12. MODELING OF THE PLANE STRESS STATE USING FEM SOFTWARE SYSTEMS. MODELING, ANALYSIS OF PROBLEM EVALUATION

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12.1 Finite element solution of plane problems

In the application of the finite element method we divide the plane domain of the whole structure into discrete elements as it is illustrated in Fig.12.1 [1].
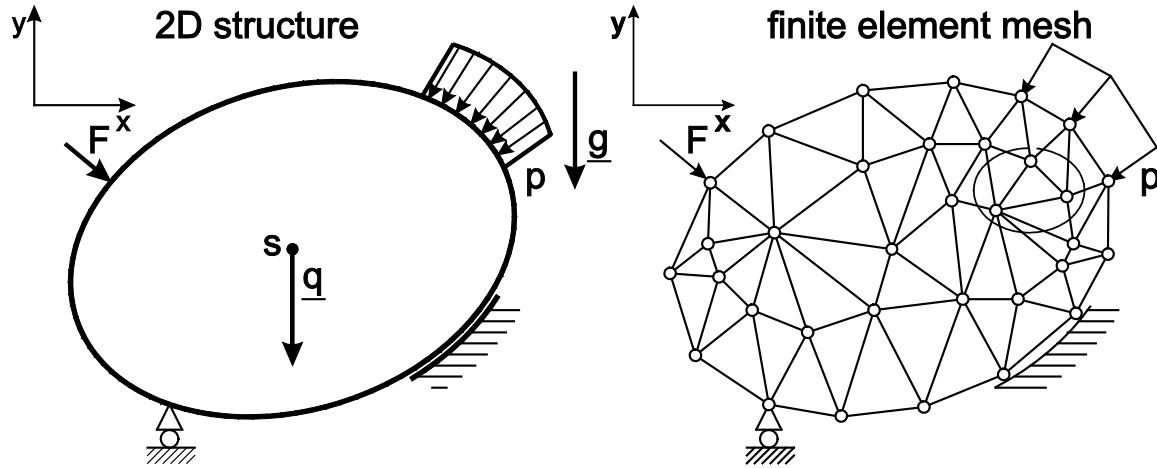

Fig.12.1. The basic concept of the finite element method in the case of plane elements.

In the FE method we apply the minimum principle of the total potential energy to develop the finite element equilibrium equations. For a single plane element the total potential energy is [2]:

$$\Pi_e = U - W = \frac{1}{2}\int_{V_e} \underline{\sigma}^T \underline{\varepsilon}\,dV - \int_{A_{pe}} \underline{u}^T \underline{p}\,dA - \int_{V_e} \underline{u}^T \underline{q}\,dV - \sum_{i=1}^{n} \underline{u}^T(x_i, y_i)\underline{F}_i,$$

(12.1)

where $\underline{\sigma}$ is the vector of stress components, $\underline{\varepsilon}$ is the vector strain components, respectively:

$$\underline{\varepsilon}^T = \left[\varepsilon_x, \varepsilon_y, \gamma_{xy}\right],$$ (12.2)

$$\underline{\sigma}^T = \left[\sigma_x, \sigma_y, \tau_{xy}\right],$$

moreover $\underline{u} = u \cdot \underline{i} + v \cdot \underline{j}$ is the displacement vector field, $\underline{p}$ is the density vector of surface forces, $\underline{q}$ is the density vector of volume (or body) forces, $\underline{F}_i$ is the vector of concentrated forces acting on the plane element with coordinates of point of action, x_i and y_i, A_{pe} is that part of the boundary curves, which is loaded by surfaces forces, V_e is the volume of the element, respectively. We provide the displacement vector field by interpolation:

$$\underline{u}(x, y) = \underline{\underline{N}}(x, y)\underline{u}_e,$$ (12.3)

where $\underline{\underline{N}}$ is the matrix of interpolation functions, its dimension depends on the degrees of freedom of the plane element, $\underline{u}_e$ is vector of nodal displacements. Referring to the basic equations of elasticity, the relationship between the strain and displacement fields in matrix form is:

$$\underline{\varepsilon} = \underline{\underline{\partial}}\,\underline{u},$$ (12.4)

where $\underline{\underline{\partial}}$ is the matrix of differential operators, it can be obtained by Eqs.(11.10) and (11.13):

$$\underline{\underline{\partial}} = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix}. \tag{12.5}$$

The combination of the latter relations gives:

$$\underline{\varepsilon} = \underline{\underline{\partial}}\,\underline{u} = \underline{\underline{\partial}}\,\underline{\underline{N}}\,\underline{u}_e = \underline{\underline{B}}\,\underline{u}_e, \tag{12.6}$$

where $\underline{\underline{B}}$ is the strain-displacement matrix. The stress field can be obtained by:

$$\underline{\sigma} = \underline{\underline{C}}\,\underline{\varepsilon}, \quad \underline{\sigma} = \underline{\underline{C}}\,\underline{\underline{B}}\,\underline{u}_e, \tag{12.7}$$

where $\underline{\underline{C}}$ is the constitutive matrix − its calculation has already been made in section 11 for plane stress and plane strain states. The strain energy for a single finite element is:

$$U_e = \frac{1}{2}\int_{V_e} \underline{\sigma}^T \underline{\varepsilon}\,dV = \frac{1}{2}\iint \underline{u}_e^T \underline{\underline{B}}^T \underline{\underline{C}}^T \underline{\underline{B}}\,\underline{u}_e\,v\,dxdy = \frac{1}{2}\underline{u}_e^T \underline{\underline{K}}_e\,\underline{u}_e, \tag{12.8}$$

where $\underline{\underline{K}}_e$ is the element stiffness matrix,

$$\underline{\underline{K}}_e = \int_{V_e} \underline{\underline{B}}^T \underline{\underline{C}}^T \underline{\underline{B}}\,dV = \iint \underline{\underline{B}}^T \underline{\underline{C}}^T \underline{\underline{B}}\,v\,dxdy, \tag{12.9}$$

its dimension depends on the degrees of freedom of the element. For plane elements the differential volume is written in the form of: $dV = v\,dA = v\,dxdy$, where v is the thickness of element. The work of external forces acting on the element by the help of Eq.(12.3) becomes:

$$W_e = \int_{A_{pe}} \underline{u}^T \underline{p}\,dA + \int_{V_e} \underline{u}^T \underline{q}\,dV + \sum_{i=1}^{n} \underline{u}^T(x_i, y_i)\underline{F}_i = \underline{u}_e^T \int_{A_{pe}} \underline{\underline{N}}^T \underline{p}\,dA + \underline{u}_e^T \int_{V_e} \underline{\underline{N}}^T \underline{q}\,dV + \underline{u}_e^T \underline{F}_{ec},$$
$$\tag{12.10}$$

where $\underline{F}_{ec}$ is the vector of concentrated forces acting in the nodes of element. Thus, the total potential energy can be written as:

$$\Pi_e = \frac{1}{2}\underline{u}_e^T \underline{\underline{K}}_e\,\underline{u}_e - \underline{u}_e^T \underline{F}_e, \tag{12.11}$$

where:

$$\underline{F}_e = \int_{A_{pe}} \underline{\underline{N}}^T \underline{p}\,dA + \int_{V_e} \underline{\underline{N}}^T \underline{q}\,dV + \underline{F}_{ec} = \underline{F}_{eb} + \underline{F}_{ep} + \underline{F}_{ec}, \tag{12.12}$$

is the vector forces acting on the element. We can formulate the equilibrium equation in the element level by means of the minimum principle of the total potential energy:

$$\underline{\underline{K}}_e\,\underline{u}_e - \underline{F}_e = \underline{0}. \tag{12.13}$$

The assembly of element stiffness matrices, vector of nodal displacements and forces leads to the structural equilibrium equation:

$$\underline{\underline{K}}\,\underline{U} - \underline{F} = \underline{0}, \tag{12.14}$$

where $\underline{\underline{K}}$ is structural stiffness matrix, $\underline{U}$ is the structural vector of nodal displacements, $\underline{F}$ is the structural vector of nodal forces, respectively. That is, the finite element equilibrium

equation is a system of algebraic equations for which the solutions are the values of nodal displacements. In terms of the nodal displacements we can calculate the nodal forces and stresses.

For the solution of plane problems there exist many types of plane elements. In the sequel we review the simplest element types.

12.2 Linear three node triangle element

The linear triangle element (Turner triangle) [1,3], which is often called the triangle membrane element or constant strain triangle (CST) element is depicted in Fig.12.2. At each node there are two degrees of freedom. Consequently the degrees of freedom are equal to six for the whole element. The arrow in the center point of the element refers to the orientation of the element, i.e. for each element we have a direction, which means how the nodes are followed by each other.

12.2.1 Interpolation of the displacement field

We collect the nodal x, y coordinates and displacement components in vectors:

$$\underline{x}_e^T = \begin{bmatrix} x_1 & y_1 & x_2 & y_2 & x_3 & y_3 \end{bmatrix}, \qquad (12.15)$$

$$\underline{u}_e^T = \begin{bmatrix} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 \end{bmatrix}.$$
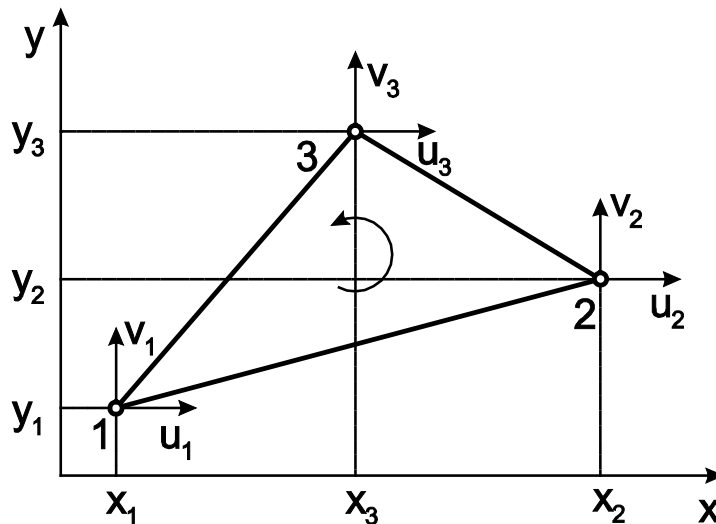

Fig.12.2. Linear triangle element. Nodal coordinates and displacements.

The triangle area can be expressed as a determinant:

$$2A_e = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix} = (x_2 y_3 - x_3 y_2) + (x_3 y_1 - x_1 y_3) + (x_1 y_2 - x_2 y_1) = \alpha_1 + \alpha_2 + \alpha_3 .$$

$$(12.16)$$

The u and v components of the displacement field are formulated as the linear function of x and y:

$$u(x, y) = a_0 + a_1 x + a_2 y , \qquad (12.17)$$

$$v(x, y) = b_0 + b_1 x + b_2 y ,$$

where a_0, a_1, a_2, b_0, b_1 and b_2 are unknown constants. The vector of strain components is:

$$\underline{\varepsilon}^T = \begin{bmatrix} \varepsilon_x, \varepsilon_y, \gamma_{xy} \end{bmatrix}, \qquad (12.18)$$

where using Eqs.(11.10) and (11.13) we have:

$$\varepsilon_x = \frac{\partial u}{\partial x} = a_1, \ \varepsilon_y = \frac{\partial v}{\partial y} = b_2, \ \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = a_2 + b_1. \qquad (12.19)$$

The nodal displacements must be obtained if we take back the nodal coordinates into the $u(x,y)$ and $v(x,y)$ functions given by Eq.(12.17), i.e.:

$$u_1 = a_0 + a_1 x_1 + a_2 y_1, \ v_1 = b_0 + b_1 x_1 + b_2 y_1, \qquad (12.20)$$

$$u_2 = a_0 + a_1 x_2 + a_2 y_2, \ v_2 = b_0 + b_1 x_2 + b_2 y_2,$$

$$u_3 = a_0 + a_1 x_3 + a_2 y_3, \ v_3 = b_0 + b_1 x_3 + b_2 y_3.$$

The solution of the system of equations above results in:

$$a_0 = \frac{\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3}{2A_e}, a_1 = \frac{\beta_1 u_1 + \beta_2 u_2 + \beta_3 u_3}{2A_e}, a_2 = \frac{\gamma_1 u_1 + \gamma_2 u_2 + \gamma_3 u_3}{2A_e},$$

$$b_0 = \frac{\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3}{2A_e}, b_1 = \frac{\beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3}{2A_e}, b_2 = \frac{\gamma_1 v_1 + \gamma_2 v_2 + \gamma_3 v_3}{2A_e},$$

$$(12.21)$$

where:

$$\alpha_1 = x_2 y_3 - x_3 y_2, \beta_1 = y_2 - y_3, \ \gamma_1 = x_3 - x_2, \qquad (12.22)$$

$$\alpha_2 = x_3 y_1 - x_1 y_3, \ \beta_2 = y_3 - y_1, \ \gamma_2 = x_1 - x_3,$$

$$\alpha_3 = x_1 y_2 - x_2 y_1, \ \beta_3 = y_1 - y_2, \ \gamma_3 = x_2 - x_1.$$

Substituting the solution above back into the components of displacement field (Eq.(12.17)) we obtain:

$$u(x, y) = \frac{1}{2A_e}[(\alpha_1 + \beta_1 x + \gamma_1 y)u_1 + (\alpha_2 + \beta_2 x + \gamma_2 y)u_2 + (\alpha_3 + \beta_3 x + \gamma_3 y)u_3],$$

$$(12.23)$$

$$v(x, y) = \frac{1}{2A_e}[(\alpha_1 + \beta_1 x + \gamma_1 y)v_1 + (\alpha_2 + \beta_2 x + \gamma_2 y)v_2 + (\alpha_3 + \beta_3 x + \gamma_3 y)v_3].$$

Considering the fact that for the triangle element we have three interpolation functions (see. Eq.(12.3)), we can write that:

$$u(x, y) = N_1 u_1 + N_2 u_2 + N_3 u_3 = \sum_{i=1}^{3} N_i(x, y)u_i, \qquad (12.24)$$

$$v(x, y) = N_1 v_1 + N_2 v_2 + N_3 v_3 = \sum_{i=1}^{3} N_i(x, y)v_i.$$

In accordance with Eq.(12.21) the interpolation functions can be derived in the following form:

$$N_i(x, y) = \frac{\alpha_i + \beta_i x + \gamma_i y}{2A_e}, i = 1, 2, 3. \qquad (12.25)$$

Based on the relation of $\underline{u}(x, y) = \underline{\underline{N}}(x, y)\underline{u}_e$ the matrix of interpolation functions becomes:

$$\underline{\underline{N}} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix}. \qquad (12.26)$$

The parameter lines of the interpolation function are shown in Fig.12.3, which implies the following properties:

- at the nodes (N_1, N_2, N_3): $(1,0,0)$, $(0,1,0)$, $(0,0,1)$,

- at the midpoints of the triangle sides $(N_1, N_2, N_3) : (1/2, 1/2, 0)$, $(1/2, 0, 1/2)$, $(0, 1/2, 1/2)$,

- at the centroid $(N_1, N_2, N_3) : (1/3, 1/3, 1/3)$,

- i.e., it is seen that at every point: $N_1 + N_2 + N_3 = 1$,

- finally:

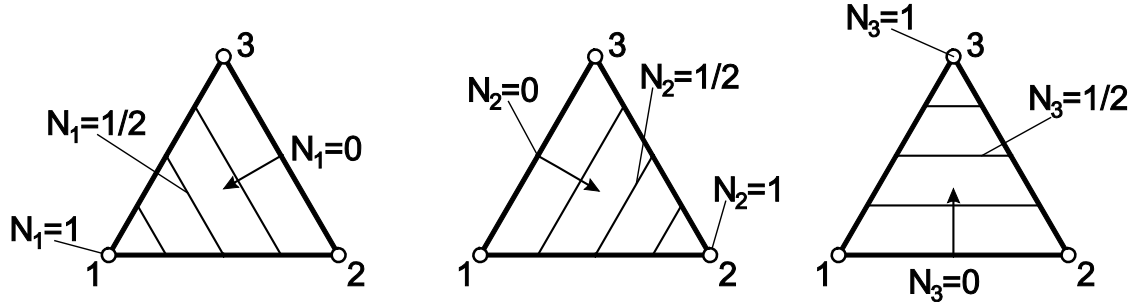$$\int_{A_e} N_1^i N_2^j N_3^k \, dA = \frac{i! \, j! \, k!}{(i+j+k+2)!} \, 2A_e . \qquad (12.27)$$

Fig.12.3. Parameter lines of the interpolation functions of linear triangle element.

