# 13. MODELING OF AXISYMMETRIC STATE BY FEM SOFTWARE SYSTEMS. MODELING, ANALYSIS OF PROBLEM EVALUATION

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# 13. MODELING OF AXISYMMETRIC STATE BY FEM SOFTWARE SYSTEMS. MODELING, ANALYSIS OF PROBLEM EVALUATION

## 13.1 Finite element solution of axisymmetric problems

For axisymmetric problems both the geometry and the load are independent of the angle coordinate, $\vartheta$. An example is shown in Fig.13.1.

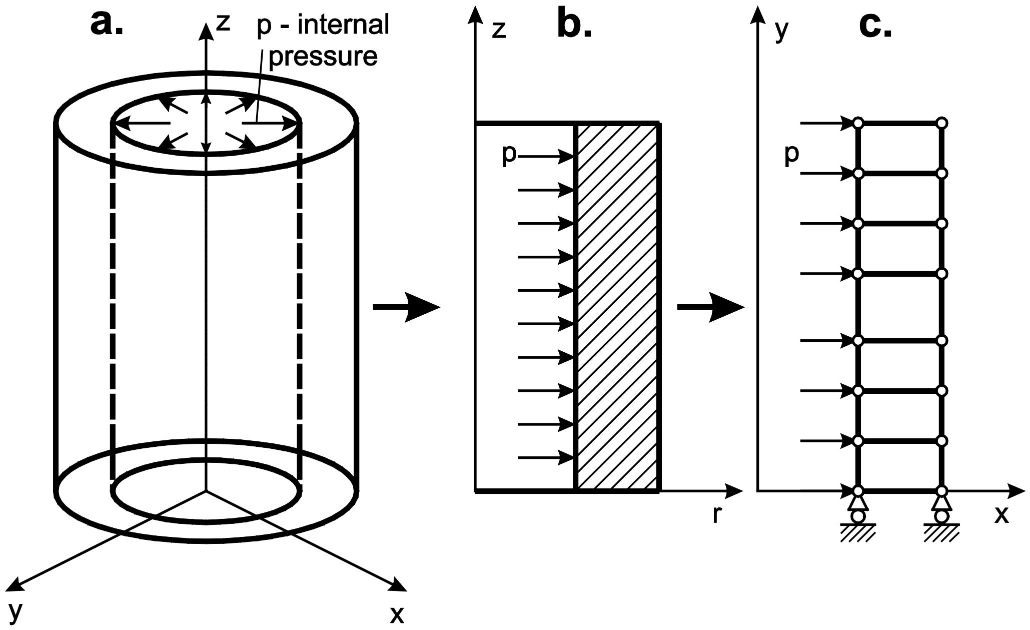


*Fig.13.1. Thick-walled tube under internal pressure (a), axisymmetric model of the tube (b), and the simplified finite element problem (c).*

Plane problems are defined in plane as the meridian section of an actual body; mathematically they can be solved as two-variable problems. The element types of axisymmetric problems are actually ring shape elements. That is why there is no concentrated force in such problems, except for the case when the force coincides with the axis of symmetry. A line load with constant intensity on the outer surface of the model defined by a radius of $r$, looks as a concentrated force. For axisymmetric problems the displacement field has the following form [1]:

$$\underline{u} = u(r,z)\underline{e}_r + w(r,z)\underline{e}_z . \qquad (13.1)$$

The strain-displacement equation is:

$$\underline{\underline{\varepsilon}} = \frac{1}{2}(\underline{u} \circ \nabla + \nabla \circ \underline{u}), \qquad (13.2)$$

where $\nabla$ is the Hamilton operator in cylindrical coordinate system (CCS). It can be derived by the help of Eq.(11.61). Based on Fig.11.7 the radial and tangential unit basis vectors become [1]:

$$\underline{e}_r = \cos\vartheta \underline{i} + \sin\vartheta \underline{j}, \quad \underline{e}_t = -\sin\vartheta \underline{i} + \cos\vartheta \underline{j} . \qquad (13.3)$$

Operator nabla in the *x*-*y*-*z* coordinate system is:

$$\nabla = \frac{\partial}{\partial x}\underline{i} + \frac{\partial}{\partial y}\underline{j} + \frac{\partial}{\partial z}\underline{k}\,. \qquad (13.4)$$

Utilizing Eq.(11.61) and substituting it into Eq.(13.4) leads to:

$$\nabla = \frac{\partial}{\partial r}\underline{e}_r + \frac{1}{r}\frac{\partial}{\partial \varphi}\underline{e}_t + \frac{\partial}{\partial z}\underline{e}_z\,. \qquad (13.5)$$

The strain components in CCS can be written as [2,3] (see Eq.(11.66)):

$$\varepsilon_r = \frac{\partial u}{\partial r}, \varepsilon_t = \frac{u}{r}, \varepsilon_z = \frac{\partial w}{\partial z}, \gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\,. \qquad (13.6)$$

In vector form:

$$\underline{\varepsilon}^T = \left[\varepsilon_r \quad \varepsilon_t \quad \varepsilon_z \quad \gamma_{rz}\right]. \qquad (13.7)$$

The vector of strain components is written in the following form:

$$\underline{\varepsilon} = \underline{\underline{\partial}}\,\underline{u}\,, \qquad (13.8)$$

where, based on Eq.(13.6) the matrix of differential operators is completed with an additional element compared to the plane stress or plane strain states:

$$\underline{\underline{\partial}} = \begin{bmatrix} \frac{\partial}{\partial r} & 0 \\ \frac{1}{r} & 0 \\ 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial r} \end{bmatrix}. \qquad .(13.9)$$

The vector of stress components is:

$$\underline{\sigma}^T = \left[\sigma_r \quad \sigma_t \quad \sigma_z \quad \tau_{rz}\right]. \qquad (13.10)$$

Independently of the coordinate system we have Hooke’s law in the form below:

$$\underline{\underline{\sigma}} = 2G\left[\underline{\underline{\varepsilon}} + \frac{\nu}{1-2\nu}\varepsilon_I \underline{\underline{E}}\right], \qquad (13.11)$$

$$\underline{\underline{\sigma}} = \begin{bmatrix} \sigma_r & 0 & \tau_{rz} \\ 0 & \sigma_t & 0 \\ \tau_{rz} & 0 & \sigma_z \end{bmatrix}, \underline{\underline{\varepsilon}} = \begin{bmatrix} \varepsilon_r & 0 & 1/2\cdot\gamma_{rz} \\ 0 & \varepsilon_t & 0 \\ 1/2\cdot\gamma_{rz} & 0 & \varepsilon_z \end{bmatrix}, \qquad (13.12)$$

from which we have:

$$\sigma_r = \frac{E}{1+\nu}\left[\varepsilon_r + \frac{\nu}{1-2\nu}(\varepsilon_r + \varepsilon_t + \varepsilon_z)\right] = \frac{E}{(1+\nu)(1-2\nu)}[\varepsilon_r(1-\nu) + \varepsilon_t\nu + \varepsilon_z\nu],$$

$$(13.13)$$

$$\sigma_t = \frac{E}{1+\nu}\left[\varepsilon_t + \frac{\nu}{1-2\nu}(\varepsilon_r + \varepsilon_t + \varepsilon_z)\right] = \frac{E}{(1+\nu)(1-2\nu)}[\varepsilon_r\nu + \varepsilon_t(1-\nu) + \varepsilon_z\nu],$$

$$\sigma_z = \frac{E}{1+\nu}\left[\varepsilon_z + \frac{\nu}{1-2\nu}(\varepsilon_r + \varepsilon_t + \varepsilon_z)\right] = \frac{E}{(1+\nu)(1-2\nu)}[\varepsilon_r\nu + \varepsilon_t\nu + \varepsilon_z(1-\nu)],$$

$$\tau_{rz} = \frac{E}{1+\nu}\gamma_{rz}\,.$$

Accordingly, the constitutive matrix based on $\underline{\sigma} = \underline{\underline{C}}\,\underline{\varepsilon}$ is [2,3]:

$$\underline{\underline{C}} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix}. \qquad (13.14)$$

The calculation of the element stiffness matrix is possible through the following definition [4]:

$$\underline{\underline{K}}_e = \int_{V_e} \underline{\underline{B}}^T \underline{\underline{C}}^T \underline{\underline{B}} dV , \qquad (13.15)$$

where the dimension of matrix $\underline{\underline{B}}$ depends on the degrees of freedom of the element. The vector of forces can be determined in the same way as it was shown for plane problems.
The domain of axisymmetric bodies can be meshed by ring shape elements. Elements can be defined in the meridian section, i.e. in plane. In the finite element softwares the same element types are available as those for plane problems; however the axisymmetric behavior should be set. In the course of the finite element analysis the same interpolation functions are applied as those presented for plane stress and plane strain states. In most of the finite element codes the plane model should be prepared in the *x*-*y* plane, where *y* is the axis of revolution (see Fig.13.1c). In the sequel we review the application of the linear triangle and the isoparametric quadrilateral elements.

