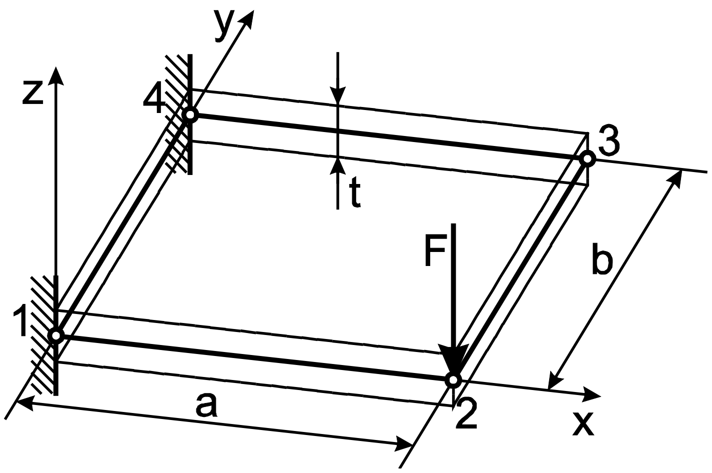

Fig.15.4. Example for the application of incompatible plate element.

Given:

E = 200 GPa, ν = 0,3, t = 1 mm, F = 5 N, a = 600 mm, b = 400 mm.

In the sequel the distances are substituted in [m], the force is given in [N]. The nodal coordinates are:

node	*x*	*y*
1	*0*	*0*
2	a	*0*
3	a	b
4	*0*	b

Considering the kinematic constraints in the construction of the vector of nodal displacement we obtain:

$$\underline{u}_e^T = \begin{bmatrix} 0 & 0 & 0 & w_2 & \alpha_2 & \beta_2 & w_3 & \alpha_3 & \beta_3 & 0 & 0 & 0 \end{bmatrix}. \quad (15.54)$$

The force vector considering the external force and the reactions is:

$$\underline{F}_{ec}^T = \begin{bmatrix} F_{z1} & M_{x1} & M_{y1} & -F & 0 & 0 & 0 & 0 & 0 & F_{z4} & M_{x1} & M_{y4} \end{bmatrix}. \quad (15.55)$$

For plane stress state the constitutive matrix is:

$$\underline{\underline{C}}^{str} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} = \begin{bmatrix} 200 & 60 & 0 \\ 60 & 200 & 0 \\ 0 & 0 & 70 \end{bmatrix} \text{GPa}. \quad (15.56)$$

Next we calculate matrix $\underline{\underline{\kappa}}$ which is required for the stiffness matrix:

$$\underline{\underline{\kappa}}^T = \begin{bmatrix}
-\frac{50}{3}(\eta-1)(2\xi-1) & -\frac{75}{2}(2\eta-1)(\xi-1) & -50(\xi^2+\eta^2-\xi-\eta+\frac{1}{6}) \\
0 & -5(3\eta-2)(\xi-1) & -\frac{10}{3}(3\eta-1)(\eta-1) \\
\frac{10}{3}(\eta-1)(3\xi-2) & 0 & 5(3\xi-1)(\xi-1) \\
\frac{50}{3}(\eta-1)(2\xi-1) & \frac{75}{2}\xi(2\eta-1) & 50(\xi^2+\eta^2-\xi-\eta+\frac{1}{6}) \\
0 & 5\xi(3\eta-2) & \frac{10}{3}(3\eta-1)(\eta-1) \\
\frac{10}{3}(\eta-1)(3\xi-1) & 0 & 5\xi(3\xi-2) \\
-\frac{50}{3}\eta(2\xi-1) & -\frac{75}{2}\xi(2\eta-1) & -50(\xi^2+\eta^2-\xi-\eta+\frac{1}{6}) \\
0 & 5\xi(3\eta-1) & \frac{10}{3}\eta(3\eta-2) \\
-\frac{10}{3}\eta(3\xi-1) & 0 & -5\xi(3\xi-2) \\
\frac{50}{3}\eta(2\xi-1) & \frac{75}{2}(2\eta-1)(\xi-1) & 50(\xi^2+\eta^2-\xi-\eta+\frac{1}{6}) \\
0 & -5(3\eta-1)(\xi-1) & -\frac{10}{3}\eta(3\eta-2) \\
-\frac{10}{3}\eta(3\xi-2) & 0 & -5(3\xi-1)(\xi-1)
\end{bmatrix}. \quad (15.57)$$