12.2.2 Calculation of the stiffness matrix

According to Eq.(12.9) the definition of the element stiffness matrix is:

$$\underline{\underline{K}}_e = \iint \underline{\underline{B}}^T \underline{\underline{C}}^T \underline{\underline{B}} v \, dx \, dy , \qquad (12.28)$$

where the previously mentioned strain-displacement matrix using Eqs.(12.5) and (12.26) becomes:

$$\underline{\underline{B}} = \underline{\underline{\partial}} \underline{\underline{N}} = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\[2mm] 0 & \dfrac{\partial}{\partial y} \\[2mm] \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 \end{bmatrix} =$$

$$= \begin{bmatrix} \dfrac{\partial N_1}{\partial x} & 0 & \dfrac{\partial N_2}{\partial x} & 0 & \dfrac{\partial N_3}{\partial x} & 0 \\[2mm] 0 & \dfrac{\partial N_1}{\partial y} & 0 & \dfrac{\partial N_2}{\partial y} & 0 & \dfrac{\partial N_3}{\partial y} \\[2mm] \dfrac{\partial N_1}{\partial y} & \dfrac{\partial N_1}{\partial x} & \dfrac{\partial N_2}{\partial y} & \dfrac{\partial N_2}{\partial x} & \dfrac{\partial N_3}{\partial y} & \dfrac{\partial N_3}{\partial x} \end{bmatrix} = \frac{1}{2A_e} \begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix} .$$

$$(12.29)$$

This formulation implies that the elements of matrix $\underline{\underline{B}}$ are independent of the x and y variables, they depend only on the nodal coordinates. Therefore, the stiffness matrix can be written as:

$$\underline{\underline{K}}_e = \underline{\underline{B}}^T \underline{\underline{C}}^T \underline{\underline{B}} v A_e = \underline{\underline{B}}^T \underline{\underline{C}}^T \underline{\underline{B}} V_e , \qquad (12.30)$$

where A_e is the element area, $V_e = A_e v$ is the element volume, respectively. As a consequence, the stiffness matrix of the linear triangle element can be computed in a relatively simple way and in closed form.

12.2.3 Definition of the loads
Body force or volume force. Let the vector of body forces be equal to:

$$\underline{q} = \begin{bmatrix} q_x \\ q_y \end{bmatrix}, \tag{12.31}$$

from which we have:

$$\underline{F}_{eb} = \iint \underline{\underline{N}}^T \underline{q} v dx dy = \int\limits_{A_e} \begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \end{bmatrix} \begin{bmatrix} q_x \\ q_y \end{bmatrix} v dA = \int\limits_{A_e} \begin{bmatrix} N_1 q_x \\ N_1 q_y \\ N_2 q_x \\ N_2 q_y \\ N_3 q_x \\ N_3 q_y \end{bmatrix} v dA .$$

$$\tag{12.32}$$

Utilizing the special properties of the interpolation functions given by Eq.(12.27), e.g. if $i = 1, j = 0$ and $k = 0$, we have:

$$\int\limits_{A_e} N_1 dA = \frac{1}{3} A_e . \tag{12.33}$$

Consequently we have:

$$\underline{F}_{eb}^T = \frac{1}{3} A_e v \begin{bmatrix} q_x & q_y & q_x & q_y & q_x & q_y \end{bmatrix}. \tag{12.34}$$

As an explanation, the body force acting on the element (e.g. the whole weight and the resulting resultant force) is divided into three equal parts and put into the nodes. The body force can be originated from gravitation or acceleration (inertia force).

Distributed force along element edges. For the calculation of force vectors as a result of line loads along element edges we should take the 1-2 edge of the element shown in Fig.12.4 into consideration. We define a dimensionless parameter, ξ along the element edge. The arc length along the element edge is then:

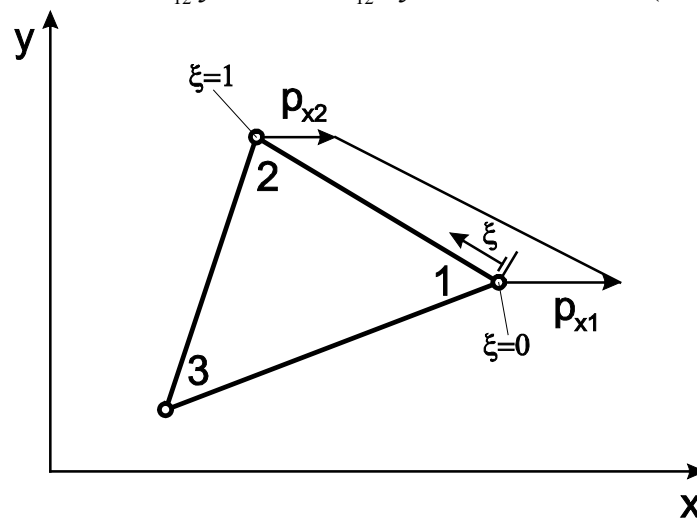$$s = l_{12}\xi \text{ and } ds = l_{12}d\xi . \tag{12.35}$$

Fig.12.4. Linear triangle element with line load in direction x along element edge 1-2.

The linearly distributed load in direction x can be described by the following function:

$$p_x(\xi) = p_{x1}(1 - \xi) + p_{x2}\xi . \qquad (12.36)$$

Similarly, the displacement function in direction x along element edge 1-2 can be written as:

$$u(\xi) = u_1(1 - \xi) + u_2\xi . \qquad (12.37)$$

The work of the distributed load is generally the integration of the load function multiplied by displacement function between the corresponding nodes:

$$W_p = \underline{u}_e^T \underline{F}_{ep} = \int_{l_{12}} \underline{u}_e^T \underline{p}_l v ds = \int_0^1 p_x(\xi)u(\xi)l_{12}v d\xi =$$

$$= \int_0^1 u_1 \left\{ p_{x1}(1 - \xi)^2 + p_{x2}(1 - \xi)\xi \right\} l_{12}v d\xi + \int_0^1 u_2 \left\{ p_{x1}(1 - \xi)\xi + p_{x2}\xi^2 \right\} l_{12}v d\xi =$$

$$= \frac{l_{12}v}{3} \left\{ u_1(p_{x1} + \frac{1}{2}p_{x2}) + u_2(\frac{1}{2}p_{x1} + p_{x2}) \right\}.$$

$$(12.38)$$

That is, the force vector from a linearly distributed line load becomes:

$$\underline{F}_{ep}^T = \frac{l_{12}v}{3} \left[p_{x1} + \frac{1}{2}p_{x2} \quad 0 \quad \frac{1}{2}p_{x1} + p_{x2} \quad 0 \quad 0 \quad 0 \right]. \qquad (12.39)$$

If the line load is constant along the element edge, then $p_{x1} = p_{x2} = p_x$ which implies:

$$\underline{F}_{ep}^T = \frac{l_{12}v}{2} \left[p_x \quad 0 \quad p_x \quad 0 \quad 0 \quad 0 \right]. \qquad (12.40)$$

The form of the vector of forces in the finite element equation is similar in the case of a linearly distributed load in direction y.

Concentrated forces. Concentrated forces can act only at nodes. The force vector can be simply formulated based on the nodes:

$$\underline{F}_{ec}^T = \left[F_{x1} \quad F_{y1} \quad F_{x2} \quad F_{y2} \quad F_{x3} \quad F_{y3} \right]. \qquad (12.41)$$

The total force vector is the sum of the vectors detailed in the previous points, i.e.:

$$\underline{F}_e = \underline{F}_{eb} + \underline{F}_{ep} + \underline{F}_{ec} . \qquad (12.42)$$

We demonstrate the solution of the finite element equation and the construction of the stiffness matrix and force vector through an example.

12.3 Example for the linear triangle element – plane stress state

The model shown in Fig.12.5a is loaded by distributed forces. Calculate the nodal displacements and forces in that case when we built-up the plate using two linear triangle elements! Calculate the strain and stress components [4]!
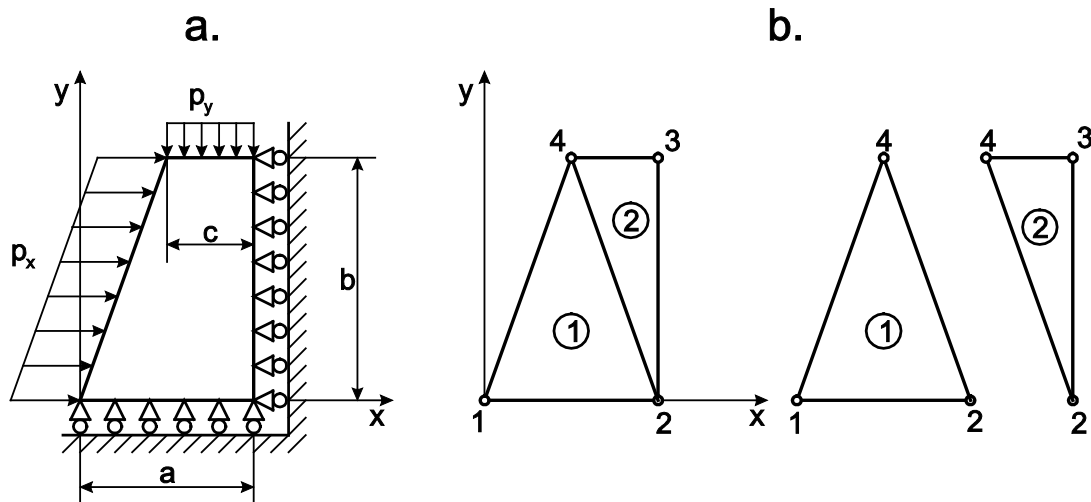
Fig.12.5. Plane model loaded by distributed forces (a), finite element model made by two linear triangle elements (b).

Given:

$p_x = 0,12$ MPa, $E = 150$ GPa, $a = 20$ mm, $c = 10$ mm, $p_y = 0,06$ MPa, $\nu = 0,25$, $b = 30$ mm
$v = 5$ mm

In the course of the computation we calculate the distances in [mm] and the force in [N]. Following Fig.12.5b, we see that the model is constructed by two triangle elements. The nodal coordinates are:

node	x [mm]	y [mm]
1	0	0
2	20	0
3	20	30
4	10	30

The so-called element-node table is:

element	nodes		
1	1	2	4
2	2	3	4

The finite element equilibrium equation to be solved is:

$$\underline{\underline{K}}\underline{U} = \underline{F}, \qquad (12.43)$$

where:

$$\underline{U}^T = \begin{bmatrix} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 & u_4 & v_4 \end{bmatrix}, \qquad (12.44)$$

is the structural vector of nodal displacements. Because of the boundary conditions ($v_1 = v_2 = u_2 = u_3 = 0$) we have:

$$\underline{U}^T = \begin{bmatrix} u_1 & 0 & 0 & 0 & 0 & v_3 & u_4 & v_4 \end{bmatrix}. \qquad (12.45)$$

In order to calculate the stiffness matrix we need the constitutive matrix for plane stress state (see section 11.23):

$$\underline{\underline{C}} = \underline{\underline{C}}^{str} = \frac{E}{1-v^2}\begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1-v}{2} \end{bmatrix} = \begin{bmatrix} 1,6 & 0,4 & 0 \\ 0,4 & 1,6 & 0 \\ 0 & 0 & 0.6 \end{bmatrix} \cdot 10^5 \text{ MPa}.$$

(12.46)

The coefficients of the interpolation functions for the first element are:

$$\beta_1 = y_2 - y_4 = -30 \text{ mm}, \gamma_1 = x_4 - x_2 = -10 \text{ mm}, \qquad (12.47)$$

$$\beta_2 = y_4 - y_1 = 30 \text{ mm}, \gamma_2 = x_1 - x_4 = -10 \text{ mm},$$

$$\beta_3 = y_1 - y_2 = 0 \text{ mm}, \gamma_3 = x_2 - x_1 = 20 \text{ mm},$$

and for the second element, respectively:

$$\beta_1 = y_3 - y_4 = 0 \text{ mm}, \gamma_1 = x_4 - x_3 = -10 \text{ mm}, \qquad (12.48)$$

$$\beta_2 = y_4 - y_2 = 30 \text{ mm}, \gamma_2 = x_2 - x_4 = 10 \text{ mm},$$

$$\beta_3 = y_2 - y_3 = -30 \text{ mm}, \gamma_3 = x_3 - x_2 = 0 \text{ mm}.$$

The triangle areas are:

$$A_{e1} = \frac{1}{2} 20 \cdot 30 = 300 \text{ mm}^2, A_{e2} = \frac{1}{2} 10 \cdot 30 = 150 \text{ mm}^2. \quad (12.49)$$

Matrix $\underline{\underline{B}}$ for the first element is:

$$\underline{\underline{B}}_1 = \frac{1}{2A_{e1}}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix} = \frac{1}{20}\begin{bmatrix} -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 & -\frac{1}{3} & 0 & \frac{2}{3} \\ -\frac{1}{3} & -1 & -\frac{1}{3} & 1 & \frac{2}{3} & 0 \end{bmatrix}\frac{1}{\text{mm}},$$

(12.50)

For the second element it is:

$$\underline{\underline{B}}_2 = \frac{1}{2A_{e2}}\begin{bmatrix} \beta_1 & 0 & \beta_2 & 0 & \beta_3 & 0 \\ 0 & \gamma_1 & 0 & \gamma_2 & 0 & \gamma_3 \\ \gamma_1 & \beta_1 & \gamma_2 & \beta_2 & \gamma_3 & \beta_3 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & -\frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 \\ -\frac{1}{3} & 0 & \frac{1}{3} & 1 & 0 & -1 \end{bmatrix}\frac{1}{\text{mm}}.$$

(12.51)

Based on Eq.(12.30) the element stiffness matrices are:

$$\underline{\underline{K}}_{e1} = \underline{\underline{B}}_1^T \underline{\underline{C}}^T \underline{\underline{B}}_1 V_{e1} = \begin{bmatrix} 6,25 & 1,25 & -5,75 & -0,25 & -0,5 & -1 \\ 1,25 & 35/12 & 0,25 & -57/36 & -1,5 & -4/3 \\ -5,75 & 0,25 & 6,25 & -1,25 & -0,5 & 1 \\ -0,25 & -57/36 & -1,25 & 35/12 & 1,5 & -4/3 \\ -0,5 & -1,5 & -0,5 & 1,5 & 1 & 0 \\ -1 & -4/3 & 1 & -4/3 & 0 & 8/3 \end{bmatrix} \cdot 10^5 \frac{\text{N}}{\text{mm}},$$

