11. INTRODUCTION TO PLANE PROBLEMS SUBJECT. APPLICATION OF PLANE STRESS, PLANE STRAIN AND REVOLUTION SYMMETRIC (AXISYMMETRIC) MODELS

Author: Dr. András Szekrényes

11. INTRODUCTION TO PLANE PROBLEMS SUBJECT. APPLICATION OF PLANE STRESS, PLANE STRAIN AND REVOLUTION SYMMETRIC (AXISYMMETRIC) MODELS

11.1 Basic types of plane problems

In the case of plane problems we have two-dimensional or two-variable problems; the basic equations of elasticity can be significantly simplified compared to spatial problems. There are two major categories of plane problems [1]:

- plane stress – a thin structure with constant thickness under in-plane loading, (Fig.11.1a),
- plane strain – a long structure with constant cross section under constant loads along the length (Fig.11.1b).

We note that the generalized plane stress state belongs also to the two-variable problems, if we relate the mechanical quantities to their average values.

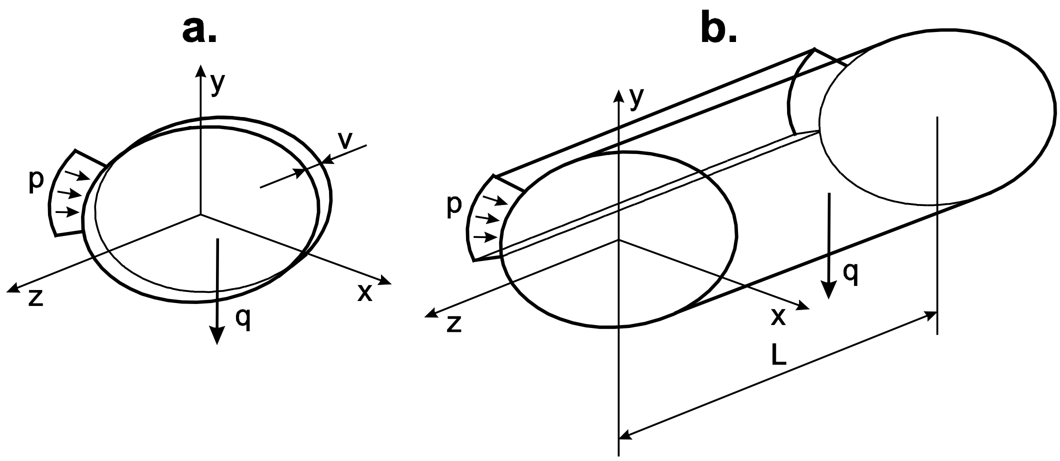

Fig.11.1. Demonstration of plane stress (a) and plane strain (b) states.

For plane problems the displacement vector field is the function of *x* and *y* only:

$$\underline{u} = \underline{u}(x, y) = \begin{bmatrix} u(x, y) \\ v(x, y) \end{bmatrix}. \qquad (11.1)$$

Consequently, even the strain and stress fields depend upon *x* and *y*:

$$\underline{\underline{\varepsilon}} = \underline{\underline{\varepsilon}}(x, y), \ \underline{\underline{\sigma}} = \underline{\underline{\sigma}}(x, y). \qquad (11.2)$$

In the followings we develop the relationship among the former mechanical quantities.

11.2 Equilibrium equation, displacement and deformation

The equilibrium equation represents the internal equilibrium of a differential plane element. Based on Fig.11.2 it is possible to express the equilibrium of the forces in directions *x* and *y* as [1,2]:

$$(\sigma_x + d\sigma_x)dy - \sigma_x dy + (\tau_{yx} + d\tau_{yx})dx - d\tau_{yx})dx + q_x dxdy = 0,$$

$$(11.3)$$

$$(\sigma_y + d\sigma_y)dx - \sigma_y dx + (\tau_{xy} + d\tau_{xy})dy - d\tau_{xy})dy + q_y dxdy = 0,$$

where σ is the normal, τ is the shear stress, q_x and q_y are the components of density vector of volume forces. The simplification of Eq.(11.3) leads to the following equations:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + q_x = 0\,, \quad \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + q_y = 0\,. \qquad (11.4)$$