## 13.2 Axisymmetric linear triangle element

The steps of the finite element discretization using linear triangle element have already been presented in section 12.2. Some modification is required considering the axisymmetric application of the triangle element. In the displacement field we change the *x* and *y* parameters to *r* and *z*, respectively [1]:

$$\underline{u}(r,z) = \begin{bmatrix} u(r,z) \\ w(r,z) \end{bmatrix} = \underline{\underline{N}}(r,z)\underline{u}_e , \qquad (13.16)$$

where the displacement components can be provided by changing the coordinate *x* to *r* and coordinate *y* to *z* in Eq.(12.24), respectively:

$$u(r,z) = N_1(r,z)u_1 + N_2(r,z)u_2 + N_3(r,z)u_3 , \qquad (13.17)$$

$$w(r,z) = N_1(r,z)w_1 + N_2(r,z)w_2 + N_3(r,z)w_3 ,$$

moreover, the matrix of interpolation functions and the vector of nodal displacements become:

$$\underline{\underline{N}}(r,z) = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix}, \qquad (13.18)$$

$$\underline{u}_e^T = [u_1 \quad w_1 \quad u_2 \quad w_2 \quad u_3 \quad w_3].$$

The calculation of the strain components is made in a similar fashion to that presented in plane problems:

$$\underline{\varepsilon} = \underline{\underline{\partial}}\underline{u} = \underline{\underline{\partial}}\underline{\underline{N}}\underline{u}_e = \underline{\underline{B}}\underline{u}_e , \qquad (13.19)$$

where the strain-displacement matrix using Eqs.(13.9) and (13.18) is:

$$\underline{\underline{B}} = \partial \underline{\underline{N}} = \begin{bmatrix} \frac{\partial}{\partial r} & 0 \\ \frac{1}{r} & 0 \\ 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial r} \end{bmatrix} \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{\partial N_1}{\partial r} & 0 & \frac{\partial N_2}{\partial r} & 0 & \frac{\partial N_3}{\partial r} & 0 \\ \frac{N_1}{r} & 0 & \frac{N_2}{r} & 0 & \frac{N_3}{r} & 0 \\ 0 & \frac{\partial N_1}{\partial z} & 0 & \frac{\partial N_2}{\partial z} & 0 & \frac{\partial N_3}{\partial z} \\ \frac{\partial N_1}{\partial z} & \frac{\partial N_1}{\partial r} & \frac{\partial N_2}{\partial z} & \frac{\partial N_2}{\partial r} & \frac{\partial N_3}{\partial z} & \frac{\partial N_3}{\partial r} \end{bmatrix}, \tag{13.20}$$

where in the second row the term $N_i/r$ appears. Considering the axisymmetric nature of the problem we can write that:

$$\underline{\underline{K}}_e = 2\pi \int_{A_e} \underline{\underline{B}}^T \underline{\underline{C}}^T \underline{\underline{B}} r dA = 2\pi \int\int \underline{\underline{B}}^T \underline{\underline{C}}^T \underline{\underline{B}} r dr dz \ . \tag{13.21}$$

The vector of forces consists of three different terms even in axisymmetric problems. For a distributed load the formula is:

$$\underline{F}_{ep} = 2\pi \int \underline{\underline{N}}^T \underline{p} r ds \ , \tag{13.22}$$

where $\underline{p}$ is the vector of pressures in the radial and axial directions:

$$\underline{p} = \begin{bmatrix} p_r \\ p_z \end{bmatrix}. \tag{13.23}$$

In the case of body force the force vector becomes:

$$\underline{F}_{eb} = 2\pi \int\int \underline{\underline{N}}^T \underline{q} r dr dz \ , \tag{13.24}$$

where:

$$\underline{q} = \begin{bmatrix} q_r \\ q_z \end{bmatrix} \tag{13.25}$$

is the density vector of volume forces. Finally, the vector of concentrated forces is:

$$\underline{F}_{ec}^T = \begin{bmatrix} F_{1r} & F_{1z} & F_{2r} & F_{2z} & F_{3r} & F_{3z} \end{bmatrix}. \tag{13.26}$$

The total force vector is the sum the following three vectors:

$$\underline{F}_e = \underline{F}_{ep} + \underline{F}_{eb} + \underline{F}_{ec} \ . \tag{13.27}$$

The problem solution involves the composition of the element and structural stiffness matrices. We calculate first the nodal displacements from the structural equation, then the reactions and strain and stress components, respectively. Let us see an example for the application of the element.

## 13.3 Example for the application of axisymmetric triangle element

Fig.13.2 shows a hollow disk with triangular cross section under internal pressure. The angular velocity of the disk is $\omega = 5$ rad/s. Consider also the own weight of the disk! Calculate the nodal displacements and reactions!

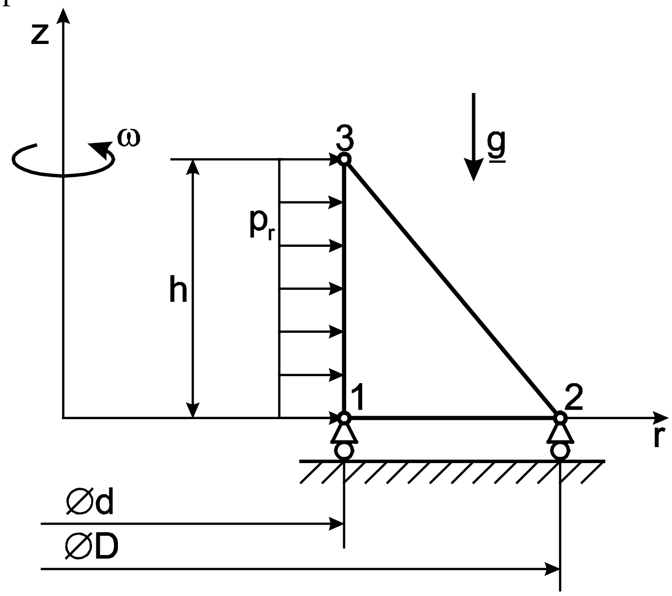


*Fig.13.2. Finite element model of a hollow disk with triangular cross section.*

*Given:*
$p_r = 20$ KPa, $E = 200$ GPa, $d = 6$ m, $D = 8$ m, $g = 9{,}81$ m/s$^2$, $\nu = 0{,}3$, $h = 1$ m

Solve the problem using a single axisymmetric triangular element [1]! The distances are given in [m], the force is given in [N]. The nodal coordinates are:

| node | *r* [m] | *z* [m] |
|---|---|---|
| 1 | 3 | 0 |
| 2 | 4 | 0 |
| 3 | 3 | 1 |

Since we have only a single element, the element equilibrium equation is the same as the structural equation:

$$\underline{\underline{K}}_e \underline{u}_e = \underline{F}_e , \tag{13.28}$$

where:

$$\underline{u}_e^T = [u_1 \quad w_1 \quad u_2 \quad w_2 \quad u_3 \quad w_3]. \tag{13.29}$$

Because of the boundary conditions only four unknowns remain, i.e.:

$$\underline{u}_e^T = [u_1 \quad 0 \quad u_2 \quad 0 \quad u_3 \quad w_3]. \tag{13.30}$$

The constitutive matrix based on Eq.(13.14) is:

$$\underline{\underline{C}} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} 1-\nu & \nu & \nu & 0 \\ \nu & 1-\nu & \nu & 0 \\ \nu & \nu & 1-\nu & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} =$$

$$= \begin{bmatrix} 269{,}2 & 115{,}38 & 115{,}38 & 0 \\ 115{,}38 & 269{,}2 & 115{,}38 & 0 \\ 115{,}38 & 115{,}38 & 269{,}2 & 0 \\ 0 & 0 & 0 & 76{,}9 \end{bmatrix} \cdot 10^9 \text{ Pa} \tag{13.31}$$

The coefficients of the interpolation functions using Eq.(12.22) and Fig.13.2 are:

$$\alpha_1 = r_2 z_3 - r_3 z_2 = 4 \text{ m}^2,\ \alpha_2 = r_3 z_1 - r_1 z_3 = -3 \text{ m}^2,\ \alpha_3 = r_1 z_2 - r_2 z_1 = 0, \tag{13.32}$$

$$\beta_1 = z_2 - z_3 = -1 \text{ m},\ \beta_2 = z_3 - z_1 = 1 \text{ m},\ \beta_3 = z_1 - z_2 = 0,$$

$$\gamma_1 = r_3 - r_2 = -1 \text{ m},\ \gamma_2 = r_1 - r_3 = 0,\ \gamma_3 = r_2 - r_1 = 1 \text{ m}.$$