(12.52)

$$
\underline{\underline{K}}_{e2} = \underline{\underline{B}}_2^T \underline{\underline{C}}^T \underline{\underline{B}}_2 V_{e2} =
\begin{bmatrix}
0,5 & 0 & -0,5 & -1,5 & 0 & 1,5 \\
0 & 4/3 & -1 & -4/3 & 1 & 0 \\
-0,5 & -1 & 12,5 & 2,5 & -12 & -1,5 \\
-1,5 & -4/3 & 2,5 & 35/6 & -1 & -4,5 \\
0 & 1 & -12 & -1 & 12 & 0 \\
1,5 & 0 & -1,5 & -4,5 & 0 & 4,5
\end{bmatrix}
\cdot 10^5 \, \frac{\text{N}}{\text{mm}},
$$

where $V_{e1} = A_{e1} \cdot v = 300 \cdot 5 = 1500 \text{ mm}^3$ and $V_{e2} = A_{e2} \cdot v = 150 \cdot 5 = 750 \text{ mm}^3$ are the element volumes. For the construction of the structural stiffness matrix we complete the element matrices with empty rows and columns corresponding to the missing degrees of freedom. On the base of Fig.12.5 and the element-node table, it is seen, that the first element includes only nodes 1, 2 and 4. Consequently those rows and columns, which belong to node 3, should be filled up with zeros:

$$
\underline{\underline{K}}_1 =
\begin{bmatrix}
k_{e11}^1 & k_{e12}^1 & k_{e13}^1 & k_{e14}^1 & 0 & 0 & k_{e15}^1 & k_{e16}^1 \\
k_{e21}^1 & k_{e22}^1 & k_{e23}^1 & k_{e24}^1 & 0 & 0 & k_{e25}^1 & k_{e26}^1 \\
k_{e31}^1 & k_{e32}^1 & k_{e33}^1 & k_{e34}^1 & 0 & 0 & k_{e35}^1 & k_{e36}^1 \\
k_{e41}^1 & k_{e42}^1 & k_{e43}^1 & k_{e44}^1 & 0 & 0 & k_{e45}^1 & k_{e46}^1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
k_{e51}^1 & k_{e52}^1 & k_{e53}^1 & k_{e54}^1 & 0 & 0 & k_{e55}^1 & k_{e56}^1 \\
k_{e61}^1 & k_{e62}^1 & k_{e63}^1 & k_{e64}^1 & 0 & 0 & k_{e65}^1 & k_{e66}^1
\end{bmatrix}.
\tag{12.53}
$$

In contrast, for the second element the rows and columns corresponding with the first node must be completed by the placement of zeros:

$$
\underline{\underline{K}}_2 =
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & k_{e11}^2 & k_{e12}^2 & k_{e13}^2 & k_{e14}^2 & k_{e15}^2 & k_{e16}^2 \\
0 & 0 & k_{e21}^2 & k_{e22}^2 & k_{e23}^2 & k_{e24}^2 & k_{e25}^2 & k_{e26}^2 \\
0 & 0 & k_{e31}^2 & k_{e32}^2 & k_{e33}^2 & k_{e34}^2 & k_{e35}^2 & k_{e36}^2 \\
0 & 0 & k_{e41}^2 & k_{e42}^2 & k_{e43}^2 & k_{e44}^2 & k_{e45}^2 & k_{e46}^2 \\
0 & 0 & k_{e51}^2 & k_{e52}^2 & k_{e53}^2 & k_{e54}^2 & k_{e55}^2 & k_{e56}^2 \\
0 & 0 & k_{e61}^2 & k_{e62}^2 & k_{e63}^2 & k_{e64}^2 & k_{e65}^2 & k_{e66}^2
\end{bmatrix}.
\tag{12.54}
$$

The structural stiffness matrix is calculated as the sum of the two former matrices:

$$
\underline{\underline{K}} = \underline{\underline{K}}_1 + \underline{\underline{K}}_2 =
\begin{bmatrix}
k_{e11}^1 & k_{e12}^1 & k_{e13}^1 & k_{e14}^1 & 0 & 0 & k_{e15}^1 & k_{e16}^1 \\
k_{e21}^1 & k_{e22}^1 & k_{e23}^1 & k_{e24}^1 & 0 & 0 & k_{e25}^1 & k_{e26}^1 \\
k_{e31}^1 & k_{e32}^1 & k_{e33}^1 + k_{e11}^2 & k_{e34}^1 + k_{e12}^2 & k_{e13}^2 & k_{e14}^2 & k_{e35}^1 + k_{e15}^2 & k_{e36}^1 + k_{e16}^2 \\
k_{e41}^1 & k_{e42}^1 & k_{e43}^1 + k_{e21}^2 & k_{e44}^1 + k_{e22}^2 & k_{e23}^2 & k_{e24}^2 & k_{e45}^1 + k_{e25}^2 & k_{e46}^1 + k_{e26}^2 \\
0 & 0 & k_{e31}^2 & k_{e32}^2 & k_{e33}^2 & k_{e34}^2 & k_{e35}^2 & k_{e36}^2 \\
0 & 0 & k_{e41}^2 & k_{e42}^2 & k_{e43}^2 & k_{e44}^2 & k_{e45}^2 & k_{e46}^2 \\
k_{e51}^1 & k_{e52}^1 & k_{e53}^1 + k_{e51}^2 & k_{e54}^1 + k_{e52}^2 & k_{e53}^2 & k_{e54}^2 & k_{e55}^1 + k_{e55}^2 & k_{e56}^1 + k_{e56}^2 \\
k_{e61}^1 & k_{e62}^1 & k_{e63}^1 + k_{e61}^2 & k_{e64}^1 + k_{e62}^2 & k_{e63}^2 & k_{e64}^2 & k_{e65}^1 + k_{e65}^2 & k_{e66}^1 + k_{e66}^2
\end{bmatrix}.
$$

(12.55)

The force vector related to the distributed load is calculated by Eq.(12.39):

$$\underline{F}^T_{ep1} = \frac{l_{14}v}{2}\begin{bmatrix} p_x & 0 & 0 & 0 & p_x & 0 \end{bmatrix}, \qquad (12.56)$$

$$\underline{F}^T_{ep2} = \frac{l_{34}v}{2}\begin{bmatrix} 0 & 0 & 0 & p_y & 0 & p_y \end{bmatrix},$$

where $l_{14} = \sqrt{10^2 + 30^2} = \sqrt{1000}$ m and $l_{34} = 10$ mm are the element edge lengths between the nodes indicated in the subscript. By completing the element vectors with zeros at the positions of the proper degrees of freedom, we get the structural force vectors:

$$\underline{F}^T_{l1} = \frac{l_{14}v}{2}\begin{bmatrix} p_x & 0 & 0 & 0 & 0 & 0 & p_x & 0 \end{bmatrix} = \begin{bmatrix} 3\sqrt{10} & 0 & 0 & 0 & 0 & 0 & 3\sqrt{10} & 0 \end{bmatrix} \text{N},$$

(12.57)

$$\underline{F}^T_{l2} = \frac{l_{34}v}{2}\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & p_y & 0 & p_y \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -1.5 & 0 & -1.5 \end{bmatrix} \text{N}.$$

We consider the reaction forces as concentrated forces at the constrained nodes:

$$\underline{F}^T_c = \begin{bmatrix} F_{x1} & F_{y1} & F_{x2} & F_{y2} & F_{x3} & F_{y3} & F_{x4} & F_{y4} \end{bmatrix}. \qquad (12.58)$$

Taking it into account, that at node 4 there is no external force and that the surfaces are frictionless, i.e.: $F_{x1} = F_{y3} = 0$, we have:

$$\underline{F}^T_c = \begin{bmatrix} 0 & F_{y1} & F_{x2} & F_{y2} & F_{x3} & 0 & 0 & 0 \end{bmatrix}. \qquad (12.59)$$

The structural force vector is:

$$\underline{F} = \underline{F}_{p1} + \underline{F}_{p2} + \underline{F}_c. \qquad (12.60)$$

The finite element equilibrium equation is $\underline{\underline{K}}\,\underline{U} = \underline{F}$, i.e. we have:

$$\begin{bmatrix}
6{,}25 & 1{,}25 & 5{,}75 & -0{,}25 & 0 & 0 & -0{,}5 & -1 \\
1{,}25 & 35/12 & 0{,}25 & -57/36 & 0 & 0 & -1{,}25 & -4/3 \\
-5{,}75 & 0{,}25 & 6{,}25 & -1{,}25 & -0{,}5 & -1{,}5 & -0{,}5 & 2{,}5 \\
-0{,}25 & -57/36 & -1{,}25 & 4{,}25 & -1 & -4/3 & 2{,}5 & -4/3 \\
0 & 0 & -0{,}5 & -1 & 12{,}5 & 2{,}5 & -12 & -1{,}5 \\
0 & 0 & -1{,}5 & -4/3 & 2{,}5 & 35/6 & -1 & -4{,}5 \\
-0{,}5 & -1{,}5 & -0{,}5 & 2{,}5 & -12 & -1 & 13 & 0 \\
-1 & -4/3 & 2{,}5 & -4/3 & -1{,}5 & -4{,}5 & 0 & 43/6
\end{bmatrix} \cdot 10^5 \begin{bmatrix} u_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix} = \begin{bmatrix} 3\sqrt{10} \\ F_{y1} \\ F_{x2} \\ F_{y2} \\ F_{x3} \\ -1.5 \\ 3\sqrt{10} \\ -1.5 \end{bmatrix}.$$

(12.61)

The nodal displacements can be determined from the system of equations constructed by the 1st, 6th, 7th and 8th component equations of the matrix equation:

$$6{,}25u_1 - 0{,}5u_4 - v_4 = 3\sqrt{10}, \qquad (12.62)$$

$$35/6 v_3 - u_4 - 4{,}5v_4 = -1{,}5,$$

$$-0{,}5u_1 - v_3 + 13u_4 = 3\sqrt{10},$$

$$-u_1 - 4{,}5v_3 + 43/6 v_4 = -1{,}5.$$

The equations above, in fact were obtained by the condensation of Eq.(12.61). When we perform the matrix condensation only those component equations remain, which contain unknowns with respect to the displacements only. On the right hand side, in the force vector there are no unknowns. The solutions are:

$$u_1 = 1,557 \cdot 10^{-5} \text{ mm}, v_3 = -0,22997 \cdot 10^{-5} \text{ mm}, \qquad (12.63)$$

$$u_4 = 0,771983 \cdot 10^{-5} \text{ mm}, v_4 = -0,13633 \cdot 10^{-5} \text{ mm}.$$

Taking the nodal displacements back into the 2nd, 3rd, 4th and 5th rows of the matrix equation, we can determine the nodal forces:

$$1,25u_1 - 1,25u_4 - 4/3v_4 = F_{y1}, \qquad (12.64)$$

$$-5,75u_1 - 1,5v_3 - 0,5u_4 + 2,5v_4 = F_{x2},$$

$$-0,25u_1 - 4/3v_3 + 2,5u_4 - 4/3v_4 = F_{y2},$$

$$2,5v_3 - 12u_4 - 1,5v_4 = F_{y3}$$

The solutions are:

$$F_{y1} = 0,971095 \text{ N}, F_{x2} = -9,339434 \text{ N}, \qquad (12.65)$$

$$F_{y2} = 2,0289 \text{ N}, F_{x3} = -9,63423 \text{ N}.$$

Using Eq.(12.19) we calculate now the strain components:

$$\varepsilon_x = \frac{\partial u}{\partial x} = a_1, \; \varepsilon_y = \frac{\partial v}{\partial y} = b_2, \; \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = a_2 + b_1. \qquad (12.66)$$

For the first element we obtain:

$$a_1 = \frac{\beta_1 u_1 + \beta_2 u_2 + \beta_3 u_4}{2A_{e1}} = \frac{-30u_1}{2 \cdot 300} = -7,7892 \cdot 10^{-7} \qquad (12.67)$$

$$b_1 = \frac{\beta_1 v_1 + \beta_2 v_2 + \beta_3 v_4}{2A_{e1}} = 0,$$

$$a_2 = \frac{\gamma_1 u_1 + \gamma_2 u_2 + \gamma_3 u_4}{2A_{e1}} = \frac{-10u_1 + 20u_4}{2 \cdot 300} = -2,3121 \cdot 10^{-9},$$

$$b_2 = \frac{\gamma_1 v_1 + \gamma_2 v_2 + \gamma_3 v_4}{2A_{e1}} = \frac{20v_4}{2 \cdot 300} = -4,54435 \cdot 10^{-8}.$$

The vector of strain components fro the first element is:

$$\underline{\varepsilon}_1 = \begin{bmatrix} \varepsilon_{x1} \\ \varepsilon_{y1} \\ \gamma_{xy1} \end{bmatrix} = \begin{bmatrix} -778,92 \\ -45,4435 \\ -2,3121 \end{bmatrix} \cdot 10^{-9} \qquad (12.68)$$

For the second element we can write:

$$a_1 = \frac{\beta_1 u_2 + \beta_2 u_3 + \beta_3 u_4}{2A_{e2}} = \frac{30u_4}{2 \cdot 150} = -7,7198 \cdot 10^{-7}, \qquad (12.69)$$

$$b_1 = \frac{\beta_1 v_2 + \beta_2 v_3 + \beta_3 v_4}{2A_{e2}} = \frac{30v_3 - 30v_4}{2 \cdot 150} = -9,36416 \cdot 10^{-8},$$

$$a_2 = \frac{\gamma_1 u_2 + \gamma_2 u_3 + \gamma_3 u_4}{2A_{e2}} = 0,$$

$$b_2 = \frac{\gamma_1 v_2 + \gamma_2 v_3 + \gamma_3 v_4}{2A_{e1}} = \frac{10v_3}{2 \cdot 150} = -7,6657 \cdot 10^{-8},$$

from which we have:

$$\underline{\varepsilon}_2 = \begin{bmatrix} \varepsilon_{x2} \\ \varepsilon_{y2} \\ \gamma_{xy2} \end{bmatrix} = \begin{bmatrix} -771,98 \\ -76,657 \\ -93,6416 \end{bmatrix} \cdot 10^{-9} . \tag{12.70}$$

Since the plate is under plane stress state we can write based on Eq.(11.19) that:

$$\varepsilon_{z1} = -\frac{\nu}{1-\nu}(\varepsilon_{x1} + \varepsilon_{y1}) = 274,787 \cdot 10^{-9} , \tag{12.71}$$

$$\varepsilon_{z2} = -\frac{\nu}{1-\nu}(\varepsilon_{x2} + \varepsilon_{y2}) = 282,88 \cdot 10^{-9} .$$

The normal and shear strains are, accordingly constants within the individual elements, we referred to this fact in the introduction of the triangle element. Incorporating the constitutive matrix we can determine the stress components too based on Eq.(12.46):

$$\underline{\sigma} = \underline{\underline{C}}^{str} \underline{\varepsilon} . \tag{12.72}$$

This equation gives the stresses of the elements, which is in general referred to as „element stress" in the commercial finite element packages. For the first element we have:

$$\sigma_{x1} = (1,6\varepsilon_{x1} + 0,4\varepsilon_{y1}) \cdot 10^5 = -0,12644 \text{ MPa} , \tag{12.73}$$

$$\sigma_{y1} = (0,4\varepsilon_{x1} + 1,6\varepsilon_{y1}) \cdot 10^5 = -0,038428 \text{ MPa} ,$$

$$\tau_{xy1} = 0,6\gamma_{xy1} \cdot 10^5 = -0,13873 \cdot 10^{-3} \text{ MPa} .$$

Similarly, for the second element the stresses are:

$$\sigma_{x2} = (1,6\varepsilon_{x2} + 0,4\varepsilon_{y2}) \cdot 10^5 = -0,12658 \text{ MPa} , \tag{12.74}$$

$$\sigma_{y2} = (0,4\varepsilon_{x2} + 1,6\varepsilon_{y2}) \cdot 10^5 = -0,043145 \text{ MPa} ,$$

$$\tau_{xy2} = 0,6\gamma_{xy2} \cdot 10^5 = -0,56185 \cdot 10^{-2} \text{ MPa} .$$

Considering the stresses it is possible to produce nodal stress solution. By computing the average stresses in the mutual nodes we obtain the so-called „nodal stress" or „average stress" solution:

Node 1: $\sigma_x = -0,12644 \text{ MPa}$, $\sigma_y = -0,038428 \text{ MPa}$, $\tau_{xy} = -0,13873 \cdot 10^{-3} \text{ MPa}$,

$$\tag{12.75}$$

Node 2: $\qquad \sigma_x = \frac{1}{2}(\sigma_{x1} + \sigma_{x2}) = -0,12651 \text{ MPa}$,

$$\sigma_y = \frac{1}{2}(\sigma_{y1} + \sigma_{y2}) = -0,0407865 \text{ MPa} ,$$

$$\tau_{xy} = \frac{1}{2}(\tau_{xy1} + \tau_{xy2}) = -0,28786 \cdot 10^{-2} \text{ MPa} .$$

Node 3: $\sigma_x = -0,12658 \text{ MPa}$, $\sigma_y = -0,043145 \text{ MPa}$, $\tau_{xy} = -0,56185 \cdot 10^{-3} \text{ MPa}$.