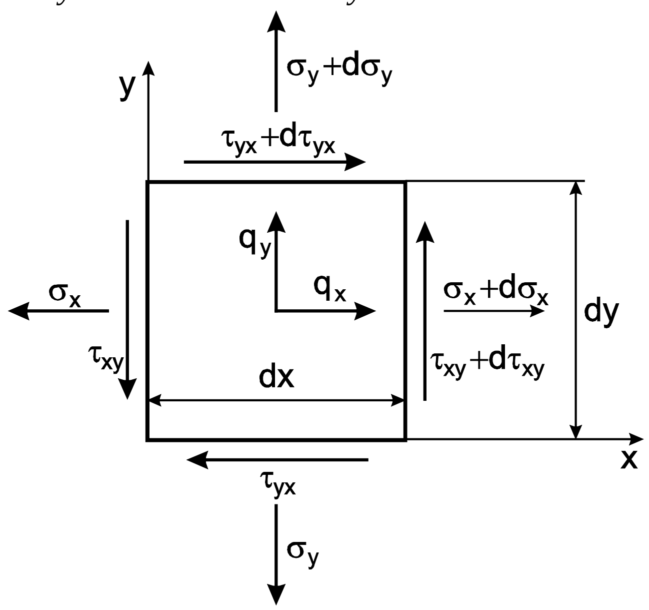

Fig.11.2. Equilibrium of a differential plane element.

The equilibrium equation can be formulated also in vector form [1,2]:

$$\underline{\underline{\sigma}} \cdot \nabla + \underline{q} = \underline{0}\,, \qquad (11.5)$$

where $\underline{q} = \underline{q}(x,y)$ is the density vector of volume forces, ∇ is the Hamiltonian differential operator (vector operator) in two dimensions:

$$\nabla = \frac{\partial}{\partial x}\underline{i} + \frac{\partial}{\partial y}\underline{j}\,. \qquad (11.6)$$

In order to establish the relationship between the strain and displacement fields we investigate the displacement and deformation of some points of the differential plane element depicted in Fig.11.3. The normal and shear strains in direction x of distance AB, and in direction y of distance AD of the element are:

$$\varepsilon_x = \frac{A'B' - AB}{AB} = \frac{A'B' - dx}{dx}\,, \varepsilon_y = \frac{A'D' - AD}{AD} = \frac{A'D' - dy}{dy}\,, \gamma_{yx} = \frac{\pi}{2} - \beta = \theta + \lambda\,. \qquad (11.7)$$

By the help of the figure we can write the following:

$$(A'B')^2 = [dx(1+\varepsilon_x)]^2 = (dx + \frac{\partial u}{\partial x}dx)^2 + (\frac{\partial v}{\partial x}dx)^2\,, \qquad (11.8)$$

from which we obtain:

$$1 + 2\varepsilon_x + \varepsilon_x^2 = 1 + 2\frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2. \qquad (11.9)$$

The expression above is applicable to calculate the normal strain in direction x in the case of the so-called large displacement. After all, within the scope of elasticity, in most of the cases we obtain reasonably accurate results by the linearization of the expression above.

The normal strain in direction *y* is derived similarly. Neglecting the higher order terms we obtain the linearized formulae:

$$\varepsilon_x = \frac{\partial u}{\partial x},\ \varepsilon_y = \frac{\partial v}{\partial y}. \qquad (11.10)$$

Utilizing Fig.11. 3 we calculate the angle denoted by θ:

$$\theta = \frac{(\partial v/\partial x)dx}{dx + (\partial u/\partial x)dx}. \qquad (11.11)$$