The area of the triangle is:

$$A_e = \frac{1}{2}(\alpha_1 + \alpha_3 + \alpha_3) = \frac{1}{2}(4-3+0) = \frac{1}{2} \text{ m}^2. \tag{13.33}$$

The interpolation functions can be calculated as:

$$N_i(r,z) = \frac{\alpha_i + \beta_i r + \gamma_i z}{2A_e}, \tag{13.34}$$

which yields:

$$N_1(r,z) = 4 - r - z,\ N_2(r,z) = -3 + r,\ N_3(r,z) = z. \tag{13.35}$$

Matrix $\underline{\underline{N}}$ becomes:

$$\underline{\underline{N}}(r,z) = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} = \begin{bmatrix} 4-r-z & 0 & -3+r & 0 & z & 0 \\ 0 & 4-r-z & 0 & -3+r & 0 & z \end{bmatrix}. \tag{13.36}$$

Accordingly, the strain-displacement matrix $\underline{\underline{B}}$ is:

$$\underline{\underline{B}} = \underline{\underline{\partial}}\,\underline{\underline{N}} = \begin{bmatrix} \frac{\partial N_1}{\partial r} & 0 & \frac{\partial N_2}{\partial r} & 0 & \frac{\partial N_3}{\partial r} & 0 \\ \frac{N_1}{r} & 0 & \frac{N_2}{r} & 0 & \frac{N_3}{r} & 0 \\ 0 & \frac{\partial N_1}{\partial z} & 0 & \frac{\partial N_2}{\partial z} & 0 & \frac{\partial N_3}{\partial z} \\ \frac{\partial N_1}{\partial z} & \frac{\partial N_1}{\partial r} & \frac{\partial N_2}{\partial z} & \frac{\partial N_2}{\partial r} & \frac{\partial N_3}{\partial z} & \frac{\partial N_3}{\partial r} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ \frac{4-r-z}{r} & 0 & \frac{-3+r}{r} & 0 & \frac{z}{r} & 0 \\ 0 & -1 & 0 & 0 & 0 & 1 \\ -1 & -1 & 0 & 1 & 1 & 0 \end{bmatrix}. \tag{13.37}$$

The stiffness matrix is given by:

$$\underline{\underline{K}}_e = 2\pi \int_0^{4-r}\int_3^4 \underline{\underline{B}}^T \underline{\underline{C}}^T \underline{\underline{B}} r dr dz = \begin{bmatrix} 3,43 & 1,89 & -2,79 & -0,81 & -0,90 & -1,09 \\ 1,89 & 3,64 & -1,33 & -0,81 & -0,93 & -2,82 \\ -2,80 & -1,33 & 3,10 & 0 & 0,14 & 1,33 \\ -0,81 & -0,81 & 0 & 0,81 & 0,81 & 0 \\ -0,90 & -0,93 & 0,14 & 0,81 & 0,85 & 0,12 \\ -1,09 & -2,82 & 1,33 & 0 & 0,12 & 2,82 \end{bmatrix} \cdot 10^{12}\ \frac{\text{N}}{\text{m}}, \qquad (13.38)$$

where all of the elements were calculated by exact integration (using the code Maple). The upper range of the first integration is the equation of the hypotenuse of the triangle: $z = 4-r$. The vector of forces is constructed as the sum of three vectors. The first one is related to distributed load along element edge 1-3, based on Eq.(13.21) it is:

$$\underline{F}_{ep} = 2\pi \int \underline{\underline{N}}^T \underline{p} r ds\,,\quad \underline{p} = \begin{bmatrix} p_r \\ p_z \end{bmatrix} = \begin{bmatrix} 20 \\ 0 \end{bmatrix} \text{KPa}\,. \qquad (13.39)$$

Obviously, the radius is $r$ = 3 m constant along element edge 1-3, furthermore the coordinate of integration is $z$, leading to:

$$\underline{F}_{ep} = 2\pi \cdot 3\int_0^1 \underline{\underline{N}}^T \underline{p} dz = 2\pi \cdot 3\int_0^1 \begin{bmatrix} 1-z & 0 \\ 0 & 1-z \\ 0 & 0 \\ 0 & 0 \\ z & 0 \\ 0 & z \end{bmatrix} \begin{bmatrix} 20000 \\ 0 \end{bmatrix} dz = \begin{bmatrix} 6\pi \\ 0 \\ 0 \\ 0 \\ 6\pi \\ 0 \end{bmatrix} \cdot 10^4\ \text{N}\,. \qquad (13.40)$$

The force vectors related to the revolution and own weight requires vector $\underline{q}$, which is calculated using $g$ and $\underline{\omega}$:

$$\underline{q} = \begin{bmatrix} q_r \\ q_z \end{bmatrix} = \begin{bmatrix} \rho r \omega^2 \\ -\rho g \end{bmatrix}. \qquad (13.41)$$

After this, we calculate $\underline{F}_{eb}$ using Eq.(13.23):

$$\underline{F}_{eb} = 2\pi \int_0^{4-r}\int_3^4 \underline{\underline{N}}^T \underline{q} r dr dz = 2\pi \int_0^{4-r}\int_3^4 \begin{bmatrix} 4-r-z & 0 \\ 0 & 4-r-z \\ -3+r & 0 \\ 0 & -3+r \\ z & 0 \\ 0 & z \end{bmatrix} \begin{bmatrix} \rho r \omega^2 \\ -\rho g \end{bmatrix} r dr dz\,, \qquad (13.42)$$

i.e., we have:

$$\underline{F}_{eb}^T = \begin{bmatrix} 6,89 & -0,829 & 7,995 & -0,893 & 6,89 & -0,829 \end{bmatrix} \pi \cdot 10^5\ \text{N}\,. \qquad (13.43)$$

Finally, the unknown reactions are collected in vector $\underline{F}_{ec}$. Considering the boundary conditions we obtain:

$$\underline{F}_{ec}^T = \begin{bmatrix} 0 & F_{1z} & 0 & F_{2z} & 0 & 0 \end{bmatrix}. \qquad (13.44)$$

Thus, the finite element equilibrium equation becomes:

$$\begin{bmatrix} 3{,}43 & 1{,}89 & -2{,}79 & -0{,}81 & -0{,}90 & -1{,}09 \\ 1{,}89 & 3{,}64 & -1{,}33 & -0{,}81 & -0{,}93 & -2{,}82 \\ -2{,}80 & -1{,}33 & 3{,}10 & 0 & 0{,}14 & 1{,}33 \\ -0{,}81 & -0{,}81 & 0 & 0{,}81 & 0{,}81 & 0 \\ -0{,}90 & -0{,}93 & 0{,}14 & 0{,}81 & 0{,}85 & 0{,}12 \\ -1{,}09 & -2{,}82 & 1{,}33 & 0 & 0{,}12 & 2{,}82 \end{bmatrix} \cdot 10^{12} \begin{bmatrix} u_1 \\ 0 \\ u_2 \\ 0 \\ u_3 \\ w_3 \end{bmatrix} = \begin{bmatrix} 7{,}49\pi \cdot 10^5 \\ -0{,}829\pi \cdot 10^5 + F_{1z} \\ 7{,}995\pi \cdot 10^5 \\ -0{,}893 \cdot 10^5 + F_{2z} \\ 7{,}49\pi \cdot 10^5 \\ -0{,}829\pi \cdot 10^5 \end{bmatrix}. \tag{13.45}$$

The solution can be obtained by the 1st, 3rd, 5th and 6th component equations. The other possibility is the application of the matrix equation using the condensed stiffness matrix, which has already been presented in section 12. The solutions are:

$$u_1 = 3{,}701 \cdot 10^{-5}\ \text{m},\ u_2 = 3{,}400 \cdot 10^{-5}\ \text{m},\ u_3 = 3{,}675 \cdot 10^{-5}\ \text{m}, w_4 = -0{,}3424 \cdot 10^{-5}\ \text{m}. \tag{13.46}$$

The reactions utilizing the 2nd and 4th component equations of the finite element equilibrium equation are:

$$F_{1z} = 7{,}272 \cdot 10^5\ \text{N},\ F_{2z} = 74144\ \text{N}. \tag{13.47}$$

The example was verified by the finite element code ANSYS 12. We note that similarly to the examples of section 12 we considered the reactions in the vector of external forces.