The dimension of the stiffness matrix is 12x12; therefore it is not detailed here. Instead of the stiffness matrix we give the resulting finite element equilibrium equation system from Eq.(15.52):

$$11{,}27w_2 + 50{,}28\alpha_2 - 42{,}87\beta_2 - 196{,}45w_3 + 58{,}61\alpha_3 - 12{,}69\beta_3 = F_{z1}, \quad (15.58)$$

$$50{,}28w_2 + 14{,}59\alpha_2 - 58{,}61w_3 + 8{,}85\alpha_3 = M_{x1},$$

$$42{,}87w_2 + 6{,}24\beta_2 + 12{,}69w_3 + 4{,}87\beta_3 = M_{y1},$$
$$926{,}23w_2 + 137{,}22\alpha_2 + 55{,}37\beta_2 - 741{,}05w_3 + 128{,}89\alpha_3 + 0{,}19\beta_3 = -5,$$
$$137{,}22w_2 + 35{,}41\alpha_2 + 5{,}00\beta_2 - 128{,}89w_3 + 16{,}15\alpha_3 = 0,$$
$$55{,}37w_2 + 5{,}00\alpha_2 + 19{,}48\beta_2 + 0{,}19w_3 + 2{,}74\beta_3 = 0,$$
$$-741{,}05w_2 - 128{,}89\alpha_2 + 0{,}19\beta_2 + 926{,}23w_3 - 137{,}22\alpha_3 + 55{,}37\beta_3 = 0,$$
$$128{,}89w_2 + 16{,}15\alpha_2 - 137{,}22w_3 + 35{,}41\alpha_3 - 5{,}0\beta_3 = 0,$$
$$0{,}19w_2 + 2{,}74\beta_2 + 55{,}37w_3 - 5{,}00\alpha_3 + 19{,}48\beta_3 = 0,$$
$$-196{,}45w_2 - 58{,}61\alpha_2 - 12{,}69\beta_2 + 11{,}27w_3 - 50{,}28\alpha_3 - 42{,}87\beta_3 = F_{z4},$$
$$58{,}61w_2 + 8{,}85\alpha_2 - 50{,}28w_3 + 14{,}59\alpha_3 = M_{x4},$$
$$12{,}69w_2 + 4{,}87\beta_2 + 42{,}87w_3 + 6{,}24\beta_3 = M_{y4}.$$

Calculating the nodal displacements from the 4^{th}, 5^{th}, 6^{th}, 7^{th}, 8^{th} and 9^{th} equations of Eq.(15.56) we have:

$$w_2 = -0{,}0655\ \text{m},\ \alpha_2 = 0{,}039\ \text{rad},\ \beta_2 = 0{,}159\ \text{rad}, \qquad (15.59)$$
$$w_3 = -0{,}0448\ \text{m},\ \alpha_3 = 0{,}0645\ \text{rad},\ \beta_3 = 0{,}122\ \text{rad}.$$

From the 1^{st}, 2^{nd}, 3^{rd} and 10^{th}, 11^{th}, 12^{th} equations of Eq.(15.56) we can determine the reactions:

$$F_{1z} = 5{,}42\ \text{N},\ M_{x1} = 0{,}47\ \text{Nm},\ M_{y1} = -1{,}79\ \text{Nm}, \qquad (15.60)$$
$$F_{z4} = -0{,}42\ \text{N},\ M_{x4} = -0{,}30\ \text{Nm},\ M_{y4} = -1{,}21\ \text{Nm}.$$

The bending and twisting moments can be obtained from Eq.(15.45), the stresses can be determined from Eq.(15.44) by taking back the nodal coordinates. Example 15.5 was verified by the finite element code ANSYS 12 and we obtained the same results.