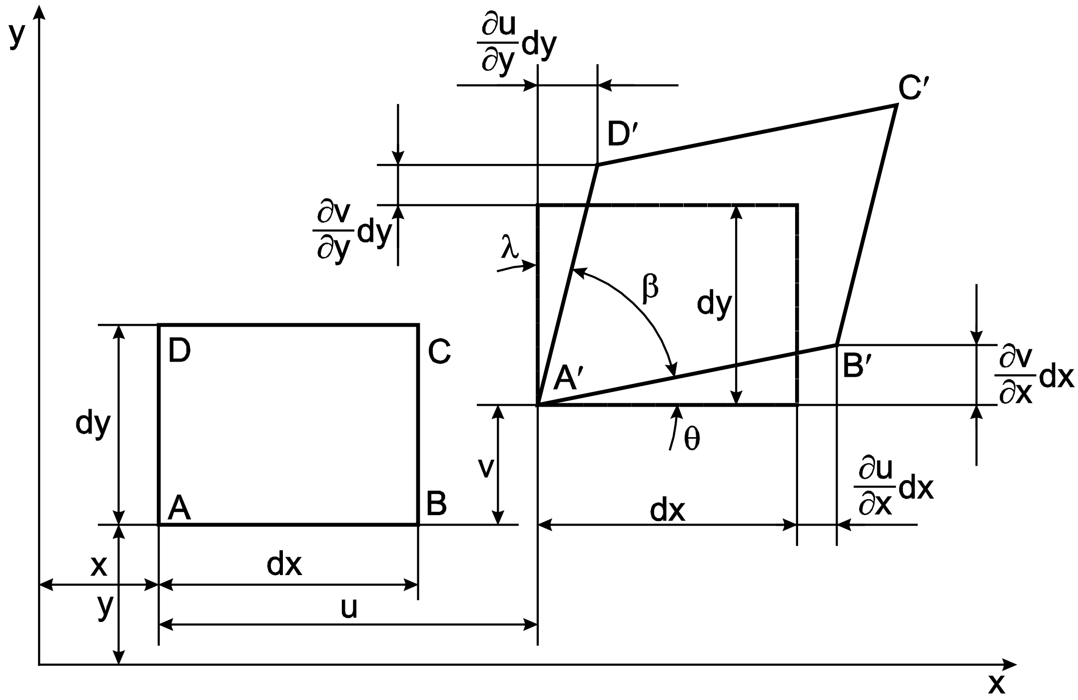

Fig.11.3. Displacement and deformation of a differential plane element.

Assuming that there are only small angles, we can write:

$$\theta = \frac{\partial v}{\partial x},\ \lambda = \frac{\partial u}{\partial y}. \qquad (11.12)$$

Based on Eq.(11.7) we obtain:

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}. \qquad (11.13)$$

We obtain the so-called strain-displacement equation by summarizing Eqs.(11.10) and (11.13) in tensorial form. The strain-displacement equation is valid also for spatial problems [1,2]:

$$\underline{\underline{\varepsilon}} = \frac{1}{2}(\underline{u} \circ \nabla + \nabla \circ \underline{u}), \qquad (11.14)$$

where the circle means dyadic product.

11.3 Constitutive equations

The material behavior, in other words the stress-strain relationship of a homogeneous, linear elastic, isotropic body is given by Hooke’s law [3]:

$$\underline{\underline{\varepsilon}} = \frac{1}{2G}\left[\underline{\underline{\sigma}} - \frac{\nu}{1+\nu}\sigma_I \underline{\underline{E}}\right], \underline{\underline{\sigma}} = 2G\left[\underline{\underline{\varepsilon}} + \frac{\nu}{1-2\nu}\varepsilon_I \underline{\underline{E}}\right], \qquad (11.15)$$

where ν is Poisson's ratio, E is the modulus of elasticity, $G = E/(2(1+\nu))$ is the shear modulus, $\underline{\underline{E}}$ is the identity tensor, σ_I and ε_I are the first scalar invariants, respectively.

11.3.1 Plane stress state

The stress components under plane stress state are:

$$\sigma_x = \sigma_x(x,y),\ \sigma_y = \sigma_y(x,y),\ \tau_{xy} = \tau_{xy}(x,y) \text{ and } \tau_{xz} = \tau_{yz} = \sigma_z = 0, \qquad (11.16)$$

i.e. the normal stress perpendicular to the *x-y* plane and the shear stresses acting on the plane with outward normal in direction *z* are zero. The stress and strain tensors have the following forms:

$$\underline{\underline{\sigma}} = \begin{bmatrix} \sigma_x & \tau_{xy} & 0 \\ \tau_{xy} & \sigma_y & 0 \\ 0 & 0 & 0 \end{bmatrix},\ \underline{\underline{\varepsilon}} = \begin{bmatrix} \varepsilon_x & 1/2\cdot\gamma_{xy} & 0 \\ 1/2\cdot\gamma_{xy} & \sigma_y & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix}. \qquad (11.17)$$

From the first of Eq.(11.15) we obtain:

$$\varepsilon_x = \frac{1+\nu}{E}\left[\sigma_x - \frac{\nu}{1+\nu}(\sigma_x+\sigma_y)\right] = \frac{1}{E}(\sigma_x - \nu\sigma_y), \qquad (11.18)$$

$$\varepsilon_y = \frac{1+\nu}{E}\left[\sigma_y - \frac{\nu}{1+\nu}(\sigma_x+\sigma_y)\right] = \frac{1}{E}(\sigma_y - \nu\sigma_x),\ \gamma_{xy} = \frac{2(1+\nu)}{E}\tau_{xy}.$$