The term $N_i/r$ appearing in the second row of matrix $\underline{\underline{B}}$ can cause trouble in the course of integration if one of the element edges lies on the axis of revolution (where $r = 0$). To avoid this problem a local coordinate system is introduced for each element, or the integration is made by approaching $r$ to zero by constructing a hole with very small diameter [1].

## 13.4 Axisymmetric isoparametric quadrilateral element

The isoparametric quadrilateral element for plane problems has been presented in section 12. The element is applicable to solve axisymmetric problems too. The functions of the local $r$ and $z$ coordinates of element edges are [4]:

$$r(\xi,\eta) = N_1(\xi,\eta)r_1 + N_2(\xi,\eta)r_2 + N_3(\xi,\eta)r_3 + N_4(\xi,\eta)r_4 = \underline{N}^T(\xi,\eta)\underline{r},$$

$$z(\xi,\eta) = N_1(\xi,\eta)z_1 + N_2(\xi,\eta)z_2 + N_3(\xi,\eta)z_3 + N_4(\xi,\eta)z_4 = \underline{N}^T(\xi,\eta)\underline{z}, \tag{13.48}$$

where in Eq.(12.77) coordinate $x$ was changed to $r$, coordinate $y$ was changed to $z$. Consequently the same interpolation functions can be used:

$$N_1(\xi,\eta) = \frac{1}{4}(1-\xi)(1-\eta),\ N_2(\xi,\eta) = \frac{1}{4}(1+\xi)(1-\eta), \tag{13.49}$$

$$N_3(\xi,\eta) = \frac{1}{4}(1+\xi)(1+\eta),\ N_4(\xi,\eta) = \frac{1}{4}(1-\xi)(1+\eta).$$

The displacement is formulated in the usual way:

$$\underline{u}(\xi,\eta) = \begin{bmatrix} u(\xi,\eta) \\ w(\xi,\eta) \end{bmatrix} = \underline{\underline{N}}(\xi,\eta)\underline{u}_e, \tag{13.50}$$

where:

$$u(\xi,\eta) = N_1(\xi,\eta)u_1 + N_2(\xi,\eta)u_2 + N_3(\xi,\eta)u_3 + N_4(\xi,\eta)u_4, \tag{13.51}$$

$$w(\xi,\eta) = N_1(\xi,\eta)w_1 + N_2(\xi,\eta)w_2 + N_3(\xi,\eta)w_3 + N_4(\xi,\eta)w_4,$$

with that the matrix of interpolation functions and the vector of nodal displacements are, respectively:

$$\underline{\underline{N}}(\xi,\eta) = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix}, \qquad (13.52)$$

$$\underline{u}_e^T = [u_1 \quad w_1 \quad u_2 \quad w_2 \quad u_3 \quad w_3 \quad u_4 \quad w_4]. \qquad (13.53)$$

The well-known strain-displacement matrix is used to calculate the strain components as:

$$\underline{\varepsilon} = \underline{\underline{\partial}}\underline{u} = \underline{\underline{\partial}}\underline{\underline{N}}\underline{u}_e = \underline{\underline{B}}\underline{u}_e, \qquad (13.54)$$

where:

$$\underline{\underline{B}} = \underline{\underline{\partial}}\underline{\underline{N}} = \begin{bmatrix} \frac{\partial}{\partial r} & 0 \\ \frac{1}{r} & 0 \\ 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial r} \end{bmatrix} \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} =$$

$$= \begin{bmatrix} \frac{\partial N_1}{\partial r} & 0 & \frac{\partial N_2}{\partial r} & 0 & \frac{\partial N_3}{\partial r} & 0 & \frac{\partial N_4}{\partial r} & 0 \\ \frac{N_1}{r} & 0 & \frac{N_2}{r} & 0 & \frac{N_3}{r} & 0 & \frac{N_4}{r} & 0 \\ 0 & \frac{\partial N_1}{\partial z} & 0 & \frac{\partial N_2}{\partial z} & 0 & \frac{\partial N_3}{\partial z} & 0 & \frac{\partial N_4}{\partial z} \\ \frac{\partial N_1}{\partial z} & \frac{\partial N_1}{\partial r} & \frac{\partial N_2}{\partial z} & \frac{\partial N_2}{\partial r} & \frac{\partial N_3}{\partial z} & \frac{\partial N_3}{\partial r} & \frac{\partial N_4}{\partial z} & \frac{\partial N_4}{\partial r} \end{bmatrix}. \qquad (13.55)$$

As it is shown, we need the derivatives of the interpolation functions with respect to $r$ and $z$. Due to the fact that the functions $N_i$ are known in terms of the natural coordinates $\xi$ and $\eta$, we need again the Jacobi matrix and its determinant, referring to Eq.(12.104) we have [4]:

$$\frac{\partial}{\partial r} = \frac{1}{J}\left(J_{22}\frac{\partial}{\partial \xi} - J_{12}\frac{\partial}{\partial \eta}\right), \qquad (13.56)$$

$$\frac{\partial}{\partial z} = \frac{1}{J}\left(-J_{21}\frac{\partial}{\partial \xi} + J_{11}\frac{\partial}{\partial \eta}\right),$$

where:

$$J_{11} = \frac{\partial r}{\partial \xi} = \sum_{i=1}^{4}\frac{\partial N_i}{\partial \xi}r_i = \frac{1}{4}\{-(1-\eta)r_1 + (1-\eta)r_2 + (1+\eta)r_3 - (1+\eta)r_4\},$$

$$(13.57)$$

$$J_{12} = \frac{\partial z}{\partial \xi} = \sum_{i=1}^{4}\frac{\partial N_i}{\partial \xi}z_i = \frac{1}{4}\{-(1-\eta)z_1 + (1-\eta)z_2 + (1+\eta)z_3 - (1+\eta)z_4\},$$

$$J_{21} = \frac{\partial r}{\partial \eta} = \sum_{i=1}^{4}\frac{\partial N_i}{\partial \eta}r_i = \frac{1}{4}\{-(1-\xi)r_1 - (1+\xi)r_2 + (1+\xi)r_3 + (1-\xi)r_4\},$$

$$J_{22} = \frac{\partial z}{\partial \eta} = \sum_{i=1}^{4} \frac{\partial N_i}{\partial \eta} z_i = \frac{1}{4}\left\{-(1-\xi)z_1 - (1+\xi)z_2 + (1+\xi)z_3 + (1-\xi)z_4\right\}.$$

Matrix $\underline{\underline{B}}$ can be produced in a similar way as it was shown by Eq.(12.105), except for the fact that we must consider the term $N_i/r$ appearing in the second row of the matrix. The calculation of the new terms is possible incorporating Eqs.(13.48)-(13.49). Coordinate $r$ in terms of $\xi$ and $\eta$ parameters is given by Eq.(13.48). The formula of the stiffness matrix is:

$$\underline{\underline{K}}_e = 2\pi \int_{-1}^{1}\int_{-1}^{1} \underline{\underline{B}}^T \underline{\underline{C}}^T \underline{\underline{B}} rJd\xi d\eta . \qquad (13.58)$$

To provide the vector of forces we need three vectors, the first one is:

$$\underline{F}_{ep} = 2\pi \int_{-1}^{1} \underline{\underline{N}}^T \underline{p} rJd\xi , \underline{F}_{ep} = 2\pi \int_{-1}^{1} \underline{\underline{N}}^T \underline{p} rJd\eta , \qquad (13.59)$$

depending on the fact that which one of the element edges is loaded by the line load, moreover the second and third vectors are:

$$\underline{F}_{eb} = 2\pi \int_{-1}^{1}\int_{-1}^{1} \underline{\underline{N}}^T \underline{q} rJd\xi d\eta , \qquad (13.60)$$

$$\underline{F}_{ec}^T = \left[F_{r1} \quad F_{z1} \quad F_{r2} \quad F_{z2} \quad F_{r3} \quad F_{z3} \quad F_{r4} \quad F_{z4}\right]. \qquad (13.61)$$

Finally the total force vector is:

$$\underline{F}_e = \underline{F}_{ep} + \underline{F}_{eb} + \underline{F}_{ec} . \qquad (13.62)$$

In the sequel we present an example for the application of the element.

## 13.5 Example for the application of axisymmetric isoparametric quadrilateral element

Solve the problem of the rotating disk of which analytical solution has been presented in section 11.6.2 using two isoparametric quadrilateral elements! The finite element model of the disk is shown in Fig.13.3.

a. *The angular velocity of the disk is $\omega$ = 880,5 rad/s, verifiy if the disk gets loose!*
b. *Calculate the stresses in that case when there is no revolution, i.e.: $\omega$ = 0, but there is an overlap of $\delta = 0{,}02 \cdot 10^{-3}$ m!*

*Given:*
$r_b$ = 0,02 m, $r_k$ = 0,2 m, $h$ = 0,04 m, $\rho$ = 7800 kg/m$^3$, $E$ = 200 GPa, $\nu$ = 0,3.