15.6 Compatible rectangular shape plate element

In that case when we want to develop a compatible plate element the interpolation function given by Eq.(15.36) has to be modified in accordance with the followings [2]:

$$w(\xi,\eta) = a_0 + a_1\xi + a_2\eta + a_3\xi^2 + a_4\xi\eta + a_5\eta^2 + a_6\xi^3 + a_7\xi^2\eta + a_8\xi\eta^2 + $$
$$+ a_9\eta^3 + a_{10}\xi^3\eta + a_{11}\xi\eta^3 + a_{12}\xi^2\eta^2 + a_{13}\xi^3\eta^2 + a_{14}\xi^2\eta^3 + a_{15}\xi^3\eta^3. \qquad (15.61)$$

However, this formulation implies 16 unknown nodal parameters. That is, at each node we must consider the mixed derivative $w_{,xy}$. The vector of nodal displacement becomes:

$$\underline{u}_e^T = [w_1 \ \alpha_1 \ \beta_1 \ w_{,xy1} \ w_2 \ \alpha_2 \ \beta_2 \ w_{,xy2} \ w_3 \ \alpha_3 \ \beta_3 \ w_{,xy3} \ w_4 \ \alpha_4 \ \beta_4 \ w_{,xy4}] \qquad (15.62)$$

The conditions for the determination of the unknown parameters are:

$$w(0,0) = w_1, \frac{1}{b}\frac{\partial w}{\partial \eta}(0,0) = \alpha_1, -\frac{1}{a}\frac{\partial w}{\partial \xi}(0,0) = \beta_1, \frac{1}{ab}\frac{\partial^2 w}{\partial \xi \partial \eta}(0,0) = w_{,xy1}, \qquad (15.63)$$
$$w(1,0) = w_2, \frac{1}{b}\frac{\partial w}{\partial \eta}(1,0) = \alpha_2, -\frac{1}{a}\frac{\partial w}{\partial \xi}(1,0) = \beta_2, \frac{1}{ab}\frac{\partial^2 w}{\partial \xi \partial \eta}(1,0) = w_{,xy2},$$
$$w(1,1) = w_3, \frac{1}{b}\frac{\partial w}{\partial \eta}(1,1) = \alpha_3, -\frac{1}{a}\frac{\partial w}{\partial \xi}(1,1) = \beta_3, \frac{1}{ab}\frac{\partial^2 w}{\partial \xi \partial \eta}(1,1) = w_{,xy3},$$

$$w(0,1) = w_4, \frac{1}{b}\frac{\partial w}{\partial \eta}(0,1) = \alpha_4, -\frac{1}{a}\frac{\partial w}{\partial \xi}(0,1) = \beta_4. \frac{1}{ab}\frac{\partial^2 w}{\partial \xi \partial \eta}(0,1) = w_{,xy4}.$$

The deflection surface is approximated by using 16 interpolation functions:

$$w(\xi,\eta) = N_1 w_1 + N_2\alpha_1 + N_3\beta_1 + N_4 w_{,xy1} + N_5 w_2 + N_6\alpha_2 + N_7\beta_2 + N_8 w_{,xy2}$$
$$+ N_9 w_3 + N_{10}\alpha_3 + N_{11}\beta_3 + N_{12} w_{,xy3} + N_{13} w_4 + N_{14}\alpha_4 + N_{15}\beta_4 + N_{16} w_{,xy4}. \qquad (15.64)$$