The normal strain in direction *z* is:

$$\varepsilon_z = \frac{1+\nu}{E}\left[-\frac{\nu}{1+\nu}(\sigma_x+\sigma_y)\right] = -\frac{\nu}{E}(\sigma_x+\sigma_y) = -\frac{\nu}{1-\nu}(\varepsilon_x+\varepsilon_y). \qquad (11.19)$$

We note, that although ε_z is not included in the equations, it can always be calculated by using the strains in the other two directions. Using the former equations we can express even the stresses:

$$\sigma_x = \frac{E}{1-\nu^2}\left[\varepsilon_x + \nu\varepsilon_y)\right],\ \sigma_y = \frac{E}{1-\nu^2}\left[\varepsilon_y + \nu\varepsilon_x)\right], \tau_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy}. \qquad (11.20)$$

An alternative formulation of the stress-strain relationship is that we collect the components in vectors:

$$\underline{\varepsilon}^T = \left[\varepsilon_x, \varepsilon_y, \gamma_{xy}\right],\ \underline{\sigma}^T = \left[\sigma_x, \sigma_y, \tau_{xy}\right]. \qquad (11.21)$$

As a result, the relationship is established through a matrix:

$$\underline{\sigma} = \underline{\underline{C}}\underline{\varepsilon}. \qquad (11.22)$$

where $\underline{\underline{C}}$ is the constitutive matrix. On the base of Eqs.(11.20)-(11.22) under plane stress state matrix $\underline{\underline{C}}$ becomes:

$$\underline{\underline{C}}^{str} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}. \qquad (11.23)$$

The inverse and the determinant of the matrix is:

$$\underline{\underline{(C}}^{str})^{-1}=\frac{1}{E}\begin{bmatrix}1 & -\nu & 0\\ -\nu & 1 & 0\\ 0 & 0 & 2(1+\nu)\end{bmatrix},\det\underline{\underline{C}}^{str}=\frac{E^3}{2(1-\nu)(1+\nu)^2}. \qquad (11.24)$$

The latter form of the stress-strain relationship is applied in finite element calculations.

11.3.2 Plane strain state

Under plane strain state the condition is: $\varepsilon_z = 0$, i.e. the normal strain perpendicular to the *x*-*y* plane is zero. In this case the stress and strain tensors are:

$$\underline{\underline{\sigma}}=\begin{bmatrix}\sigma_x & \tau_{xy} & 0\\ \tau_{xy} & \sigma_y & 0\\ 0 & 0 & \sigma_z\end{bmatrix},\ \underline{\underline{\varepsilon}}=\begin{bmatrix}\varepsilon_x & 1/2\cdot\gamma_{xy} & 0\\ 1/2\cdot\gamma_{xy} & \varepsilon_y & 0\\ 0 & 0 & 0\end{bmatrix}. \qquad (11.25)$$

According to Hooke's law we obtain:

$$\sigma_x=\frac{E}{1+\nu}\left[\varepsilon_x+\frac{\nu}{1-2\nu}(\varepsilon_x+\varepsilon_y)\right],\sigma_y=\frac{E}{1+\nu}\left[\varepsilon_y+\frac{\nu}{1-2\nu}(\varepsilon_x+\varepsilon_y)\right], \qquad (11.26)$$

$$\tau_{xy}=\frac{E}{2(1+\nu)}\gamma_{xy},\qquad \sigma_z=\nu(\sigma_x+\sigma_y).$$

Developing the stress-strain relationship from $\underline{\sigma}=\underline{\underline{C}}\underline{\varepsilon}$ we get:

$$\underline{\underline{C}}^{stn}=\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & 0\\ \nu & 1-\nu & 0\\ 0 & 0 & \frac{1-2\nu}{2}\end{bmatrix}, \qquad (11.27)$$

and:

$$(\underline{\underline{C}}^{stn})^{-1}=\frac{1-\nu^2}{E}\begin{bmatrix}1 & -\frac{\nu}{1-\nu} & 0\\ -\frac{\nu}{1-\nu} & 1 & 0\\ 0 & 0 & \frac{2}{1-\nu}\end{bmatrix},\ \det\underline{\underline{C}}^{stn}=\frac{E^3}{2(1-2\nu)(1+\nu)^3}. \qquad (11.28)$$

11.4 Basic equations of plane elasticity

The number of unknowns in case of plane problems is always eight: σ_x, σ_y, τ_{xy}, ε_x, ε_y, γ_{xy}, *u* and *v*. Under plane stress ε_z, under plane strain σ_z component can always be calculated by the help of the components in directions *x* and *y*.