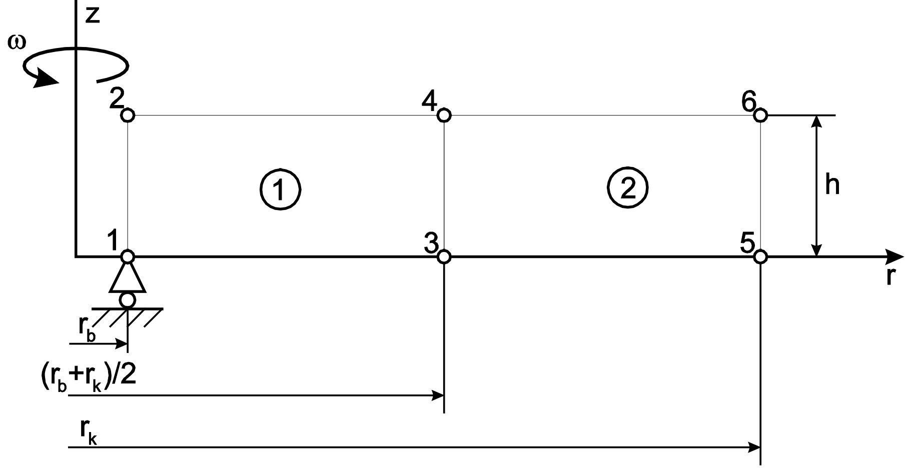

*Fig.13.3. A simple finite element model of a rotating disk.*

We give the distances in [m] and the force in [N]. The nodal coordinates are:

| node | *r* [m] | *z* [m] | node | *r* [m] | *z* [m] |
|---|---|---|---|---|---|
| 1 | 0,02 | 0 | 4 | 0,11 | 0,04 |
| 2 | 0,02 | 0,04 | 5 | 0,2 | 0 |
| 3 | 0,11 | 0 | 6 | 0,2 | 0,04 |

The element-node table becomes:

| element | node | | | |
|---|---|---|---|---|
| 1 | 1 | 3 | 4 | 2 |
| 2 | 3 | 5 | 6 | 4 |

In the knowledge of the boundary conditions the structural vector of nodal displacements is:

$$\underline{U}^T = [u_1 \quad 0 \quad u_2 \quad w_2 \quad u_3 \quad w_3 \quad u_4 \quad w_4 \quad u_5 \quad w_5 \quad u_6 \quad w_6]. \tag{13.63}$$

The constitutive matrix using Eq.(13.14) is:

$$\underline{\underline{C}} = \begin{bmatrix} 269,2 & 115,38 & 115,38 & 0 \\ 115,38 & 269,2 & 115,38 & 0 \\ 115,38 & 115,38 & 269,2 & 0 \\ 0 & 0 & 0 & 76,9 \end{bmatrix} \cdot 10^9 \text{ Pa}. \tag{13.64}$$

The elements of the Jacobi matrix must be produced for both elements based on Eq.(13.57):

$$J_{11}^{(1)} = \frac{1}{4}\{-(1-\eta)r_1 + (1-\eta)r_3 + (1+\eta)r_4 - (1+\eta)r_2\} = 0{,}045,$$

$$(13.65)$$

$$J_{12}^{(1)} = \frac{1}{4}\{-(1-\eta)z_1 + (1-\eta)z_3 + (1+\eta)z_4 - (1+\eta)z_2\} = 0,$$

$$J_{21}^{(1)} = \frac{1}{4}\{-(1-\xi)r_1 - (1+\xi)r_3 + (1+\xi)r_4 + (1-\xi)r_2\} = 0,$$

$$J_{22}^{(1)} = \frac{1}{4}\{-(1-\xi)z_1 - (1+\xi)z_3 + (1+\xi)z_4 + (1-\xi)z_2\} = 0{,}02.$$

and:

$$J_{11}^{(2)} = \frac{1}{4}\{-(1-\eta)r_3 + (1-\eta)r_5 + (1+\eta)r_6 - (1+\eta)r_4\} = 0{,}045,$$

$$(13.66)$$

$$J_{12}^{(2)} = \frac{1}{4}\{-(1-\eta)z_3 + (1-\eta)z_5 + (1+\eta)z_6 - (1+\eta)z_4\} = 0,$$

$$J_{21}^{(2)} = \frac{1}{4}\{-(1-\xi)r_3 - (1+\xi)r_5 + (1+\xi)r_6 + (1-\xi)r_4\} = 0,$$

$$J_{22}^{(2)} = \frac{1}{4}\left\{-(1-\xi)z_3 - (1+\xi)z_5 + (1+\xi)z_6 + (1-\xi)z_4\right\} = 0{,}02\,.$$

The elements of the Jacobi matrix, and so the determinant is constant and identical for both elements:

$$J^{(1)} = J^{(2)} = J = 0{,}0009\,. \qquad (13.67)$$

Continuing the calculation we compute the derivatives of interpolation functions with respect to $r$ and $z$ in accordance with Eq.(13.56). Due to the identical Jacobi determinants of the elements, the derivatives of the interpolation functions will be identical too. Therefore, we can omit the superscripts of the elements of Jacobi matrix:

$$\frac{\partial N_1}{\partial r}^{(1)} = \frac{\partial N_1}{\partial r}^{(2)} = \frac{1}{J}(J_{22}\frac{\partial N_1}{\partial \xi} - J_{12}\frac{\partial N_1}{\partial \eta}) = -5{,}55555 + 5{,}55555\eta\,,$$

$$(13.68)$$

$$\frac{\partial N_2}{\partial r}^{(1)} = \frac{\partial N_2}{\partial r}^{(2)} = \frac{1}{J}(J_{22}\frac{\partial N_2}{\partial \xi} - J_{12}\frac{\partial N_2}{\partial \eta}) = 5{,}55555 - 5{,}55555\eta\,,$$

$$\frac{\partial N_3}{\partial r}^{(1)} = \frac{\partial N_3}{\partial r}^{(2)} = \frac{1}{J}(J_{22}\frac{\partial N_3}{\partial \xi} - J_{12}\frac{\partial N_3}{\partial \eta}) = 5{,}55555 + 5{,}55555\eta\,,$$

$$\frac{\partial N_4}{\partial r}^{(1)} = \frac{\partial N_4}{\partial r}^{(2)} = \frac{1}{J}(J_{22}\frac{\partial N_4}{\partial \xi} - J_{12}\frac{\partial N_4}{\partial \eta}) = -5{,}55555 - 5{,}55555\eta\,,$$

$$\frac{\partial N_1}{\partial z}^{(1)} = \frac{\partial N_1}{\partial z}^{(2)} = \frac{1}{J}(-J_{21}\frac{\partial N_1}{\partial \xi} + J_{11}\frac{\partial N_1}{\partial \eta}) = -12{,}5 + 12{,}5\xi\,,$$

$$(13.69)$$

$$\frac{\partial N_2}{\partial z}^{(1)} = \frac{\partial N_2}{\partial z}^{(2)} = \frac{1}{J}(-J_{21}\frac{\partial N_2}{\partial \xi} + J_{11}\frac{\partial N_2}{\partial \eta}) = -12{,}5 - 12{,}5\xi\,,$$

$$\frac{\partial N_3}{\partial z}^{(1)} = \frac{\partial N_3}{\partial z}^{(2)} = \frac{1}{J}(-J_{21}\frac{\partial N_3}{\partial \xi} + J_{11}\frac{\partial N_3}{\partial \eta}) = 12{,}5 + 12{,}5\xi\,,$$

$$\frac{\partial N_4}{\partial z}^{(1)} = \frac{\partial N_4}{\partial z}^{(2)} = \frac{1}{J}(-J_{21}\frac{\partial N_4}{\partial \xi} + J_{11}\frac{\partial N_4}{\partial \eta}) = 12{,}5 - 12{,}5\xi\,,$$