The interpolation functions can be written by the help of the Hermitian polynomials, which are presented in the beam finite elements (see Fig.15.5) [2]:

$$f_1(\lambda) = 2\lambda^3 - 3\lambda^2 + 1,\; f_2(\lambda) = -2\lambda^3 + 3\lambda^2, \qquad (15.65)$$
$$f_3(\lambda) = \lambda^3 - 2\lambda^2 + \lambda,\; f_4(\lambda) = \lambda^3 - \lambda^2,$$

with that the 16 interpolation functions become:

$$N_1 = f_1(\xi)\cdot f_1(\eta), N_9 = f_2(\xi)\cdot f_2(\eta), \qquad (15.66)$$
$$N_2 = b\cdot f_1(\xi)\cdot f_3(\eta), N_{10} = b\cdot f_2(\xi)\cdot f_4(\eta),$$
$$N_3 = -a\cdot f_3(\xi)\cdot f_1(\eta), N_{11} = -a\cdot f_4(\xi)\cdot f_2(\eta),$$
$$N_4 = a\cdot b\cdot f_3(\xi)\cdot f_3(\eta), N_{12} = a\cdot b\cdot f_4(\xi)\cdot f_4(\eta),$$
$$N_5 = f_2(\xi)\cdot f_1(\eta), N_{13} = f_1(\xi)\cdot f_2(\eta),$$
$$N_6 = b\cdot f_2(\xi)\cdot f_3(\eta), N_{14} = b\cdot f_1(\xi)\cdot f_4(\eta),$$
$$N_7 = -a\cdot f_4(\xi)\cdot f_1(\eta), N_{15} = -a\cdot f_3(\xi)\cdot f_2(\eta),$$
$$N_8 = a\cdot b\cdot f_4(\xi)\cdot f_3(\eta), N_{16} = a\cdot b\cdot f_3(\xi)\cdot f_4(\eta).$$

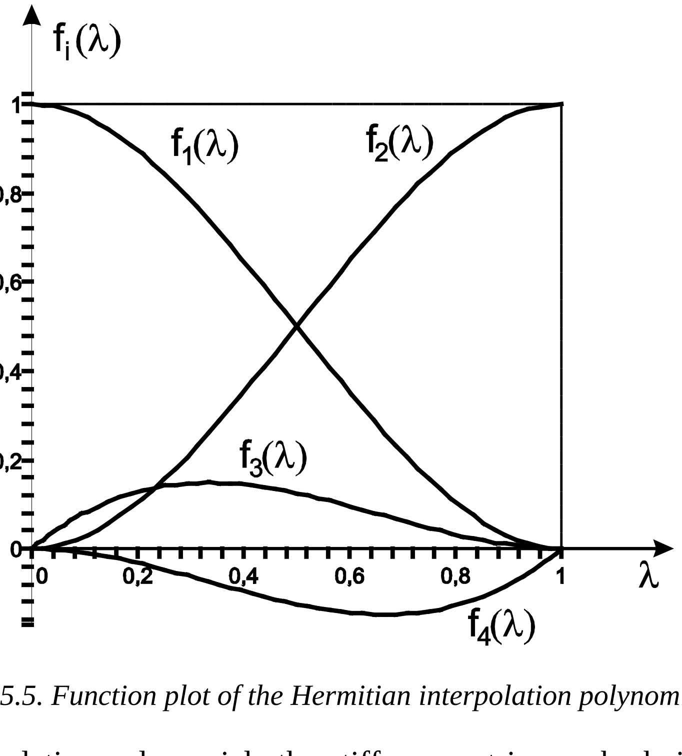

Fig.15.5. Function plot of the Hermitian interpolation polynomials.

By using the interpolation polynomials the stiffness matrix can be built-up by the same methodology as that presented in the incompatible plate element. The only difference is that we obtain a matrix with dimension of 16x16. Assuming a constant distributed force, the relevant term in the force vector is:

$$\underline{F}_{ep}=\int_0^1\int_0^1 p\cdot ab\underline{N}d\eta d\xi=$$

$$=\frac{pab}{4}\begin{bmatrix}1 & \frac{b}{6} & -\frac{a}{6} & \frac{ab}{36} & 1 & \frac{b}{6} & \frac{a}{6} & -\frac{ab}{36} & 1 & -\frac{b}{6} & \frac{a}{6} & \frac{ab}{36} & 1 & -\frac{b}{6} & -\frac{a}{6} & -\frac{ab}{36}\end{bmatrix}^T, \tag{15.67}$$