Node 4: $\qquad \sigma_x = \frac{1}{2}(\sigma_{x1} + \sigma_{x2}) = -0,12651 \text{ MPa}$,

$$\sigma_y = \frac{1}{2}(\sigma_{y1} + \sigma_{y2}) = -0,0407865 \text{ MPa} ,$$

$$\tau_{xy} = \frac{1}{2}(\tau_{xy1} + \tau_{xy2}) = -0,28786 \cdot 10^{-2} \text{ MPa} .$$

The problem presented in section 12.3 was verified by the finite element code ANSYS 12, resulting in the same results. The solution with the above applied low mesh resolution is naturally very inaccurate.

12.4 Quadratic six node triangle element

The more advanced version of the linear triangle element is the six node quadratic triangle element, in which there are additional nodes in the midpoints of the element sides [2,5]. Because of the additional nodes we need displacement functions including six unknowns, which are:

$$u(x, y) = a_0 + a_1 x + a_2 y + a_3 xy + a_4 x^2 + a_5 y^2, \qquad (12.76)$$

$$v(x, y) = b_0 + b_1 x + b_2 y + b_3 xy + b_4 x^2 + b_5 y^2.$$

The calculation of the stiffness matrix and force vector can be performed in the same fashion as it was done in the linear triangle element. Within the individual elements the strain and stress components vary linearly. As a consequence, using identical mesh resolution, the quadratic triangle element provides a better approximation of the problem than the linear one.

12.5 Isoparametric four node quadrilateral

The isoparametric quadrilateral (see Fig.12.6a) is one of the most important finite element type for plane problems [2,4,5]. An element is called isoparametric if we formulate the local geometry and displacement field by the same set of functions.

12.5.1 Interpolation of the geometry

For the sake of simplicity we map the quadrilateral element to a regular square into the ξ-η natural coordinate system, as it is shown in Fig 12.6b. We give the functions of the x and y coordinates of element edges in the following form:

$$x(\xi,\eta) = N_1(\xi,\eta)x_1 + N_2(\xi,\eta)x_2 + N_3(\xi,\eta)x_3 + N_4(\xi,\eta)x_4 = \underline{N}^T(\xi,\eta)\underline{x},$$
$$(12.77)$$
$$y(\xi,\eta) = N_1(\xi,\eta)y_1 + N_2(\xi,\eta)y_2 + N_3(\xi,\eta)y_3 + N_4(\xi,\eta)y_4 = \underline{N}^T(\xi,\eta)\underline{y},$$

where:

$$\underline{x}^T = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}, \ \underline{y}^T = \begin{bmatrix} y_1 & y_2 & y_3 & y_4 \end{bmatrix}. \qquad (12.78)$$
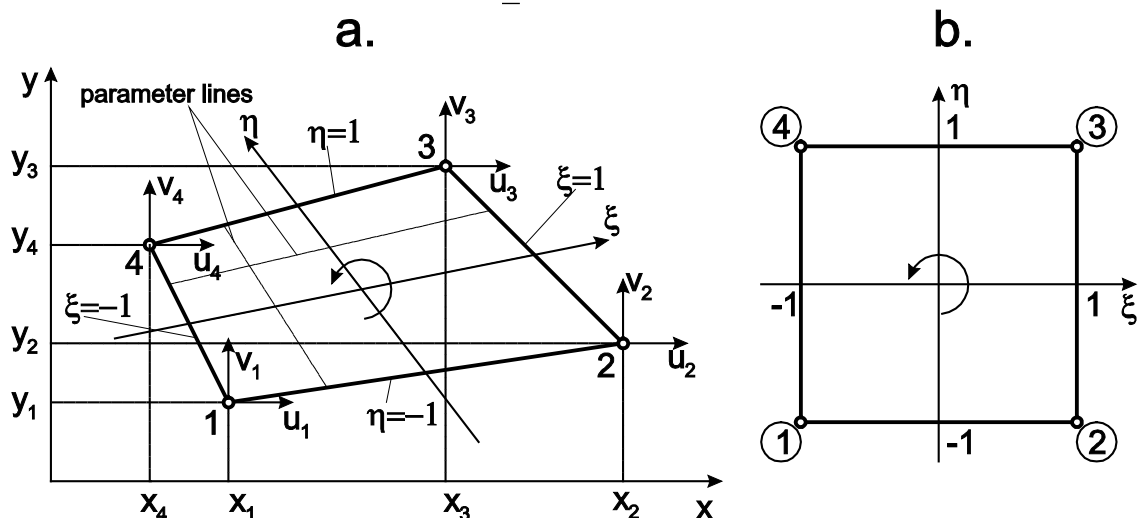
Fig.12.6. Isoparametric quadrilateral in the global (a) and natural (b) coordinate systems.

Due to the fact that we have four nodes, the interpolation function may contain to a maximum four unknowns:

$$x(\xi,\eta) = a_0 + a_1\xi + a_2\eta + a_3\xi\eta = \underline{P}^T \underline{A}, \qquad (12.79)$$

where $\underline{A}$ is the vector of coefficients, $\underline{P}$ is the vector of basis polynomials, respectively:

$$\underline{A}^T = \begin{bmatrix} a_0 & a_1 & a_2 & a_3 \end{bmatrix}, \; \underline{P}^T = \begin{bmatrix} 1 & \xi & \eta & \xi\eta \end{bmatrix}. \qquad (12.80)$$

The function given by Eq.(12.79) must satisfy the following conditions:

$$x(-1,-1) = a_0 - a_1 - a_2 + a_3 = x_1, \qquad (12.81)$$

$$x(1,-1) = a_0 + a_1 - a_2 - a_3 = x_2,$$

$$x(1,1) = a_0 + a_1 + a_2 + a_3 = x_3,$$

$$x(-1,1) = a_0 - a_1 + a_2 - a_3 = x_4.$$

In matrix form it is:

$$\underline{\underline{M}}\,\underline{A} = \underline{x}, \qquad (12.82)$$

where:

$$\underline{\underline{M}} = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}. \qquad (12.83)$$

Then the coefficients can be determined by using Eq.(12.82):

$$\underline{A} = \underline{\underline{M}}^{-1}\underline{x} \text{ and: } x(\xi,\eta) = \underline{P}^T \underline{A} = \underline{P}^T \underline{\underline{M}}^{-1}\underline{x}. \qquad (12.84)$$

The solutions for the coefficients are:

$$a_0 = \frac{1}{4}(x_1 + x_2 + x_3 + x_4), \; a_1 = \frac{1}{4}(-x_1 + x_2 + x_3 - x_4), \quad (12.85)$$

$$a_2 = \frac{1}{4}(-x_1 - x_2 + x_3 + x_4), \; a_3 = \frac{1}{4}(x_1 - x_2 + x_3 - x_4).$$

Taking them back into Eq.(12.79) we get:

$$x(\xi,\eta) = \frac{1}{4}(x_1 + x_2 + x_3 + x_4) + \frac{1}{4}(-x_1 + x_2 + x_3 - x_4)\xi +$$

$$+ \frac{1}{4}(-x_1 - x_2 + x_3 + x_4)\eta + \frac{1}{4}(x_1 - x_2 + x_3 - x_4)\xi\eta =$$

$$\frac{1}{4}(1 - \xi - \eta + \xi\eta)x_1 + \frac{1}{4}(1 + \xi - \eta - \xi\eta)x_2 + \frac{1}{4}(1 + \xi + \eta + \xi\eta)x_3 + \frac{1}{4}(1 - \xi + \eta - \xi\eta)x_4.$$

$$(12.86)$$

The interpolation polynomials on the base of Eq.(12.86) are:

$$N_1(\xi,\eta) = \frac{1}{4}(1-\xi)(1-\eta), \; N_2(\xi,\eta) = \frac{1}{4}(1+\xi)(1-\eta), \quad (12.87)$$

$$N_3(\xi,\eta) = \frac{1}{4}(1+\xi)(1+\eta), \; N_4(\xi,\eta) = \frac{1}{4}(1-\xi)(1+\eta).$$

Performing the same computation for coordinate *y* we obtain the same interpolation functions. The three dimensional plot of the $N_i(\xi,\eta)$ interpolation functions represents line sur-

faces, of which value in the location of the ith node is equal to unity, while in the location of the other nodes it is equal to zero, as it is demonstrated in Fig.12.7.
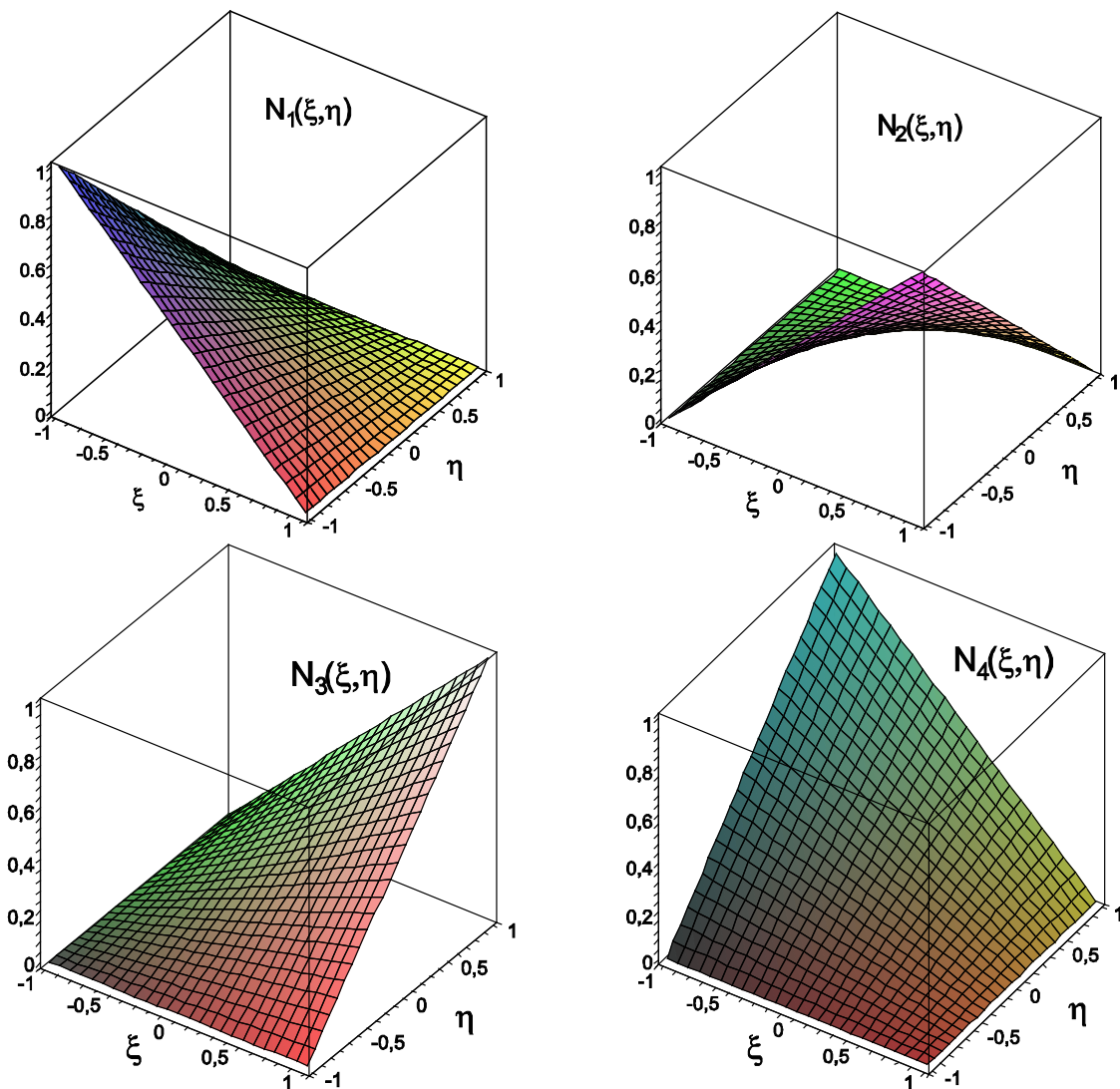

Fig.12.7. Interpolation functions of the isoparametric quadrilateral element.

The summary of the geometry is given by the formulae below:

$$\begin{bmatrix} x(\xi,\eta) \\ y(\xi,\eta) \end{bmatrix} = \underline{\underline{N}}(\xi,\eta)\underline{R}_e \,, \qquad (12.88)$$

where:

$$\underline{R}_e^T = \begin{bmatrix} x_1 & y_1 & x_2 & y_2 & x_3 & y_3 & x_4 & y_4 \end{bmatrix}, \qquad (12.89)$$

is the vector of nodal coordinates, and:

$$\underline{\underline{N}}(\xi,\eta) = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix}.$$

The compact form of the interpolation functions is:

$$N_i(\xi,\eta) = \frac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i) \,, \qquad (12.90)$$

where ξ_i and η_i are the corner node coordinates according to Fig.12.6b.