Coordinate $r$ should be given for both elements separately based on Eq.(13.48):

$$r^{(1)} = N_1 r_1 + N_2 r_3 + N_3 r_4 + N_4 r_2 =$$

$$= 0{,}005(1-\eta)(1-\xi) + 0{,}0275(1-\eta)(1+\xi) + 0{,}0275(1+\eta)(1+\xi) + 0{,}005(1+\eta)(1-\xi),$$

$$(13.70)$$

$$r^{(2)} = N_1 r_3 + N_2 r_5 + N_3 r_6 + N_4 r_4 =$$

$$= 0{,}0275(1-\eta)(1-\xi) + 0{,}005(1-\eta)(1+\xi) + 0{,}005(1+\eta)(1+\xi) + 0{,}0275(1+\eta)(1-\xi),$$

where we considered also the element orientation (the local numbering of the nodes of element). As a next step, we provide the strain-displacement matrix for each element using Eq.(13.55). The elements of matrices are the functions of $\xi$ and $\eta$, which are extremely complicated, therefore we do not give them here. The element stiffness matrices can be calculated using the $\underline{\underline{B}}$ matrices:

$$\underline{\underline{K}}_{e1} = 2\pi \int_{-1}^{1}\int_{-1}^{1} \underline{\underline{B}}^{(1)T} \underline{\underline{C}}^{T} \underline{\underline{B}}^{(1)} r^{(1)} J d\xi d\eta =$$

$$= \begin{bmatrix} 40{,}62 & 4{,}23 & 1{,}34 & 3{,}02 & -17{,}00 & -15{,}10 & -2{,}80 & 7{,}85 \\ . & 58{,}57 & -4{,}83 & 36{,}58 & -24{,}17 & -43{,}56 & -7{,}85 & -51{,}59 \\ . & . & 65{,}83 & -35{,}04 & -14{,}66 & 15{,}71 & -17{,}00 & 24{,}17 \\ . & . & . & 115{,}67 & -15{,}71 & -108{,}69 & 15{,}10 & -43{,}56 \\ . & . & . & . & 65{,}84 & 35{,}04 & 1{,}34 & 4{,}83 \\ . & . & . & . & . & 115{,}67 & -3{,}02 & 36{,}58 \\ . & . & . & . & . & . & 40{,}62 & -4{,}23 \\ . & . & . & . & . & . & . & 58{,}57 \end{bmatrix} \cdot 10^{9}\,\frac{\mathrm{N}}{\mathrm{m}}, \tag{13.71}$$

$$\underline{\underline{K}}_{e2} = 2\pi \int_{-1}^{1}\int_{-1}^{1} \underline{\underline{B}}^{(2)T} \underline{\underline{C}}^{T} \underline{\underline{B}}^{(2)} r^{(2)} J d\xi d\eta =$$

$$= \begin{bmatrix} 82{,}40 & 31{,}42 & -8{,}53 & 8{,}46 & -46{,}41 & -42{,}29 & -30{,}85 & 2{,}42 \\ . & 179{,}20 & -10{,}27 & 87{,}23 & -51{,}35 & -103{,}87 & -2{,}42 & -162{,}56 \\ . & . & 116{,}71 & -62{,}23 & -38{,}16 & 21{,}15 & -46{,}40 & 51{,}35 \\ . & . & . & 236{,}30 & -21{,}15 & -219{,}65 & 42{,}29 & -103{,}88 \\ . & . & . & . & 116{,}71 & 62{,}23 & -8{,}53 & 10{,}23 \\ . & . & . & . & . & 236{,}30 & -8{,}46 & 87{,}23 \\ . & . & . & . & . & . & 82{,}40 & -31{,}42 \\ . & . & . & . & . & . & . & 179{,}20 \end{bmatrix} \cdot 10^{9}\,\frac{\mathrm{N}}{\mathrm{m}}.$$

As the node numbering does not correspond to the element orientations we need to rearrange the element stiffness matrices in accordance with the numerals of degrees of freedom. Let the vector of nodal displacements be equal to:

$$\underline{u}_{e1}^{T} = [u_1 \quad w_1 \quad u_2 \quad w_2 \quad u_3 \quad w_3 \quad u_4 \quad w_4], \tag{13.72}$$

$$\underline{u}_{e2}^{T} = [u_3 \quad w_3 \quad u_4 \quad w_4 \quad u_5 \quad w_5 \quad u_6 \quad w_6].$$

Corresponding to the former, the original element stiffness matrices are rearranged as:

$$\underline{\underline{K}}_{e1} = \begin{bmatrix} k_{e11}^{1} & k_{e12}^{1} & k_{e17}^{1} & k_{e18}^{1} & k_{e13}^{1} & k_{e14}^{1} & k_{e15}^{1} & k_{e16}^{1} \\ k_{e21}^{1} & k_{e22}^{1} & k_{e27}^{1} & k_{e28}^{1} & k_{e23}^{1} & k_{e24}^{1} & k_{e25}^{1} & k_{e26}^{1} \\ k_{e71}^{1} & k_{e72}^{1} & k_{e77}^{1} & k_{e78}^{1} & k_{e73}^{1} & k_{e74}^{1} & k_{e75}^{1} & k_{e76}^{1} \\ k_{e81}^{1} & k_{e82}^{1} & k_{e87}^{1} & k_{e88}^{1} & k_{e83}^{1} & k_{e84}^{1} & k_{e85}^{1} & k_{e86}^{1} \\ k_{e31}^{1} & k_{e32}^{1} & k_{e37}^{1} & k_{e38}^{1} & k_{e33}^{1} & k_{e34}^{1} & k_{e35}^{1} & k_{e36}^{1} \\ k_{e41}^{1} & k_{e42}^{1} & k_{e47}^{1} & k_{e48}^{1} & k_{e43}^{1} & k_{e44}^{1} & k_{e45}^{1} & k_{e46}^{1} \\ k_{e51}^{1} & k_{e52}^{1} & k_{e57}^{1} & k_{e58}^{1} & k_{e53}^{1} & k_{e54}^{1} & k_{e55}^{1} & k_{e56}^{1} \\ k_{e61}^{1} & k_{e62}^{1} & k_{e67}^{1} & k_{e68}^{1} & k_{e63}^{1} & k_{e64}^{1} & k_{e65}^{1} & k_{e66}^{1} \end{bmatrix}. \tag{13.73}$$

Based on the nodes of the second element the rearrangement is made as:

$$\underline{\underline{K}}_{e2} = \begin{bmatrix} k_{e11}^2 & k_{e12}^2 & k_{e17}^2 & k_{e18}^2 & k_{e13}^2 & k_{e14}^2 & k_{e15}^2 & k_{e16}^2 \\ k_{e21}^2 & k_{e22}^2 & k_{e27}^2 & k_{e28}^2 & k_{e23}^2 & k_{e24}^2 & k_{e25}^2 & k_{e26}^2 \\ k_{e71}^2 & k_{e72}^2 & k_{e77}^2 & k_{e78}^2 & k_{e73}^2 & k_{e74}^2 & k_{e75}^2 & k_{e76}^2 \\ k_{e81}^2 & k_{e82}^2 & k_{e87}^2 & k_{e88}^2 & k_{e83}^2 & k_{e84}^2 & k_{e85}^2 & k_{e86}^2 \\ k_{e31}^2 & k_{e32}^2 & k_{e37}^2 & k_{e38}^2 & k_{e33}^2 & k_{e34}^2 & k_{e35}^2 & k_{e36}^2 \\ k_{e41}^2 & k_{e42}^2 & k_{e47}^2 & k_{e48}^2 & k_{e43}^2 & k_{e44}^2 & k_{e45}^2 & k_{e46}^2 \\ k_{e51}^2 & k_{e52}^2 & k_{e57}^2 & k_{e58}^2 & k_{e53}^2 & k_{e54}^2 & k_{e55}^2 & k_{e56}^2 \\ k_{e61}^2 & k_{e62}^2 & k_{e67}^2 & k_{e68}^2 & k_{e63}^2 & k_{e64}^2 & k_{e65}^2 & k_{e66}^2 \end{bmatrix}. \quad (13.74)$$