11.4.1 Compatibility equation

The combination of Eqs.(11.10) and (11.13) leads to the so-called compatibility equation [1,2]:

$$\frac{\partial^2\varepsilon_x}{\partial y^2}+\frac{\partial^2\varepsilon_y}{\partial x^2}=\frac{\partial^2\gamma_{yx}}{\partial x\partial y}. \qquad (11.29)$$

The equation above is equally true for plane stress and plane strain states. It is possible to formulate the compatibility equation in terms of stresses. Let us express Eq.(11.29) in terms of stresses for plane stress state by utilizing Eq.(11.19):

$$\frac{1}{E}\left(\frac{\partial^2\sigma_x}{\partial y^2}-\nu\frac{\partial^2\sigma_y}{\partial y^2}+\frac{\partial^2\sigma_y}{\partial x^2}-\nu\frac{\partial^2\sigma_x}{\partial x^2}\right)=\frac{1}{G}\frac{\partial^2\tau_{yx}}{\partial x\partial y}. \qquad (11.30)$$

We express the mixed derivative of the shear stress from Eq.(11.4):

$$\frac{\partial^2\tau_{xy}}{\partial x\partial y}=-\frac{1}{2}\left(\frac{\partial q_x}{\partial x}+\frac{\partial q_y}{\partial y}+\frac{\partial^2\sigma_x}{\partial x^2}+\frac{\partial^2\sigma_y}{\partial y^2}\right). \qquad (11.31)$$

The combination of the two former equations results in:

$$\nabla^2(\sigma_x+\sigma_y)=-(1+\nu)\left(\frac{\partial q_x}{\partial x}+\frac{\partial q_y}{\partial y}\right), \qquad (11.32)$$

where:

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}. \qquad (11.33)$$

In a similar way we can develop the following equation for plane strain state:

$$\nabla^2(\sigma_x+\sigma_y)=-\frac{1}{1-\nu}\left(\frac{\partial q_x}{\partial x}+\frac{\partial q_y}{\partial y}\right). \qquad (11.34)$$

It can be seen, that if there is no volume force, then the compatibility equation has the same form under plane stress as that under plane strain. In that case, when the force field is conservative, then a potential function, U exists, of which gradient gives the components of the density vector of volume force, i.e.:

$$q_x=\frac{\partial U}{\partial x} \text{ and } q_y=\frac{\partial U}{\partial y}. \qquad (11.35)$$

11.4.2 Airy's stress function

The equilibrium and the compatibility equations can be reduced to one equation by introducing the Airy's stress function. Let $\chi = \chi(x,y)$ be the Airy's stress function, which is defined in the following way [1,2]:

$$\sigma_x+U=\frac{\partial^2\chi}{\partial y^2},\ \sigma_y+U=\frac{\partial^2\chi}{\partial x^2},\ \tau_{xy}=-\frac{\partial^2\chi}{\partial x\partial y}. \qquad (11.36)$$

Taking them back into the equilibrium equations given by Eq.(11.4), it is seen that the equations are identically satisfied. The stress function can be derived for every stress field, which satisfies the equilibrium equations and the body force field is conservative. In terms of the stresses the compatibility equation given by Eq.(11.34) becomes:

$$\nabla^4\chi=(1-\nu)\nabla^2 U, \qquad (11.37)$$

where:

$$\nabla^4=\nabla^2(\nabla^2)=\frac{\partial^4}{\partial x^4}+2\frac{\partial^4}{\partial x^2\partial y^2}+\frac{\partial^4}{\partial y^4} \qquad (11.38)$$

is called the biharmonic operator. Eq.(11.37) is the governing field equation for plane stress problems in which the body forces are conservative. If a function $\chi = \chi(x,y)$ is found such that it satisfies Eq.(11.37) and the proper prescribed boundary conditions, then it represents the solution of the problem. The corresponding stresses and strains may be determined from Eqs.(11.36) and (11.19), respectively. If the body forces are constant, or if *U* is a harmonic function, then the governing equation is:

$$\nabla^4\chi = 0, \qquad (11.39)$$

which is a partial differential equation called biharmonic equation.