12.5.2 Interpolation of the displacement field

The displacement vector field of the isoparametric quadrilateral element can be written as:

$$\underline{u}(\xi,\eta) = \begin{bmatrix} u(\xi,\eta) \\ v(\xi,\eta) \end{bmatrix} = \underline{\underline{N}}(\xi,\eta)\underline{u}_e, \qquad (12.91)$$

where:

$$u(\xi,\eta) = N_1(\xi,\eta)u_1 + N_2(\xi,\eta)u_2 + N_3(\xi,\eta)u_3 + N_4(\xi,\eta)u_4, \qquad (12.92)$$

$$v(\xi,\eta) = N_1(\xi,\eta)v_1 + N_2(\xi,\eta)v_2 + N_3(\xi,\eta)v_3 + N_4(\xi,\eta)v_4,$$

moreover, the matrix of interpolation functions and the vector of nodal displacements are:

$$\underline{\underline{N}}(\xi,\eta) = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix}, \qquad (12.93)$$

$$\underline{u}_e^T = \begin{bmatrix} u_1 & v_1 & u_2 & v_2 & u_3 & v_3 & u_4 & v_4 \end{bmatrix}.$$

The displacement field must result in the nodal displacements if we substitute the coordinates of the proper nodes back, i.e. it must satisfy the following conditions:

$$u(-1,-1) = u_1, u(1,-1) = u_2, u(1,1) = u_3, \ u(-1,1) = u_4 \qquad (12.94)$$

Mathematically this is the same set of conditions for the displacements as that formulated in the case of the geometrical parameters. Consequently the computation leads to the same interpolation functions as those given by Eq.(12.87). The quadrilateral element is called isoparametric element because of the fact, that the same interpolation functions are applied for the displacement field and local geometry.

12.5.3 Calculation of strain components, Jacobi matrix and Jacobi determinant

The vector of strain components using Eq.(12.2) is the following:

$$\underline{\varepsilon} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} u_{,x} \\ v_{,y} \\ u_{,y} + v_{,x} \end{bmatrix} = \underline{\partial}\underline{u} = \underline{\partial}\underline{\underline{N}}u_e = \underline{\underline{B}}u_e, \qquad (12.95)$$

where $u_{,x}$ is the partial derivative of u with respect to x, $v_{,y}$ is the partial derivative of v with respect to y. Moreover:

$$\underline{\underline{B}} = \underline{\partial}\underline{\underline{N}} = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} =$$

$$= \begin{bmatrix} \dfrac{\partial N_1}{\partial x} & 0 & \dfrac{\partial N_2}{\partial x} & 0 & \dfrac{\partial N_3}{\partial x} & 0 & \dfrac{\partial N_4}{\partial x} & 0 \\ 0 & \dfrac{\partial N_1}{\partial y} & 0 & \dfrac{\partial N_2}{\partial y} & 0 & \dfrac{\partial N_3}{\partial y} & 0 & \dfrac{\partial N_4}{\partial y} \\ \dfrac{\partial N_1}{\partial y} & \dfrac{\partial N_1}{\partial x} & \dfrac{\partial N_2}{\partial y} & \dfrac{\partial N_2}{\partial x} & \dfrac{\partial N_3}{\partial y} & \dfrac{\partial N_3}{\partial x} & \dfrac{\partial N_4}{\partial y} & \dfrac{\partial N_4}{\partial x} \end{bmatrix}.$$

$$(12.96)$$

Apparently, matrix $\underline{B}$ contains the first derivatives of the interpolation functions with respect to x and y. it can be elaborated based on Eq.(12.87) that the N_i interpolation functions are known in terms of ξ and η. We refer to the chain rule of differentiation:

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial x}, \frac{\partial}{\partial y} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial y}, \qquad (12.97)$$

$$\frac{\partial}{\partial \xi} = \frac{\partial}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \xi}, \frac{\partial}{\partial \eta} = \frac{\partial}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial}{\partial y}\frac{\partial y}{\partial \eta}.$$

Utilizing Eq.(12.77) the local geometry and the first derivative of the functions with respect to ξ and η are:

$$x(\xi,\eta) = \sum_{i=1}^{4} N_i(\xi,\eta)x_i, \frac{\partial x}{\partial \xi} = \sum_{i=1}^{4}\frac{\partial N_i}{\partial \xi}x_i, \frac{\partial x}{\partial \eta} = \sum_{i=1}^{4}\frac{\partial N_i}{\partial \eta}x_i, \quad (12.98)$$

$$y(\xi,\eta) = \sum_{i=1}^{4} N_i(\xi,\eta)y_i, \frac{\partial y}{\partial \xi} = \sum_{i=1}^{4}\frac{\partial N_i}{\partial \xi}y_i, \frac{\partial y}{\partial \eta} = \sum_{i=1}^{4}\frac{\partial N_i}{\partial \eta}y_i.$$

Writing it in matrix form we have:

$$\begin{bmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} \begin{bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{bmatrix} = \underline{\underline{J}} \begin{bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{bmatrix}, \qquad (12.99)$$

where $\underline{\underline{J}}$ is the so-called Jacobi matrix:

$$\underline{\underline{J}} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}. \qquad (12.100)$$

The Jacobi determinant is:

$$J = J_{11}J_{22} - J_{12}J_{21} = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \xi}\frac{\partial x}{\partial \eta}. \qquad (12.101)$$

The derivatives with respect to x and y are provided by the help of the inverse Jacobi matrix:

$$\underline{\underline{J}}^{-1} = \frac{1}{J}\begin{bmatrix} \dfrac{\partial y}{\partial \eta} & -\dfrac{\partial y}{\partial \xi} \\ -\dfrac{\partial x}{\partial \eta} & \dfrac{\partial x}{\partial \xi} \end{bmatrix} = \frac{1}{J}\begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix}, \qquad (12.102)$$

furthermore:

$$\begin{bmatrix} \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial y} \end{bmatrix} = \underline{\underline{J}}^{-1} \begin{bmatrix} \dfrac{\partial}{\partial \xi} \\ \dfrac{\partial}{\partial \eta} \end{bmatrix}, \qquad (12.103)$$

From which we obtain the followings:

$$\frac{\partial}{\partial x} = \frac{1}{J}(J_{22}\frac{\partial}{\partial \xi} - J_{12}\frac{\partial}{\partial \eta}), \qquad (12.104)$$

$$\frac{\partial}{\partial y} = \frac{1}{J}(-J_{21}\frac{\partial}{\partial \xi} + J_{11}\frac{\partial}{\partial \eta}).$$

With the aid of the former matrix $\underline{\underline{B}}$ becomes:

$$\underline{\underline{B}} = \frac{1}{J}\begin{bmatrix} J_{22}N_{1,\xi} - J_{12}N_{1,\eta} & 0 & J_{22}N_{2,\xi} - J_{12}N_{2,\eta} & 0 & .... \\ 0 & -J_{21}N_{1,\xi} + J_{11}N_{1,\eta} & 0 & J_{21}N_{2,\xi} - J_{11}N_{2,\eta} & .... \\ -J_{21}N_{1,\xi} + J_{11}N_{1,\eta} & J_{22}N_{1,\xi} - J_{12}N_{1,\eta} & J_{21}N_{2,\xi} - J_{11}N_{2,\eta} & J_{22}N_{2,\xi} - J_{12}N_{2,\eta} & .... \end{bmatrix}$$

$$\begin{bmatrix} .... & J_{22}N_{3,\xi} - J_{12}N_{3,\eta} & 0 & J_{22}N_{4,\xi} - J_{12}N_{4,\eta} & 0 \\ .... & 0 & -J_{21}N_{3,\xi} + J_{11}N_{3,\eta} & 0 & -J_{21}N_{4,\xi} + J_{11}N_{4,\eta} \\ .... & -J_{21}N_{3,\xi} + J_{11}N_{3,\eta} & J_{22}N_{3,\xi} - J_{12}N_{3,\eta} & -J_{21}N_{4,\xi} + J_{11}N_{4,\eta} & J_{22}N_{4,\xi} - J_{12}N_{4,\eta} \end{bmatrix}$$

(12.105)

We need the derivatives of the interpolations functions and the elements of the Jacobi matrix, which are:

$$\frac{\partial N_1}{\partial \xi} = -\frac{1}{4}(1-\eta), \ \frac{\partial N_2}{\partial \xi} = \frac{1}{4}(1-\eta), \ \frac{\partial N_3}{\partial \xi} = \frac{1}{4}(1+\eta), \ \frac{\partial N_4}{\partial \xi} = -\frac{1}{4}(1+\eta),$$

(12.106)

$$\frac{\partial N_1}{\partial \eta} = -\frac{1}{4}(1-\xi), \ \frac{\partial N_2}{\partial \eta} = -\frac{1}{4}(1+\xi), \ \frac{\partial N_3}{\partial \eta} = \frac{1}{4}(1+\xi), \ \frac{\partial N_4}{\partial \eta} = \frac{1}{4}(1-\xi),$$

respectively, and:

$$J_{11} = \frac{\partial x}{\partial \xi} = \sum_{i=1}^{4}\frac{\partial N_i}{\partial \xi}x_i = \frac{1}{4}\{-(1-\eta)x_1 + (1-\eta)x_2 + (1+\eta)x_3 - (1+\eta)x_4\},$$

(12.107)

$$J_{12} = \frac{\partial y}{\partial \xi} = \sum_{i=1}^{4}\frac{\partial N_i}{\partial \xi}y_i = \frac{1}{4}\{-(1-\eta)y_1 + (1-\eta)y_2 + (1+\eta)y_3 - (1+\eta)y_4\},$$

$$J_{21} = \frac{\partial x}{\partial \eta} = \sum_{i=1}^{4}\frac{\partial N_i}{\partial \eta}x_i = \frac{1}{4}\{-(1-\xi)x_1 - (1+\xi)x_2 + (1+\xi)x_3 + (1-\xi)x_4\},$$

$$J_{22} = \frac{\partial y}{\partial \eta} = \sum_{i=1}^{4}\frac{\partial N_i}{\partial \eta}y_i = \frac{1}{4}\{-(1-\xi)y_1 - (1+\xi)y_2 + (1+\xi)y_3 + (1-\xi)y_4\}.$$

Based on the former equations we can formulate the Jacobi matrix either in the following form:

$$\underline{\underline{J}} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} = \begin{bmatrix} N_{1,\xi} & N_{2,\xi} & N_{3,\xi} & N_{4,\xi} \\ N_{1,\eta} & N_{2,\eta} & N_{3,\eta} & N_{4,\eta} \end{bmatrix}\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix}. \text{ (12.108)}$$

12.5.4 The importance of the Jacobi determinant, example

Calculate the elements of the Jacobi matrix for the quadrilateral shown in Fig.12.8! The nodal coordinates are:

$$x_1 = 0, \ y_1 = 0, \ x_2 = a, \ y_2 = 0, \ x_3 = a, \ y_3 = a, \ x_4 = \frac{2}{3}a, \ y_4 = \frac{1}{3}a.$$
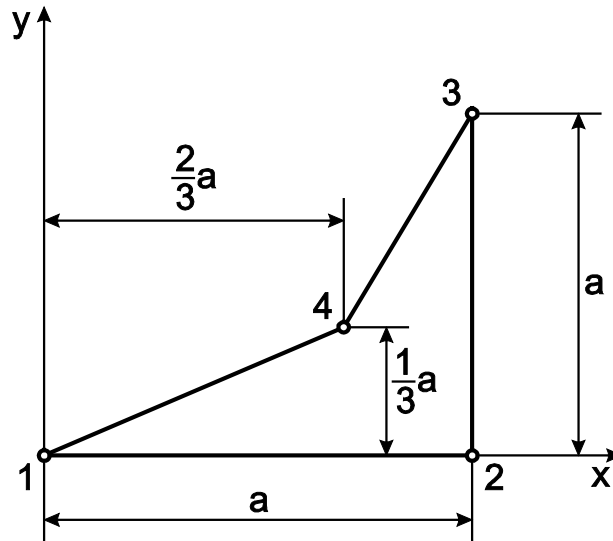
(12.109)

Fig.12.8. Isoparametric quadrilateral element with excessive distortion.

The elements of the Jacobi matrix based on Eq.(12.107) are:

$$J_{11} = \frac{1}{4}\left\{-(1-\eta)0 + (1-\eta)a + (1+\eta)a - (1+\eta)\frac{2}{3}a\right\} = \frac{1}{3}a - \frac{1}{6}a\eta,$$

(12.110)

$$J_{12} = \frac{1}{4}\left\{-(1-\eta)0 + (1-\eta)0 + (1+\eta)a - (1+\eta)\frac{1}{3}a\right\} = \frac{1}{6}a + \frac{1}{6}a\eta,$$

$$J_{21} = \frac{1}{4}\left\{-(1-\xi)0 - (1+\xi)a + (1+\xi)a + (1-\xi)\frac{2}{3}a\right\} = \frac{1}{6}a - \frac{1}{6}a\xi,$$

$$J_{22} = \frac{1}{4}\left\{-(1-\xi)0 - (1+\xi)0 + (1+\xi)a + (1-\xi)\frac{1}{3}a\right\} = \frac{1}{3}a + \frac{1}{6}a\xi,$$

from which the Jacobi determinant is:

$$J = J_{11}J_{22} - J_{12}J_{21} = \frac{1}{12}a^2(1 + \xi - \eta).$$ (12.111)

The Jacobi determinant is 0, if e.g. $\xi = -1$ and $\eta = 0$, or $\xi = 0$ and $\eta = -1$. This case is said to be excessive distortion, it means that we have degenerate element. If $J = 0$, then the inverse Jacobi matrix does not exist at the point under consideration. Moreover, the parameter lines intersect each other outside the domain of the quadrilateral. That is why the sum of the inner angles of quadrilateral must be less than 180°, in other words the quadrilateral can not be concave.

12.5.5 Calculation of the stress field
The vector of stress components can be obtained from Eq.(12.7):

$$\underline{\sigma} = \underline{\underline{C}}\underline{\varepsilon} = \underline{\underline{C}}\underline{\underline{B}}\underline{u}_e,$$ (12.112)

where $\underline{\underline{C}} = \underline{\underline{C}}^{str}$ for plane stress and $\underline{\underline{C}} = \underline{\underline{C}}^{stn}$ for plane strain (see section 11.)

12.5.6 Calculation of the stiffness matrix
The stiffness matrix for plane problems is calculated by Eq.(12.9):

$$\underline{\underline{K}}_e = \int\int \underline{\underline{B}}^T \underline{\underline{C}}^T \underline{\underline{B}}v dx dy.$$ (12.113)

In the case of the isoparametric quadrilateral the elements of matrix $\underline{\underline{B}}$ contains the derivatives of the interpolation functions. Consequently, for the stiffness matrix calculation the transformation of surface integrals must be performed. The vectors and parameters, which are required for the analysis, are shown in Fig.12.9. The ranges of parameters c_1 and c_2 are:

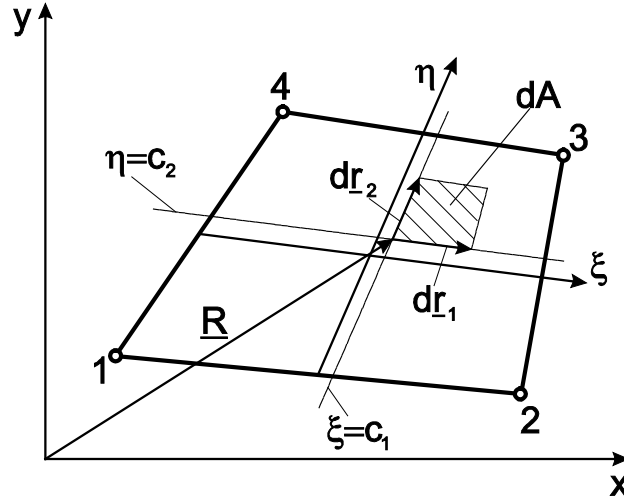$$-1 \le c_1 \le 1, \ -1 \le c_2 \le 1. \tag{12.114}$$

Fig.12.9. Transformation of surface integral in the isoparametric quadrilateral.