Now, we can construct the structural stiffness matrix. The mutual nodes are the third and fourth ones. Accordingly, the combination of the two matrices results in:

$$\underline{\underline{K}} = \begin{bmatrix} k_{e11}^1 & k_{e12}^1 & k_{e17}^1 & k_{e18}^1 & k_{e13}^1 & k_{e14}^1 & k_{e15}^1 & k_{e16}^1 & 0 & 0 & 0 & 0 \\ k_{e21}^1 & k_{e22}^1 & k_{e27}^1 & k_{e28}^1 & k_{e23}^1 & k_{e24}^1 & k_{e25}^1 & k_{e26}^1 & 0 & 0 & 0 & 0 \\ k_{e71}^1 & k_{e72}^1 & k_{e77}^1 & k_{e78}^1 & k_{e73}^1 & k_{e74}^1 & k_{e75}^1 & k_{e76}^1 & 0 & 0 & 0 & 0 \\ k_{e81}^1 & k_{e82}^1 & k_{e87}^1 & k_{e88}^1 & k_{e83}^1 & k_{e84}^1 & k_{e85}^1 & k_{e86}^1 & 0 & 0 & 0 & 0 \\ k_{e31}^1 & k_{e32}^1 & k_{e37}^1 & k_{e38}^1 & k_{e33}^1+k_{e11}^2 & k_{e34}^1+k_{e12}^2 & k_{e35}^1+k_{e17}^2 & k_{e36}^1+k_{e18}^2 & k_{e13}^2 & k_{e14}^2 & k_{e15}^2 & k_{e16}^2 \\ k_{e41}^1 & k_{e42}^1 & k_{e47}^1 & k_{e48}^1 & k_{e43}^1+k_{e21}^2 & k_{e44}^1+k_{e22}^2 & k_{e45}^1+k_{e27}^2 & k_{e46}^1+k_{e28}^2 & k_{e23}^2 & k_{e24}^2 & k_{e25}^2 & k_{e26}^2 \\ k_{e51}^1 & k_{e52}^1 & k_{e57}^1 & k_{e58}^1 & k_{e53}^1+k_{e71}^2 & k_{e54}^1+k_{e72}^2 & k_{e55}^1+k_{e77}^2 & k_{e56}^1+k_{e78}^2 & k_{e73}^2 & k_{e74}^2 & k_{e75}^2 & k_{e76}^2 \\ k_{e61}^1 & k_{e62}^1 & k_{e67}^1 & k_{e68}^1 & k_{e63}^1+k_{e81}^2 & k_{e64}^1+k_{e82}^2 & k_{e65}^1+k_{e87}^2 & k_{e66}^1+k_{e88}^2 & k_{e83}^2 & k_{e84}^2 & k_{e85}^2 & k_{e86}^2 \\ 0 & 0 & 0 & 0 & k_{e31}^2 & k_{e32}^2 & k_{e37}^2 & k_{e38}^2 & k_{e33}^2 & k_{e34}^2 & k_{e35}^2 & k_{e36}^2 \\ 0 & 0 & 0 & 0 & k_{e41}^2 & k_{e42}^2 & k_{e47}^2 & k_{e48}^2 & k_{e43}^2 & k_{e44}^2 & k_{e45}^2 & k_{e46}^2 \\ 0 & 0 & 0 & 0 & k_{e51}^2 & k_{e52}^2 & k_{e57}^2 & k_{e58}^2 & k_{e53}^2 & k_{e54}^2 & k_{e55}^2 & k_{e56}^2 \\ 0 & 0 & 0 & 0 & k_{e61}^2 & k_{e62}^2 & k_{e67}^2 & k_{e68}^2 & k_{e63}^2 & k_{e64}^2 & k_{e65}^2 & k_{e66}^2 \end{bmatrix}.$$

(13.75)

We note that the finite element codes provide the structural stiffness matrix using the element-node table. The numerical values can be obtained using Eq.(13.71). The force vector consists of the vectors of body and concentrated forces. The density vector of the body force is:

$$\underline{q} = \begin{bmatrix} q_r \\ 0 \end{bmatrix} = \begin{bmatrix} \rho r \omega^2 \\ 0 \end{bmatrix}, \text{ and: } \underline{q}^{(1)} = \begin{bmatrix} \rho r^{(1)} \omega^2 \\ 0 \end{bmatrix}, \ \underline{q}^{(2)} = \begin{bmatrix} \rho r^{(2)} \omega^2 \\ 0 \end{bmatrix},$$

(13.76)

from which we have:

$$\underline{F}_{eb}^{(1)} = 2\pi \int_{-1}^{1}\int_{-1}^{1} \underline{\underline{N}}^T \underline{q}^{(1)} r^{(1)} J d\xi d\eta = [1{,}01 \quad 0 \quad 2{,}34 \quad 0 \quad 2{,}34 \quad 0 \quad 1{,}01 \quad 0]^T \cdot 10^5 \text{ N},$$

(13.77)

$$\underline{F}_{eb}^{(2)} = 2\pi \int_{-1}^{1}\int_{-1}^{1} \underline{\underline{N}}^T \underline{q}^{(2)} r^{(2)} J d\xi d\eta = [6{,}86 \quad 0 \quad 10{,}04 \quad 0 \quad 10{,}04 \quad 0 \quad 6{,}86 \quad 0]^T \cdot 10^5 \text{ N}.$$

Similarly to the stiffness matrices, the rearrangement is required also in the force vectors according to the local node numbering:

$$\underline{F}_{eb}^{(1)}=\begin{bmatrix}F_{eb1}^{(1)}\\F_{eb2}^{(1)}\\F_{eb7}^{(1)}\\F_{eb8}^{(1)}\\F_{eb3}^{(1)}\\F_{eb4}^{(1)}\\F_{eb5}^{(1)}\\F_{eb6}^{(1)}\end{bmatrix}=\begin{bmatrix}1{,}01\\0\\1{,}01\\0\\2{,}34\\0\\2{,}34\\0\end{bmatrix}\cdot 10^5\ \text{N and } \underline{F}_{eb}^{(1)}=\begin{bmatrix}F_{eb1}^{(2)}\\F_{eb2}^{(2)}\\F_{eb7}^{(2)}\\F_{eb8}^{(2)}\\F_{eb3}^{(2)}\\F_{eb4}^{(2)}\\F_{eb5}^{(2)}\\F_{eb6}^{(2)}\end{bmatrix}=\begin{bmatrix}6{,}86\\0\\6{,}86\\0\\10{,}04\\0\\10{,}04\\0\end{bmatrix}\cdot 10^5\ \text{N}. \tag{13.78}$$

The structural force vector is calculated as the sum the two former vectors:

$$\underline{F}_b^T=\begin{bmatrix}F_{eb1}^{(1)} & F_{eb2}^{(1)} & F_{eb7}^{(1)} & F_{eb8}^{(1)} & \begin{matrix}F_{eb3}^{(1)}+\\+F_{eb1}^{(2)}\end{matrix} & \begin{matrix}F_{eb4}^{(1)}+\\+F_{eb2}^{(2)}\end{matrix} & \begin{matrix}F_{eb5}^{(1)}+\\+F_{eb7}^{(2)}\end{matrix} & \begin{matrix}F_{eb6}^{(1)}+\\+F_{eb8}^{(2)}\end{matrix} & F_{eb3}^{(2)} & F_{eb4}^{(2)} & F_{eb5}^{(2)} & F_{eb6}^{(2)}\end{bmatrix}=$$

$$=[1{,}01\quad 0\quad 1{,}01\quad 0\quad 9{,}20\quad 0\quad 9{,}20\quad 0\quad 10{,}04\quad 0\quad 10{,}04\quad 0]\cdot 10^5\ \text{N}. \tag{13.79}$$

The vector containing the reaction is:

$$\underline{F}_c^T=[0\quad F_{1z}\quad 0\quad 0\quad 0\quad 0\quad 0\quad 0\quad 0\quad 0\quad 0\quad 0]. \tag{13.80}$$

The structural force vector is:

$$\underline{F}=\underline{F}_b+\underline{F}_c. \tag{13.81}$$

Finally, the structural equation is:

$$\underline{\underline{K}}\underline{U}=\underline{F}. \tag{13.82}$$

The system of equations consists of twelve equations. From the 1st and 3rd-12th equations we determine the nodal displacements. The solutions are:

$$u_1=u_2=0{,}0168\cdot 10^{-3}\ \text{m},\ w_1=0,\ w_2=-0{,}0149\cdot 10^{-3}\ \text{m} \tag{13.83}$$

$$u_3=u_4=0{,}0368\cdot 10^{-3}\ \text{m},\ w_3=-0{,}00365\cdot 10^{-3}\ \text{m},\ w_4=-0{,}0113\cdot 10^{-3}\ \text{m},$$

$$u_5=u_6=0{,}0440\cdot 10^{-3}\ \text{m},\ w_5=-0{,}0051\cdot 10^{-3}\ \text{m},\ w_6=-0{,}0098\cdot 10^{-3}\ \text{m},$$

It is seen that if the disk rotates with maximal angular velocity, then in accordance with the finite element model we do not reach the overlap value of $0{,}02\cdot 10^{-3}$ m calculated from the analytical model, i.e. the disk will not get loose. This disagreement can be explained by the coarse mesh of the finite element model, which consists of only two elements. The deformed shape of the structure compared to the original state is shown in Fig.13.4. Based on the displacement solutions we construct the nodal displacement vectors of the elements:

$$\underline{u}_{e1}^T=[u_1\quad 0\quad u_3\quad w_3\quad u_4\quad w_4\quad u_2\quad w_2], \tag{13.84}$$

$$\underline{u}_{e2}^T=[u_3\quad w_3\quad u_5\quad w_5\quad u_6\quad w_6\quad u_4\quad w_4].$$