i.e., similarly to the plane beam element subjected to bending the distributed load is represented by concentrated forces and moments in the nodes. As usual, we have to consider the case of concentrated loads, the relevant vector term is:

$$\underline{F}_{ec}^T=\left[F_{z1}\quad M_{x1}\quad M_{y1}\quad M_{xy1}\quad F_{z2}\quad M_{x2}\quad M_{y2}\quad M_{xy2}\ldots\ldots\ldots \right.$$
$$\left. \ldots\ldots\ldots F_{z3}\quad M_{x3}\quad M_{y3}\quad M_{xy3}\quad F_{z4}\quad M_{x4}\quad M_{y4}\quad M_{xy4}\right] \tag{15.68}$$

Example 15.5 was solved by using the compatible plate element too. In this case the nodal displacements are:

$$w_2=-0{,}0658\text{ m}, \alpha_2=0{,}062\text{ rad}, \beta_2=0{,}165\text{ rad}, w_{,xy2}=0{,}2414\frac{\text{rad}}{\text{m}}, \tag{15.69}$$

$$w_3=-0{,}0450\text{ m}, \alpha_3=0{,}043\text{ rad}, \beta_3=0{,}128\text{ rad}, w_{,xy2}=-0{,}0415\frac{\text{rad}}{\text{m}}.$$

The reactions are given below:

$$F_{z1}=5{,}512\text{ N}, M_{x1}=0{,}676\text{ Nm}, M_{y1}=-1{,}84\text{ Nm}, M_{xy1}=0{,}138\text{ Nm}^2. \tag{15.70}$$

$$F_{z4}=-0{,}512\text{ N}, M_{x4}=-0{,}471\text{ Nm}, M_{y4}=-1{,}155\text{ Nm}, M_{xy4}=0{,}115\text{ Nm}^2.$$

It is seen, that the difference between the results of the two solutions is not significant.

15.7 Plates under in-plane and transverse load

If the plate is loaded by in-plane and transverse forces simultaneously, then we have to produce an element by having both in-plane and bending load-carrying capability, i.e. it means the superposition of plane membrane and plate bending elements. This problem can be solved based on sections 12 and 15 in a relatively simple way. First we collect the corresponding nodal displacements into a vector. Second, we create the stiffness matrix of the combined element by placing the stiffness matrix components corresponding to the membrane and bending deformation into the right positions. The vector of forces is obtained by a similar combination of the element vectors. This technique is suitable to model in-plane plate structures too. However, if we connect the elements by containing an angle differing from 180° among the surfaces, then it is possible to approximate curved surfaces. In other words the combined membrane-plate element is suitable to model spatial shells and shell structures too. Since in the modeling of plane and spatial shells similar steps are required, these issues will be detailed in section 16.

15.8 Bibliography

[1] Singiresu S. Rao, *The finite element method in engineering – fourth edition*. Elsevier Science & Technology Books, 2004.

[2] Imre Bojtár ~~Imre~~, Gábor Vörös ~~Gábor~~, *Application of the finite element method to plate and shell structures.* ~~végeselem módszer alkalmazása lemez- és~~

~~*héjszerkezetekre.*~~ Technical Book Publisher~~Műszaki Könyvkiadó~~, 1986, Budapest (in Hungarian).

[3] Ali Ozgul. *Edge Nodes of Triangular Kirchhoff Bending Thin Plate F.E.M. (Adini, Tocher, BCIZ)*, 31 Aug 2006 (Updated 15 Sep 2006). http://www.mathworks.com/matlabcentral/fileexchange/12113-edge-nodes-of-triangular-kirchoff-bending-thin-plate-f-e-m-adinitocherbciz

[4] O.C. Zienkiewicz, R.L. Taylor, *The finite element method – fifth edition, Volume 1: The basis*. Butterworth-Heinemann, 2000, Oxford, Auckland, Boston, Johannesburg, Melbourne, New Delhi.