11.4.3 Navier's equation

Now let us formulate the governing equations in terms of displacement field for plane stress state! The combination of Eqs.(11.10), (11.13) and (11.19) provides the followings [1,2]:

$$\frac{\partial u}{\partial x} = \frac{1}{E}(\sigma_x - \nu\sigma_y), \; \frac{\partial v}{\partial y} = \frac{1}{E}(\sigma_y - \nu\sigma_x), \; \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{1}{G}\tau_{yx}. \qquad (11.40)$$

After a simple rearrangement we obtain:

$$\sigma_x = \frac{E}{1-\nu^2}\left[\frac{\partial u}{\partial x} + \nu\frac{\partial v}{\partial y}\right], \; \sigma_y = \frac{E}{1-\nu^2}\left[\frac{\partial v}{\partial y} + \nu\frac{\partial u}{\partial x}\right], \tau_{xy} = \frac{E}{2(1+\nu)}\left[\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right]. \qquad (11.41)$$

Substitution of the above stresses into the equilibrium equation given by Eq.(11.4) gives the Navier's equation:

$$G\nabla^2 u + \frac{E}{2(1-\nu)}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + q_x = 0, \qquad (11.42)$$

$$G\nabla^2 v + \frac{E}{2(1-\nu)}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + q_y = 0.$$

We can develop Navier's equation for plane strain state in a similar way, the result is:

$$G\nabla^2 u + \frac{E}{2(1+\nu)(1-2\nu)}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + q_x = 0, \qquad (11.43)$$

$$G\nabla^2 v + \frac{E}{2(1+\nu)(1-2\nu)}\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + q_y = 0.$$

Under plane stress state the first scalar invariant of the stress tensor is:

$$\sigma_I = \sigma_x + \sigma_y = \nabla^2\chi. \qquad (11.44)$$

11.4.4 Boundary value problems

It can be shown that for plates under symmetrically distributed external forces with respect to the plane *z* = 0, the exact solution satisfying all of the equilibrium and compatibility equations is [2]:

$$\chi = \chi_0 - \frac{1}{2}\frac{\nu}{1+\nu}(\nabla^2\chi_0)z^2, \qquad (11.45)$$

where:

$$\chi_0 = \chi_0(x, y), \qquad (11.46)$$

which satisfies

$$\nabla^4\chi_0 = 0. \qquad (11.47)$$

The second term in Eq.(11.45), however, depends on *z* and may be neglected for thin plates, in which case we have:

$$\nabla^4 \chi \cong \nabla^4 \chi_0 = 0 . \qquad (11.48)$$

That is, for thin plates, solutions by Eq.(11.48) very closely approximate the stress distributions by Eq.(11.45).

Let us summarize what kind of requirements should be met of plane stress state! The actual elastic body must be a thin plate, the two *z* surfaces of the plate must be free from load, the external forces can have only *x* and *y* components, the external forces should be distributed symmetrically with respect to the *x* and *y* axes.

The governing equation system of plane problems is a system of partial differential equations (equilibrium equation, strain-displacement equation and material law) with corresponding boundary conditions. The dynamic boundary condition is the relationship between the stress tensor and the vector of external load at certain points of the lateral boundary curve:

$$\underline{\underline{\sigma}}\,\underline{n} = \underline{p} , \qquad (11.49)$$

where $\underline{p}$ is traction vector of the corresponding boundary surface, $\underline{n}$ is the outward normal of the boundary surface or the outward normal of a certain part of it, which is parallel to the *x-y* plane. The kinematic boundary condition represents the imposed displacement of a point (or certain points):

$$\underline{u}(x_0, y_0) = \underline{u}_b , \qquad (11.50)$$

where $\underline{u}_b$ is the imposed displacements vector, x_0 and y_0 are the coordinates of the actual point. The system of governing partial differential equations together with relevant dynamic and kinematic boundary conditions built a boundary value problem.

We note that closed form solutions of the governing partial differential equations of plane problems with prescribed boundary conditions which occur in elasticity problems are very difficult to obtain directly, and they are frequently impossible to achieve. Because of this fact the inverse and semi-inverse methods may be attempted in the solution of certain problems [1]. In the inverse method we select a specific solution which satisfies the governing equations, and then search for the boundary conditions which can be satisfied by this solution, i.e., we have the solution first and then ask what problem it can solve. In the semi-inverse method, we assume a partial solution to a given problem. A partial solution consists of an assumed form for each dependent variable in terms of known and unknown functions. The assumed partial solution is then substituted into the original set of governing equations. As a result, these equations will be reduced to a set of simplified differential equations, which govern the remaining unknown functions. This simplified set of equations, together with proper boundary conditions, is then solved by direct methods.