In the former two vectors we followed the original order of the local node numbering, because matrix $\underline{\underline{B}}$ was constructed in accordance with this fact. The vectors of strain components for both elements are calculated using matrix $\underline{\underline{B}}$:

$$\underline{\varepsilon}^{(1)}=\underline{\underline{B}}^{(1)}\underline{u}_{e1},\ \underline{\varepsilon}^{(2)}=\underline{\underline{B}}^{(2)}\underline{u}_{e2}. \tag{13.85}$$

The vector of stress components are:

$$\underline{\sigma}^{(1)}=\underline{\underline{C}}\underline{\varepsilon}^{(1)},\ \underline{\sigma}^{(2)}=\underline{\underline{C}}\underline{\varepsilon}^{(2)}. \tag{13.86}$$

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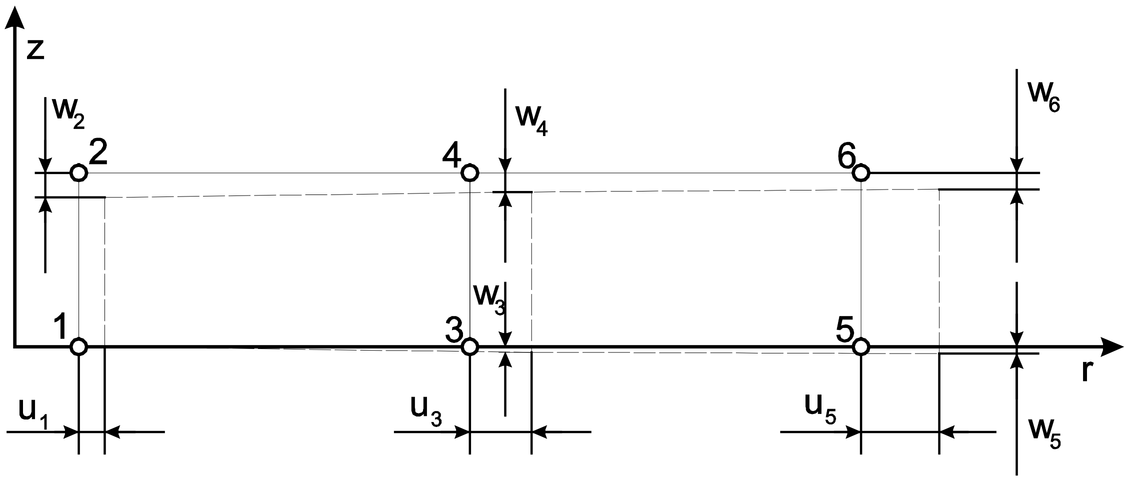


*Fig.13.4. Deformed shape of the finite element model of rotating disk.*

The results are summarized in Tables 13.1 and 13.2. In the tables we listed the nodal solutions. Element solutions are possible to calculate only at mutual nodes 3 and 4 by averaging the nodal solution. According to Table 13.2 it is seen that the dynamic boundary conditions are violated, concretely speaking the radial stress at nodes 1, 2, 5 and 6 is not zero. The reason for that is the low resolution of the mesh and the linear interpolation. On the contrary, the tangential stress agrees quite well at the inner and outer boundaries with the results presented in Fig.11.10a. The example was verified by the code ANSYS 12.

| element | node | $\varepsilon_r$ [$\cdot 10^{-3}$] | $\varepsilon_t$ [$\cdot 10^{-3}$] | $\varepsilon_z$ [$\cdot 10^{-3}$] | $\gamma_{rz}$ [$\cdot 10^{-3}$] |
|---|---|---|---|---|---|
| 1 | 1 | 0,222 | 0,840 | -0,373 | -0,041 |
| | 2 | 0,222 | 0,840 | -0,373 | 0,041 |
| | 3 | 0,222 | 0,335 | -0,191 | -0,041 |
| | 4 | 0,222 | 0,335 | -0,191 | 0,041 |
| 2 | 3 | 0,080 | 0,335 | -0,191 | -0,016 |
| | 4 | 0,080 | 0,335 | -0,191 | 0,016 |
| | 5 | 0,080 | 0,220 | -0,118 | -0,016 |
| | 6 | 0,080 | 0,220 | -0,118 | 0,016 |

*Table 13.1. Strain components in the rotating disk in the case of $\omega$ = 880,5 rad/s.*

| element | node | $\sigma_r$ [MPa] | $\sigma_t$ [MPa] | $\sigma_z$ [MPa] | $\tau_{rz}$ [MPa] |
|---|---|---|---|---|---|
| 1 | 1 | 113,7 | 208,7 | 22,1 | -3,1 |
| | 2 | 113,7 | 208,7 | 22,1 | 3,1 |
| | 3 | 76,4 | 93,7 | 12,9 | -3,1 |
| | 4 | 76,4 | 93,7 | 12,9 | 3,1 |
| 2 | 3 | 38,1 | 77,3 | -3,5 | -1,25 |
| | 4 | 38,1 | 77,3 | -3,53 | 1,25 |
| | 5 | 33,4 | 54,9 | 2,97 | -1,25 |
| | 6 | 33,4 | 54,9 | 2,97 | 1,25 |

*Table 13.2. Stresses in the rotating disk in the case of $\omega$ = 880,5 rad/s.*

In that case, when there is no rotation the structural vector of nodal displacements becomes:

$$\underline{U}^T = [\delta \quad 0 \quad \delta \quad w_2 \quad u_3 \quad w_3 \quad u_4 \quad w_4 \quad u_5 \quad w_5 \quad u_6 \quad w_6]. \tag{13.87}$$

The stiffness matrix remains the same, the vector of forces is:

$$\underline{F}_c^T = [F_{1r} \quad F_{1z} \quad F_{2r} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0]. \tag{13.88}$$

The solutions are:

$$u_1 = u_2 = 0{,}02 \cdot 10^{-3}\ \mathrm{m},\ w_1 = 0,\ w_2 = -0{,}0049 \cdot 10^{-3}\ \mathrm{m}, \tag{13.89}$$

$$u_3 = u_4 = 0{,}0055 \cdot 10^{-3}\ \mathrm{m},\ w_3 = 0{,}0056 \cdot 10^{-3}\ \mathrm{m},\ w_4 = -0{,}0020 \cdot 10^{-3}\ \mathrm{m},$$

$$u_5 = u_6 = 0{,}0038 \cdot 10^{-3}\ \mathrm{m},\ w_5 = -0{,}0022 \cdot 10^{-3}\ \mathrm{m},\ w_6 = -0{,}0027 \cdot 10^{-3}\ \mathrm{m}.$$

Table 13.3 contains the stresses in the disk when there is no rotation. Compared to the results of the analytical solution the differences are quite large, which can be explained again by the low resolution finite element mesh and the linear interpolation.

| element | node | $\sigma_r$ [MPa] | $\sigma_t$ [MPa] | $\sigma_z$ [MPa] | $\tau_{rz}$ [MPa] |
|---|---|---|---|---|---|
| 1 | 1 | 58,1 | 236,6 | 63,9 | -2,5 |
| | 2 | 58,1 | 236,6 | 63,9 | 2,5 |
| | 3 | -34,8 | -2,3 | -6,7 | -2,5 |
| | 4 | -34,8 | -2,3 | 6,7 | 2,5 |
| 2 | 3 | 3,2 | 13,9 | 9,6 | 0,6 |
| | 4 | 3,2 | 13,9 | 9,6 | -0,6 |
| | 5 | -4,5 | 1,4 | -3,6 | 0,6 |
| | 6 | -4,5 | 1,4 | -3,6 | -0,6 |

*Table 13.3. Stresses in the disk in the case of $\omega$ = 0.*

## 13.6 Bibliography

[1] József Uj, *Lectures and practices of the subject Numerical methods in mechanics*, PhD formation, Budapest University of Technology and Economics, Faculty of Mechanical Engineering, Department of Applied Mechanics, 2002/2003 spring semester (in Hungarian).

[2] David V. Hutton, *Fundamentals of finite element analysis, 1st edition*. McGraw-Hill, 2004, New York.

[3] O.C. Zienkiewicz, R.L. Taylor, *The finite element method – fifth edition*, *Volume 1: The basis*. Butterworth-Heinemann, 2000, Oxford, Auckland, Boston, Johannesburg, Melbourne, New Delhi.

[4] Erdogan Madenci, Ibrahim Guven, *The finite element method and applications in engineering using ANSYS*. Springer Science+Business Media Inc., 2006, New York, USA.