The differential vectors written by lowercase letters, can be formulated by utilizing Eq. (12.100):

$$d\underline{r}_1 = \begin{bmatrix} dx \\ dy \end{bmatrix}_{\eta=konst} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} d\xi + \dfrac{\partial x}{\partial \eta} d\eta \\ \dfrac{\partial y}{\partial \xi} d\xi + \dfrac{\partial y}{\partial \eta} d\eta \end{bmatrix}_{\eta=konst} = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} d\xi \\ \dfrac{\partial x}{\partial \xi} d\xi \end{bmatrix} = \begin{bmatrix} J_{11} \\ J_{12} \end{bmatrix} d\xi ,$$

$$\tag{12.115}$$

and similarly:

$$d\underline{r}_2 = \begin{bmatrix} dx \\ dy \end{bmatrix}_{\xi=konst} = \begin{bmatrix} \dfrac{\partial x}{\partial \eta} d\eta \\ \dfrac{\partial x}{\partial \eta} d\eta \end{bmatrix} = \begin{bmatrix} J_{21} \\ J_{22} \end{bmatrix} d\eta . \tag{12.116}$$

The definition of the elementary area is:

$$dA = \left| d\underline{r}_1 \times d\underline{r}_2 \right| = abs \left(\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ J_{11}d\xi & J_{12}d\xi & 0 \\ J_{21}d\eta & J_{22}d\eta & 0 \end{vmatrix} \right) = (J_{11}J_{22} - J_{12}J_{21})d\xi d\eta = dxdy ,$$

$$\tag{12.117}$$

this yields:

$$dA = dxdy = Jd\xi d\eta \ \text{ and: } \int_A dxdy = \int_{-1}^{1}\int_{-1}^{1} Jd\xi d\eta . \tag{12.118}$$

The stiffness matrix becomes:

$$\underline{\underline{K}}_e = \int_A \underline{\underline{B}}^T \underline{\underline{C}}^T \underline{\underline{B}} v \, dx dy = \int_{-1}^{1}\int_{-1}^{1} \underline{\underline{B}}^T \underline{\underline{C}}^T \underline{\underline{B}} v J \, d\xi d\eta \,, \qquad (12.119)$$

i.e., the stiffness matrix can be computed by the help of an area integral. For the calculation we can apply analytical or numerical method. The commercial finite element packages, in general, implement the Gaussian quadrature to perform the integration. This method will be presented in section 12.6.

12.5.7 Calculation of the force vector

Distributed load along the element edge. The force vector resulting from the distributed load along element edge 1-2 shown in Fig.12.10 can be defined as:

$$\underline{F}_{ep} = v\int \underline{\underline{N}}^T \underline{p}_{12} \, ds \,, \qquad (12.120)$$

where:

$$s = \frac{1}{2}l_{12}\xi \ \ \text{and} \ \ ds = \frac{1}{2}l_{12}d\xi \,, \qquad (12.121)$$

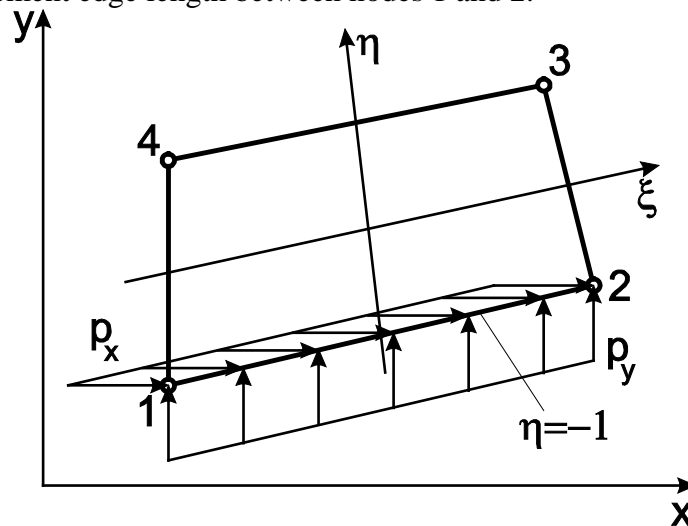where l_{12} is the element edge length between nodes 1 and 2.

Fig.12.10. Distributed load along the element edge of an isoparametric quadrilateral element.

Moreover, we know that along edge 1-2 $\eta = -1$ and $-1 \le \xi \le 1$ (see. Fig.12.6). We can write after all, that:

$$\underline{F}_{ep} = v\int_{l_{12}} \underline{\underline{N}}^T \underline{p}_{12} \, ds \bigg|_{\eta=-1} = \frac{1}{2}v\int_{-1}^{1}
\begin{bmatrix} N_1 & 0 \\ 0 & N_1 \\ N_2 & 0 \\ 0 & N_2 \\ N_3 & 0 \\ 0 & N_3 \\ N_4 & 0 \\ 0 & N_4 \end{bmatrix}
\begin{bmatrix} p_x \\ p_y \end{bmatrix} l_{12} d\xi \bigg|_{\eta=-1} = \frac{1}{2}vl_{12}\int_{-1}^{1}
\begin{bmatrix} p_x N_1 \\ p_y N_1 \\ p_x N_2 \\ p_y N_2 \\ p_x N_3 \\ p_y N_3 \\ p_x N_4 \\ p_y N_4 \end{bmatrix} d\xi \bigg|_{\eta=-1}.$$

$$(12.122)$$

For further calculation we must evaluate the interpolation functions along the parameter line, for which $\eta = -1$:

$$N_1\big|_{\eta=-1} = \frac{1}{4}(1-\xi)(1-\eta)\bigg|_{\eta=-1} = \frac{1}{2}(1-\xi), \qquad (12.123)$$

$$N_2\big|_{\eta=-1} = \frac{1}{4}(1+\xi)(1-\eta)\bigg|_{\eta=-1} = \frac{1}{2}(1+\xi),$$

$$N_3\big|_{\eta=-1} = \frac{1}{4}(1+\xi)(1+\eta)\bigg|_{\eta=-1} = 0,$$

$$N_4\big|_{\eta=-1} = \frac{1}{4}(1-\xi)(1+\eta)\bigg|_{\eta=-1} = 0.$$

This yields:

$$\int_{-1}^{1} p_x N_1\big|_{\eta=-1} d\xi = \frac{1}{2} p_x \int_{-1}^{1}(1-\xi)d\xi = \frac{1}{2} p_x \left[\xi - \frac{\xi^2}{2}\right]_{-1}^{1} = \frac{1}{2} p_x \left[1 - \frac{1}{2} - (-1 - \frac{1}{2})\right] = p_x,$$

$$(12.124)$$

$$\int_{-1}^{1} p_y N_2\big|_{\eta=-1} d\xi = \frac{1}{2} p_y \int_{-1}^{1}(1+\xi)d\xi = \frac{1}{2} p_y \left[\xi + \frac{\xi^2}{2}\right]_{-1}^{1} = \frac{1}{2} p_y \left[1 + \frac{1}{2} - (-1 + \frac{1}{2})\right] = p_y.$$

By taking the results back into the force vector we obtain:

$$\underline{F}_{ep}^T = \frac{1}{2} v l_{12} \begin{bmatrix} p_x & p_y & p_x & p_y & 0 & 0 & 0 & 0 \end{bmatrix}. \qquad (12.125)$$

The resultant of the uniformly distributed load is divided into two parts and (similarly to the beam and linear triangle elements) put into the nodes of element edge. The calculation can be made also in the case of linearly distributed load; naturally it results in a different force vector.

Body force. The force vector calculated from the body force is:

$$\underline{F}_{eb} = v \int_{-1}^{1}\int_{-1}^{1} \underline{\underline{N}}^T \underline{q} J d\xi d\eta, \qquad (12.126)$$

for which we need again the evaluation of surface integral. Similarly to the stiffness matrix, the Gaussian quadrature will be applied to evaluate the integral.

Concentrated loads. For plane problems there are concentrated forces acting in the nodes and there are no moments. The *x* and *y* components of the concentrated forces are collected in a vector:

$$\underline{F}_{ec}^T = \begin{bmatrix} F_{x1} & F_{y1} & F_{x2} & F_{y2} & F_{x3} & F_{y3} & F_{x4} & F_{y4} \end{bmatrix}. \qquad (12.127)$$

The total vector of forces is the sum of vectors presented in the last three points:

$$\underline{F}_e = \underline{F}_{ep} + \underline{F}_{eb} + \underline{F}_{ec}. \qquad (12.128)$$

12.6   Numerical integration, the Gauss rule

For the calculation of the element stiffness matrix and the body force vector of isoparametric quadrilaterals there are numerical integration schemes implemented in the finite element packages. Commonly, the Gauss rule is applied because it uses minimal number of sample points and it is relatively accurate [1,2,6].

12.6.1 One dimensional Gauss rule

The main aim is the approximate but relatively accurate calculation of the area under the curve shown by Fig.12.11 using the one dimensional rule.
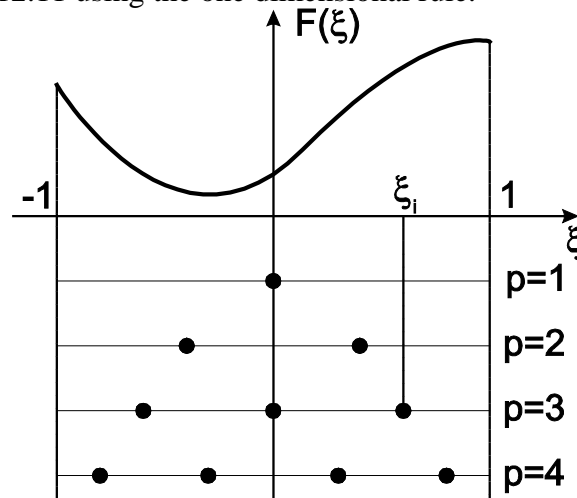

Fig.12.11. Sample points of the one dimensional Gauss rule.

The approximate area under the curve is calculated by:

$$\int_{-1}^{1} F(\xi) d\xi \cong \sum_{i=1}^{p} w_i F(\xi_i).$$

(12.129)

The sample (or integration) point coordinates, ξ_i and the integration weights, w_i are listed in table 12.1. The one dimensional rule provides the exact solution for a polynomial up to the order of $2p$-1.

p	ξ_i	w_i
1	*0*	*2*
2	$-1/\sqrt{3}$ $1/\sqrt{3}$	*1* *1*
3	$-\sqrt{3/5}$ *0* $\sqrt{3/5}$	*5/9* *8/9* *5/9*
4	$-\sqrt{(3-2\sqrt{6/5})/7}$ $-\sqrt{(3+2\sqrt{6/5})/7}$ $\xi_3 = -\xi_2$ $\xi_4 = -\xi_1$	$1/2-(\sqrt{5/6})/6$ $1/2+(\sqrt{5/6})/6$ $w_3 = w_2$ $w_4 = w_1$

Table 12.1. Parameters of the one dimensional Gauss rule.

Let us solve an example for the application of the Gauss rule! Calculate the exact value of the integral:

$$I = \int_{1}^{3} \frac{1}{x} dx$$

(12.130)

as well as its approximate value using one, two and three integration points!
Exact solution:

$$I = \left[\ln x\right]_1^3 = \ln 3 - \ln 1 = 1{,}098612.\qquad(12.131)$$

Gauss rule, p = 1. Let $\xi = x\text{-}2$. If $x = 3$, then $\xi = 1$, on the other hand if $x = 1$, then $\xi = \text{-}1$, consequently:

$$I = \int_1^3 \frac{1}{x}\,dx = \int_{-1}^1 \frac{1}{\xi+2}\,d\xi, \text{ and: } F(\xi) = \frac{1}{\xi+2}.\qquad(12.132)$$

The approximate value of the integral is:

$$I_1 \cong w_1 F(0) = 2\frac{1}{2} = 1.\qquad(12.133)$$

That means an error of 9,9% compared to the exact solution.
Gauss rule, p = 2. In this case:

$$I_2 \cong w_1 F(-\frac{1}{\sqrt{3}}) + w_2 F(\frac{1}{\sqrt{3}}) = 1\frac{1}{-\dfrac{1}{\sqrt{3}+2}} + 1\frac{1}{\dfrac{1}{\sqrt{3}+2}} = 1{,}090909.$$
$$(12.134)$$

The value of the integral differs with 0,7 % from the exact solution.
Gauss rule, p = 3.

$$I_3 \cong w_1 F(-\sqrt{\frac{3}{5}}) + w_2 F(0) + w_3 F(\sqrt{\frac{3}{5}}) = \frac{5}{9}\frac{1}{-\sqrt{\dfrac{3}{5}}+2} + \frac{8}{9}\frac{1}{2} + \frac{5}{9}\frac{1}{\sqrt{\dfrac{3}{5}}+2} = 1{,}0980387.$$
$$(12.135)$$

The error of approximation is only 0,052%.

12.6.2 Two dimensional Gauss rule

The two dimensional Gauss rule makes it possible to evaluate the approximate value of surface integrals. The integral is approximated by the expression below:

$$\int_a^b\int_c^d f(x,y)\,dx\,dy = \int_{-1}^1\int_{-1}^1 f(\xi,\eta)J\,d\xi\,d\eta \cong \sum_{j=1}^n \sum_{i=1}^n w_i w_j f(\xi_i,\eta_j)\left|J(\xi_i,\eta_j)\right|,$$
$$(12.136)$$

where w_i and w_j are the integration weights, ξ_i and η_j are the integration point coordinates, moreover, the ranges are $\text{-}1 \le \xi_i \le 1$, $\text{-}1 \le \eta_j \le 1$, respectively. Depending on the number of integration points we can define different Gaussian quadratures, as it is demonstrated in Fig.12.12.
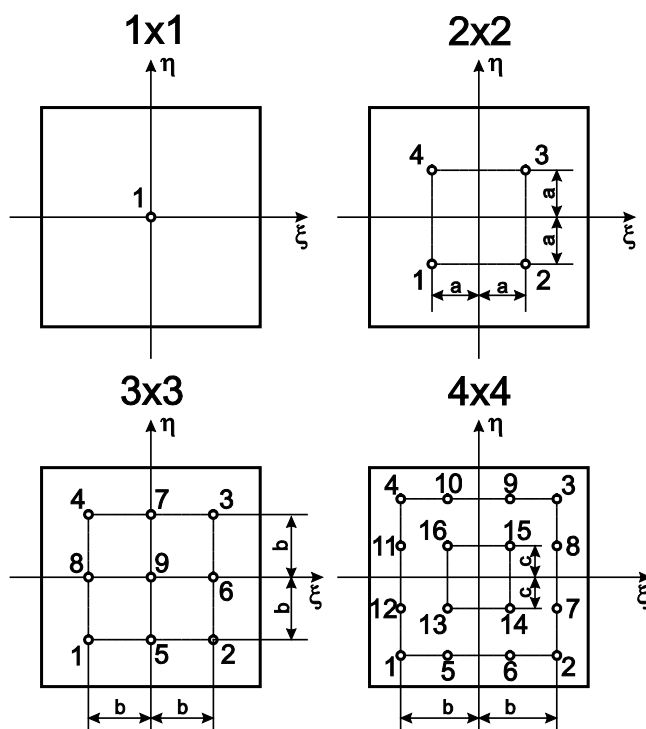
Fig.12.12. Integration points of the 1x1, 2x2, 3x3 and 4x4 Gaussian quadratures.

For the 1x1 quadrature there is only a single integration point, for the 2x2 we have 4, etc. The parameters of the 1x1, 2x2 and 3x3 Gaussian quadratures are summarized in Table 12.2.

1x1	2x2		3x3			
	$a = 1/\sqrt{3}$		$b = \sqrt{3/5}$		w_i	w_j
$\xi_1 = 0$	$\xi_1 = -a$	$\eta_1 = -a$	$\xi_1 = -b$	$\eta_1 = -b$	5/9	5/9
$\eta_1 = 0$	$\xi_2 = a$	$\eta_2 = -a$	$\xi_2 = b$	$\eta_2 = -b$	5/9	5/9
$w_1 = 2$	$\xi_3 = a$	$\eta_3 = a$	$\xi_3 = b$	$\eta_3 = b$	5/9	5/9
	$\xi_4 = -a$	$\eta_4 = a$	$\xi_4 = -b$	$\eta_4 = b$	5/9	5/9
			$\xi_5 = 0$	$\eta_5 = -b$	8/9	5/9
	$w_1 = 1$	$w_2 = 1$	$\xi_6 = b$	$\eta_6 = 0$	5/9	8/9
	$w_3 = 1$	$w_4 = 1$	$\xi_7 = 0$	$\eta_7 = b$	8/9	5/9
			$\xi_8 = -b$	$\eta_8 = 0$	5/9	8/9
			$\xi_9 = 0$	$\eta_9 = 0$	8/9	8/9

Table 12.2. Parameters of the Gaussian quadratures.

Example for the application of Gaussian quadrature. Calculate the approximate value of the integral:

$$I = \int_A xy \, dA \qquad (12.137)$$

using the 2x2 Gauss quadrature for the domain of parallelogram depicted in Fig.12.13! Compare the result to that of the exact integration [4]!
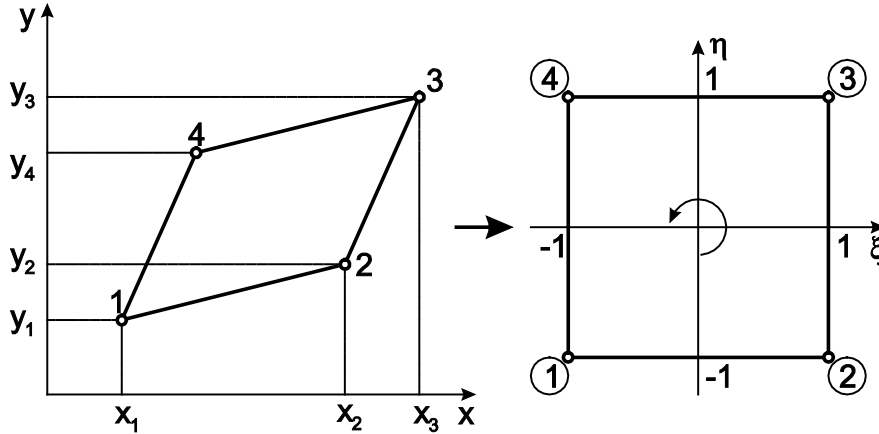
Fig.12.13. Example for the application of Gaussian quadrature.

The nodal coordinates are:

$$x_1 = 1,\ x_2 = 3,\ x_3 = 4,\ x_4 = 2,\ y_1 = 1,\ y_2 = 2,\ y_3 = 4,\ y_4 = 3. \quad (12.138)$$

Based on the approximate expression of the 2x2 Gaussian quadrature we can write:

$$I = \int_A xy\,dA \cong \sum_{i=1}^{2}\sum_{j=1}^{2} w_i w_j f(\xi_i, \eta_j) \left| J(\xi_i, \eta_j) \right|. \quad (12.139)$$

The calculation requires the elements of the Jacobi matrix. We need the nodal coordinates and also the derivatives of the interpolation functions (see Eq.(12.110)):

$$J_{11} = \frac{1}{4}\left\{-(1-\eta)1 + (1-\eta)3 + (1+\eta)4 - (1+\eta)2\right\} = 1, \quad (12.140)$$

$$J_{12} = \frac{1}{4}\left\{-(1-\eta)1 + (1-\eta)2 + (1+\eta)4 - (1+\eta)3\right\} = \frac{1}{2},$$

$$J_{21} = \frac{1}{4}\left\{-(1-\xi)1 - (1+\xi)3 + (1+\xi)4 + (1-\xi)2\right\} = \frac{1}{2},$$

$$J_{22} = \frac{1}{4}\left\{-(1-\xi)1 - (1+\xi)2 + (1+\xi)4 + (1-\xi)3\right\} = 1.$$

The Jacobi matrix is:

$$\underline{\underline{J}} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} = \begin{bmatrix} 1 & \dfrac{1}{2} \\ \dfrac{1}{2} & 1 \end{bmatrix}, \quad (12.141)$$

and the Jacobi determinant is:

$$J = 1\cdot 1 - \frac{1}{2}\cdot\frac{1}{2} = \frac{3}{4} = \text{const}. \quad (12.142)$$

The x and y parameters utilizing the interpolated form given by Eq.(12.77) are:

$$x(\xi,\eta) = \sum_{i=1}^{4} N_i x_i = \frac{1}{4}\big\{(1-\xi)(1-\eta)1 + (1+\xi)(1-\eta)3 +$$
$$+ (1+\xi)(1+\eta)4 + (1-\xi)(1+\eta)2\big\}, \quad (12.143)$$

$$y(\xi,\eta) = \sum_{i=1}^{4} N_i y_i = \frac{1}{4}\big\{(1-\xi)(1-\eta)1 + (1+\xi)(1-\eta)2 +$$
$$+ (1+\xi)(1+\eta)4 + (1-\xi)(1+\eta)3\big\}.$$

The function, $f(\xi,\eta)$ is:

$$f(\xi,\eta) = xy = \frac{1}{4}\{(5 + 2\xi + \eta) \cdot (5 + \xi + 2\eta)\}. \qquad (12.144)$$

We can calculate the approximate value of the integral based on the figure and table above:

$$I = [f(-a,-a) + f(-a,a) + f(a,-a) + f(a,a)]\frac{3}{4} = 19{,}75. \qquad (12.145)$$

The exact value of the integral is:

$$I = \int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)d\xi d\eta = xy = \int_{-1}^{1}\int_{-1}^{1}\frac{1}{4}\{(5 + 2\xi + \eta) \cdot (5 + \xi + 2\eta)\}\frac{3}{4}d\xi d\eta = \frac{79}{4} = 19{,}75. \qquad (12.146)$$

It is shown apparently, that the Gaussian quadrature provides the exact value in this case. Most of the commercial finite element packages implements 2x2 quadrature.

12.7   Example for the isoparametric quadrilateral

Solve the example presented in section 12.3 using one isoparametric quadrilateral element! The data are the same as those given in the linear triangle element. Apply a single finite element by following Fig.12.14 [4]. Determine the nodal displacements and the reactions!
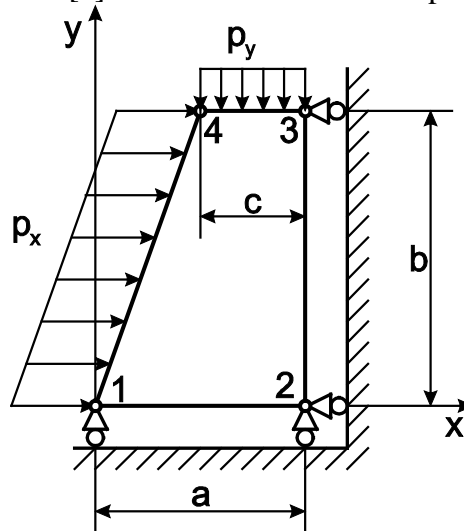

Fig.12.14. Example for the application of the isoparametric quadrilateral element.

The finite element equilibrium equation to be solved is:

$$\underline{\underline{K}}\,\underline{U} = \underline{F}. \qquad (12.147)$$

Since we have only a single element, in this case Eq.(12.147) corresponds to the equilibrium equation in the element level:

$$\underline{\underline{K}}_e\,\underline{u}_e = \underline{F}_e, \qquad (12.148)$$

where:

$$\underline{U}^T = [u_1 \quad v_1 \quad u_2 \quad v_2 \quad u_3 \quad v_3 \quad u_4 \quad v_4] \qquad (12.149)$$

is the vector of nodal displacements. Due to the boundary conditions ($v_1 = v_2 = u_2 = u_3 = 0$) we have:

$$\underline{U}^T = [u_1 \quad 0 \quad 0 \quad 0 \quad 0 \quad v_3 \quad u_4 \quad v_4]. \qquad (12.150)$$

Similarly to the linear triangle element, we have a system of equations including four unknowns. The element stiffness matrix is calculated by the Gaussian quadrature, i.e. we can write, that:

$$\underline{\underline{K}}_e = \int\limits_{-1}^{1}\int\limits_{-1}^{1} \underline{\underline{B}}^T \underline{\underline{C}}^T \underline{\underline{B}} v J d\xi d\eta \cong v\sum_{i=1}^{p}\sum_{j=1}^{p} w_i w_j \underline{\underline{B}}^T(\xi_i,\eta_j)\underline{\underline{C}}^T\underline{\underline{B}}(\xi_i,\eta_j)\left|J(\xi_i,\eta_j)\right|.$$

(12.151)

The elements of the Jacobi matrix are equally required:

$$J_{11} = \frac{\partial x}{\partial \xi} = \sum_{i=1}^{4}\frac{\partial N_i}{\partial \xi}x_i = \frac{1}{4}\left\{-(1-\eta)x_1 + (1-\eta)x_2 + (1+\eta)x_3 - (1+\eta)x_4\right\} =$$

$$= \frac{1}{4}\left\{-(1-\eta)0 + (1-\eta)20 + (1+\eta)20 - (1+\eta)10\right\} = \frac{15}{2} - \frac{5}{2}\eta,$$

(12.152)

$$J_{12} = \frac{\partial y}{\partial \xi} = \sum_{i=1}^{4}\frac{\partial N_i}{\partial \xi}y_i = \frac{1}{4}\left\{-(1-\eta)y_1 + (1-\eta)y_2 + (1+\eta)y_3 - (1+\eta)y_4\right\} =$$

$$= \frac{1}{4}\left\{-(1-\eta)0 + (1-\eta)0 + (1+\eta)30 - (1+\eta)30\right\} = 0,$$

$$J_{21} = \frac{\partial x}{\partial \eta} = \sum_{i=1}^{4}\frac{\partial N_i}{\partial \eta}x_i = \frac{1}{4}\left\{-(1-\xi)x_1 - (1+\xi)x_2 + (1+\xi)x_3 + (1-\xi)x_4\right\} =$$

$$= \frac{1}{4}\left\{-(1-\xi)0 - (1+\xi)20 + (1+\xi)20 + (1-\xi)10\right\} = \frac{5}{2} - \frac{5}{2}\xi,$$

$$J_{22} = \frac{\partial y}{\partial \eta} = \sum_{i=1}^{4}\frac{\partial N_i}{\partial \eta}y_i = \frac{1}{4}\left\{-(1-\xi)y_1 - (1+\xi)y_2 + (1+\xi)y_3 + (1-\xi)y_4\right\} =$$

$$= \frac{1}{4}\left\{-(1-\xi)0 - (1+\xi)0 + (1+\xi)30 + (1-\xi)30\right\} = 15.$$

Constructing the Jacobi matrix we have:

$$\underline{\underline{J}} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} = \begin{bmatrix} \dfrac{15}{2} - \dfrac{5}{2}\eta & 0 \\ \dfrac{5}{2} - \dfrac{5}{2}\eta & 15 \end{bmatrix},$$

(12.153)

and the Jacobi determinant is:

$$J = \frac{225}{2} - \frac{75}{2}\eta.$$

(12.154)

It can be seen, that if $-1 \le \eta \le 1$, then $J > 0$ for each case, consequently the element is not degenerate, which is obviously seen based on Fig.12.5. The inverse Jacobi matrix is:

$$\underline{\underline{J}}^{-1} = \frac{1}{J}\begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} = \begin{bmatrix} -\dfrac{2}{5(-3+\eta)} & 0 \\ \dfrac{1-\xi}{15(\eta-3)} & \dfrac{1}{15} \end{bmatrix}.$$

(12.155)

As a next step, we calculate matrix $\underline{\underline{B}}$ (see Eq.(12.96)), where referring to Eq.(12.105) we have:

$$\frac{\partial N_1}{\partial x} = \frac{1}{J}(J_{22}N_{1,\xi} - J_{12}N_{1,\eta}) = -\frac{1-\eta}{10(3-\eta)},$$

(12.156)

$$\frac{\partial N_1}{\partial y} = \frac{1}{J}(-J_{21}N_{1,\xi} + J_{11}N_{1,\eta}) = -\frac{1-\xi}{30(3-\eta)},$$

$$\frac{\partial N_2}{\partial x} = \frac{1}{J}(J_{22}N_{2,\xi} - J_{12}N_{2,\eta}) = -\frac{1-\eta}{10(3-\eta)},$$

$$\frac{\partial N_2}{\partial y} = \frac{1}{J}(-J_{21}N_{2,\xi} + J_{11}N_{2,\eta}) = -\frac{2+\xi-\eta}{30(3-\eta)},$$

$$\frac{\partial N_3}{\partial x} = \frac{1}{J}(J_{22}N_{3,\xi} - J_{12}N_{3,\eta}) = \frac{1+\eta}{10(3-\eta)},$$

$$\frac{\partial N_3}{\partial y} = \frac{1}{J}(-J_{21}N_{3,\xi} + J_{11}N_{3,\eta}) = \frac{1+2\xi-\eta}{30(3-\eta)},$$

$$\frac{\partial N_4}{\partial x} = \frac{1}{J}(J_{22}N_{4,\xi} - J_{12}N_{4,\eta}) = -\frac{1+\eta}{10(3-\eta)},$$

$$\frac{\partial N_4}{\partial y} = \frac{1}{J}(-J_{21}N_{4,\xi} + J_{11}N_{4,\eta}) = \frac{1-\xi}{15(3-\eta)}.$$

Matrix $\underline{\underline{B}}$ becomes:

$$\underline{\underline{B}} = \begin{bmatrix} -\dfrac{1-\eta}{10(3-\eta)} & 0 & -\dfrac{1-\eta}{10(3-\eta)} & 0 & \cdots \\[2ex] 0 & -\dfrac{1-\xi}{30(3-\eta)} & 0 & -\dfrac{2+\xi-\eta}{30(3-\eta)} & \cdots \\[2ex] -\dfrac{1-\xi}{30(3-\eta)} & -\dfrac{1-\eta}{10(3-\eta)} & -\dfrac{2+\xi-\eta}{30(3-\eta)} & -\dfrac{1-\eta}{10(3-\eta)} & \cdots \end{bmatrix}$$

$$\begin{bmatrix} \cdots & \dfrac{1+\eta}{10(3-\eta)} & 0 & -\dfrac{1+\eta}{10(3-\eta)} & 0 \\[2ex] \cdots & 0 & \dfrac{1+2\xi-\eta}{30(3-\eta)} & 0 & \dfrac{1-\xi}{15(3-\eta)} \\[2ex] \cdots & \dfrac{1+2\xi-\eta}{30(3-\eta)} & \dfrac{1+\eta}{10(3-\eta)} & \dfrac{1-\xi}{15(3-\eta)} & -\dfrac{1+\eta}{10(3-\eta)} \end{bmatrix}.$$

(12.157)

We calculate the element [1,1] of the stiffness matrix by the Gaussian quadrature. For that, let us calculate the following:

$$\left[\underline{\underline{B}}^T \underline{\underline{C}}^T \underline{\underline{B}}\right]_{1,1} = \frac{1600(1-\eta)^2}{(3-\eta)^2} + \frac{200(1-\xi)^2}{3(3-\xi)^2}, \qquad (12.158)$$

and:

$$\left[\underline{\underline{K}}_e\right]_{1,1} = v\int_{-1}^{1}\int_{-1}^{1}\left[\underline{\underline{B}}^T \underline{\underline{C}}^T \underline{\underline{B}}\right]_{1,1} Jd\xi d\eta =$$

$$= v\int_{-1}^{1}\int_{-1}^{1}\left[\frac{1600(1-\eta)^2}{(3-\eta)^2} + \frac{200(1-\xi)^2}{3(3-\xi)^2}\right]\left[\frac{225}{2} - \frac{75}{2}\eta\right]d\xi d\eta = v\int_{-1}^{1}\int_{-1}^{1}f(\xi,\eta)J(\xi,\eta)d\xi d\eta,$$

(12.159)

where $v = 5$ mm is the thickness of the plate. We carry out the calculation in three ways, by using the 2x2, 3x3 Gaussian quadratures and the exact integration, respectively.

I. 2x2 Gaussian quadrature:

$$[\underline{\underline{K}}_e]_{1,1} \cong v \cdot \{f(-a,-a)|J(-a,-a)| + f(a,-a)|J(a,-a)| + f(a,a)|J(a,a)|f(-a,a)|J(-a,a)|\} =$$

$$= (2.1734 + 2.0927 + 0.2304 + 0.3496) \cdot 10^5 = 4,8462 \cdot 10^5 \frac{\text{N}}{\text{mm}}.$$

(12.160)

II. 3x3 Gaussian quadrature:

$$[\underline{\underline{K}}_e]_{1,1} \cong v \cdot \{f(-b,-b)|J(-b,-b)| + f(b,-b)|J(b,-b)| + f(b,b)|J(b,b)|f(-b,b)|J(-b,b)|\}\frac{5}{9}\frac{5}{9} +$$

$$+ v \cdot \{f(0,-b)|J(0,-b)| + f(b,0)|J(b,0)| + f(0,b)|J(0,b)|f(-b,0)|J(-b,0)|\}\frac{5}{9}\frac{8}{9} +$$

$$+ v \cdot f(0,0)|J(0,0)|\frac{8}{9}\frac{8}{9} = (2.6072 + 2.5046 + 0.07134 + 0.2454)\frac{25}{81} \cdot 10^5 +$$

$$+ (2.5360 + 1.0021 + 0.1247 + 1.1312)\frac{40}{81} \cdot 10^5 + 1.0417\frac{64}{81} \cdot 10^5 = 4,8660 \cdot 10^5 \frac{\text{N}}{\text{mm}}.$$

(12.161)

III. Exact integration:

$$[\underline{\underline{K}}_e]_{1,1} = 4,8666 \cdot 10^5 \frac{\text{N}}{\text{mm}}.$$

(12.162)

Calculating all of the components of the element stiffness matrix we obtain:

$$\underline{\underline{K}}_e = \begin{bmatrix} 4,8666 & 0,76713 & -4,3666 & 0,23287 & -2,7668 & -0,9657 & 2,2668 & -0,03426 \\ . & 2,3545 & 0,7329 & -1,0211 & -0,9657 & -1,1244 & -0,5343 & -0,20891 \\ . & . & 5,3666 & -1,7329 & 2,2668 & -0,53426 & -3,2668 & 1,5343 \\ . & . & . & 3,6878 & -0,03426 & -0,20891 & 1,5343 & -2,4578 \\ . & . & . & . & 6,9663 & 0,5685 & -6,4663 & 0,43147 \\ . & . & . & . & . & 3,5845 & 0,93147 & -2,2512 \\ . & . & . & . & . & . & 7,4663 & -1,9315 \\ . & . & . & . & . & . & . & 4,9178 \end{bmatrix} \cdot 10^5 \frac{\text{N}}{\text{mm}}.$$

(12.163)

The vector of forces can be constructed in a similar way to that shown in the triangle element:

$$\underline{F}_{p1}^T = \frac{l_{14}v}{2}[p_x \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad p_x \quad 0] = [3\sqrt{10} \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 3\sqrt{10} \quad 0]\text{N},$$

(12.164)

$$\underline{F}_{p2}^T = \frac{l_{34}v}{2}[0 \quad 0 \quad 0 \quad 0 \quad 0 \quad p_y \quad 0 \quad p_y] = [0 \quad 0 \quad 0 \quad 0 \quad 0 \quad -1,5 \quad 0 \quad -1,5]\text{N}.$$

We consider the reactions as concentrated forces in the kinematically constrained nodes:

$$\underline{F}_c^T = [F_{x1} \quad F_{y1} \quad F_{x2} \quad F_{y2} \quad F_{x3} \quad F_{y3} \quad F_{x4} \quad F_{y4}].$$ (12.165)

Considering the fact that the surfaces are frictionless and that at node 4 there is no external force, we have $F_{x1} = F_{y3} = F_{x4} = F_{y4} = 0$, which leads to:

$$\underline{F}_c^T = [0 \quad F_{y1} \quad F_{x2} \quad F_{y2} \quad F_{x3} \quad 0 \quad 0 \quad 0].$$ (12.166)

The structural force vector becomes:

$$\underline{F} = \underline{F}_{p1} + \underline{F}_{p2} + \underline{F}_c.$$ (12.167)

The construction of the finite element equilibrium equation results in:

$$\begin{bmatrix} 4,8666 & 0,76713 & -4,3666 & 0,23287 & -2,7668 & -0,9657 & 2,2668 & -0,03426 \\ . & 2,3545 & 0,7329 & -1,0211 & -0,9657 & -1,1244 & -0,5343 & -0,20891 \\ . & . & 5,3666 & -1,7329 & 2,2668 & -0,53426 & -3,2668 & 1,5343 \\ . & . & . & 3,6878 & -0,03426 & -0,20891 & 1,5343 & -2,4578 \\ . & . & . & . & 6,9663 & 0,5685 & -6,4663 & 0,43147 \\ . & . & . & . & . & 3,5845 & 0,93147 & -2,2512 \\ . & . & . & . & . & . & 7,4663 & -1,9315 \\ . & . & . & . & . & . & . & 4,9178 \end{bmatrix} \cdot 10^5 \begin{bmatrix} u_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix} = \begin{bmatrix} 3\sqrt{10} \\ F_{y1} \\ F_{x2} \\ F_{y2} \\ F_{x3} \\ -1,5 \\ 3\sqrt{10} \\ -1.5 \end{bmatrix}.$$

(12.168)

In the stiffness matrix we eliminate those rows and columns, for which the corresponding displacement component is a prescribed (here constrained) value. This way we obtain the so-called condensed stiffness matrix, which is used to expand the system of equations, of which solutions are the nodal displacements:

$$\begin{bmatrix} 4,8666 & -0,9657 & 2,2668 & -0,03426 \\ . & 3,5845 & 0,93147 & -2,2512 \\ . & . & 7,4663 & -1,9315 \\ . & . & . & 4,9178 \end{bmatrix} \cdot 10^5 \begin{bmatrix} u_1 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix} = \begin{bmatrix} 3\sqrt{10} \\ -1,5 \\ 3\sqrt{10} \\ -1,5 \end{bmatrix}.$$

(12.169)

The solutions are:

$$u_1 = 1,5078 \cdot 10^{-5} \text{ mm}, \quad v_3 = -0,29199 \cdot 10^{-5} \text{ mm}, \quad (12.170)$$

$$u_4 = 0,822016 \cdot 10^{-5} \text{ mm}, \quad v_4 = -0,10532 \cdot 10^{-5} \text{ mm}.$$

Then, the reactions are calculated by the 2nd, 3rd, 4th and 5th component equations of Eq. (12.168). The solutions are:

$$F_{y1} = 1,0678 \text{ N}, F_{x2} = -9,27494 \text{ N}, \quad (12.171)$$

$$F_{y2} = 1,93216 \text{ N}, F_{x3} = -9,6987 \text{ N}.$$

The strain and stress components of the element can be expressed in parametric form (as the function of ξ and η) from Eqs.(12.95) and (12.105). Taking the coordinates of the corresponding node back, the strain and stress components can be calculated. The example above was verified by the finite element code ANSYS 12.

12.8   Quadratic isoparametric quadrilateral

The advanced version of the linear quadrilateral is the quadratic quadrilateral, in which the curves of the element sides as well as the displacements are approximated by a second order function of the ξ and η coordinates [2,7]. On each element edge we provide a midside node, as it is shown in Fig.12.15, implying 8 nodes and 8 unknown coefficients in the approximate function of e.g. the x coordinate:

$$x(\xi,\eta) = a_0 + a_1\xi, + a_2\eta + a_3\xi\eta + a_4\xi^2 + a_5\eta^2 + a_6\xi^2\eta + a_7\xi\eta^2.$$

(12.172)

Using the nodal conditions we can derive the interpolation functions of the quadratic element in a similar way to that shown in the four node quadrilateral. The interpolation functions become:

$$N_1(\xi,\eta) = -\frac{1}{4}(1-\xi)(1-\eta)(1+\xi+\eta) \,, \quad N_2(\xi,\eta) = \frac{1}{2}(1-\xi^2)(1-\eta) \,,$$

$$(12.173)$$

$$N_3(\xi,\eta) = -\frac{1}{4}(1+\xi)(1-\eta)(1-\xi+\eta) \,, \quad N_4(\xi,\eta) = \frac{1}{2}(1+\xi)(1-\eta^2) \,,$$

$$N_5(\xi,\eta) = -\frac{1}{4}(1+\xi)(1+\eta)(1-\eta-\xi) \,, \quad N_6(\xi,\eta) = \frac{1}{2}(1-\xi^2)(1+\eta) \,,$$

$$N_7(\xi,\eta) = -\frac{1}{4}(1-\xi)(1+\eta)(1+\xi-\eta) \,, \quad N_8(\xi,\eta) = \frac{1}{2}(1-\xi)(1-\eta^2) \,.$$

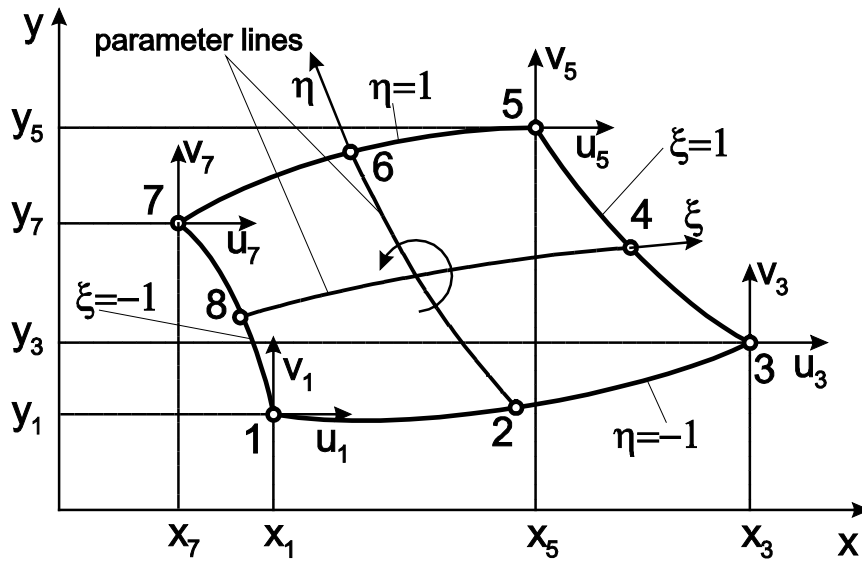*Fig.12.15. Quadratic isoparametric quadrilateral.*

The interpolation functions can be formulated also in compact from:

$$N_i = \frac{1}{4}(1+\xi\xi_i)(1+\eta\eta_i)(\xi\xi_i+\eta\eta_i-1) \,, \; i = 1, 3, 5, 7, \quad (12.174)$$

$$N_i = \frac{1}{2}\xi_i^2(1+\xi\xi_i)(1-\eta^2) + \frac{1}{2}\eta_i^2(1+\eta\eta_i)(1-\xi^2) \,, \; i = 2, 4, 6, 8.$$

where ξ_i and η_i are the coordinates of the nodes. Fig.12.16 shows the function plot of the interpolation functions N_5 and N_8.
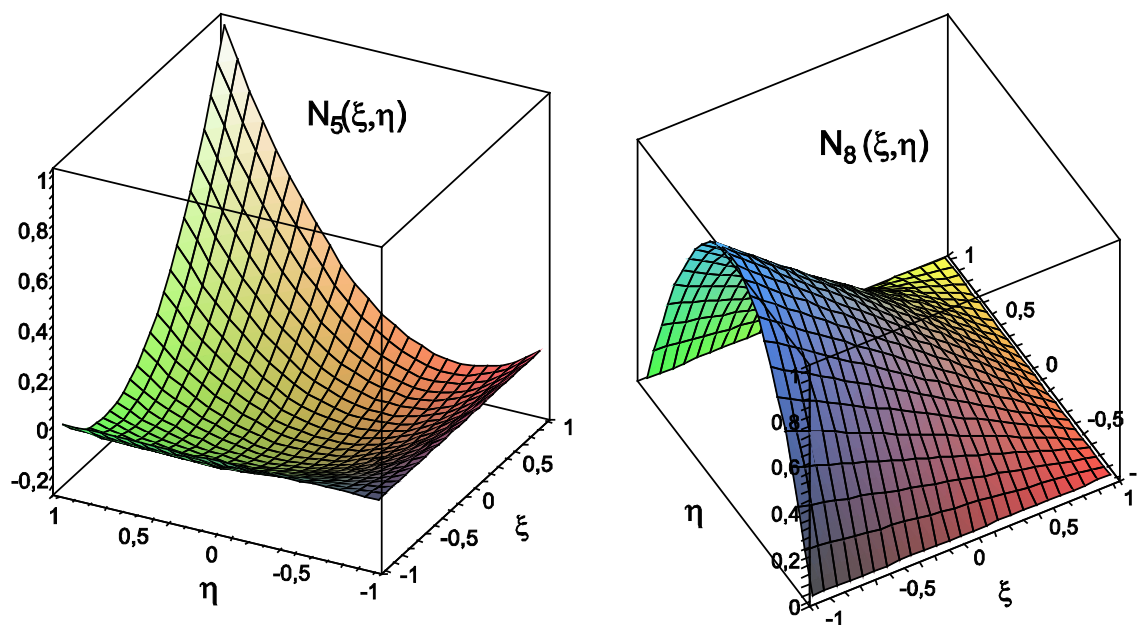
Fig.12.16. interpolation functions of the quadratic isoparametric quadrilateral.

The value of N_i corresponding to the ith node is equal to unity, in the other nodes it is zero. The calculation of the stiffness matrix and the vector of forces can be made in the same way as that shown in the quadrilateral with straight edges. The Jacobi determinant and the Gaussian quadrature is equally required.

12.9 Bibliography

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