11.5 Examples for plane stress

11.5.1 Determination of the traction on the boundaries of a square shape plate

For the square shape plate shown in Fig.11.4 we know the Airy's stress function in the *x-y* coordinate system [3]:

$$\chi(x, y) = \frac{p_0}{a^2}\left(\frac{1}{2}x^2 y^2 - \frac{1}{6}y^4\right). \qquad (11.51)$$

where p_0 is the intensity of the distributed line load. The body force is negligible; we assume that the plate is in plane stress state.

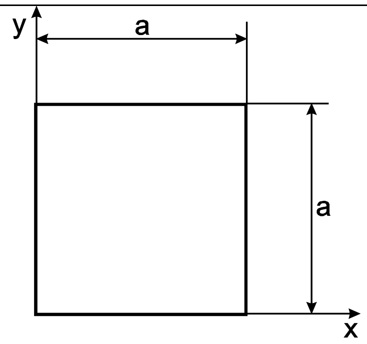

Fig.11.4. Square shape plate under plane stress.

What kind of system of forces loads the boundary curves of the plate?

First, we produce the stress field:

$$\sigma_x = \frac{\partial^2 \chi}{\partial y^2} = \frac{p_0}{a^2}(x^2 - 2y^2),\ \sigma_y = \frac{\partial^2 \chi}{\partial x^2} = \frac{p_0}{a^2} y^2, \qquad (11.52)$$

$$\tau_{xy} = \tau_{yx} = -\frac{\partial^2 \chi}{\partial x \partial y} = -\frac{p_0}{a^2} 2xy, \sigma_z = 0.$$

The traction vectors can be calculated by the help of the definition of dynamic boundary condition and the localization of it into the boundary curves. Therefore, we need the outward normal of each boundary curve:

boundary curve	***constant coordinate***	***outward normal (n)***
1	*x = 0*	-*i*
2	*x =a*	*i*
3	*y =0*	-*j*
4	*y = a*	*j*

Furthermore, we need Eqs.(11.49) and (11.52). We obtain the traction vectors by applying the former equations:

$$\underline{p}_1 = -\underline{\underline{\sigma}}\,\underline{i} = \begin{bmatrix} \sigma_x(0,y) & \tau_{xy}(0,y) & 0 \\ \tau_{xy}(0,y) & \sigma_y(0,y) & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -\sigma_x(0,y) \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{p_0}{a^2} 2y^2 \\ 0 \\ 0 \end{bmatrix}, \qquad (11.53)$$

$$\underline{p}_2 = \underline{\underline{\sigma}}\,\underline{i} = \begin{bmatrix} \sigma_x(a,y) & \tau_{xy}(a,y) & 0 \\ \tau_{xy}(a,y) & \sigma_y(a,y) & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \sigma_x(a,y) \\ \tau_{xy}(a,y) \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{p_0}{a^2}(a^2 - 2y^2) \\ -\frac{p_0}{a} 2y \\ 0 \end{bmatrix},$$

$$\underline{p}_3 = -\underline{\underline{\sigma}}\,\underline{j} = \begin{bmatrix} \sigma_x(x,0) & \tau_{xy}(x,0) & 0 \\ \tau_{xy}(x,0) & \sigma_y(x,0) & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\tau_{xy}(x,0) \\ -\sigma_y(x,0) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\underline{p}_4 = \underline{\underline{\sigma}}\,\underline{j} = \begin{bmatrix} \sigma_x(x,a) & \tau_{xy}(x,a) & 0 \\ \tau_{xy}(x,a) & \sigma_y(x,a) & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \tau_{xy}(x,a) \\ \sigma_y(x,a) \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{p_0}{a}2x \\ p_0 \\ 0 \end{bmatrix}.$$

The system of forces acting on the boundary curves can be obtained by plotting the components of the vectors above along the corresponding boundary curve. Fig.11.5 demonstrates the function plots, where Fig.11.5a depicts the loads in the normal direction (perpendicularly to the boundary curve), Fig.11.5b represents the tangential (with respect to the boundary curve) stress distributions.

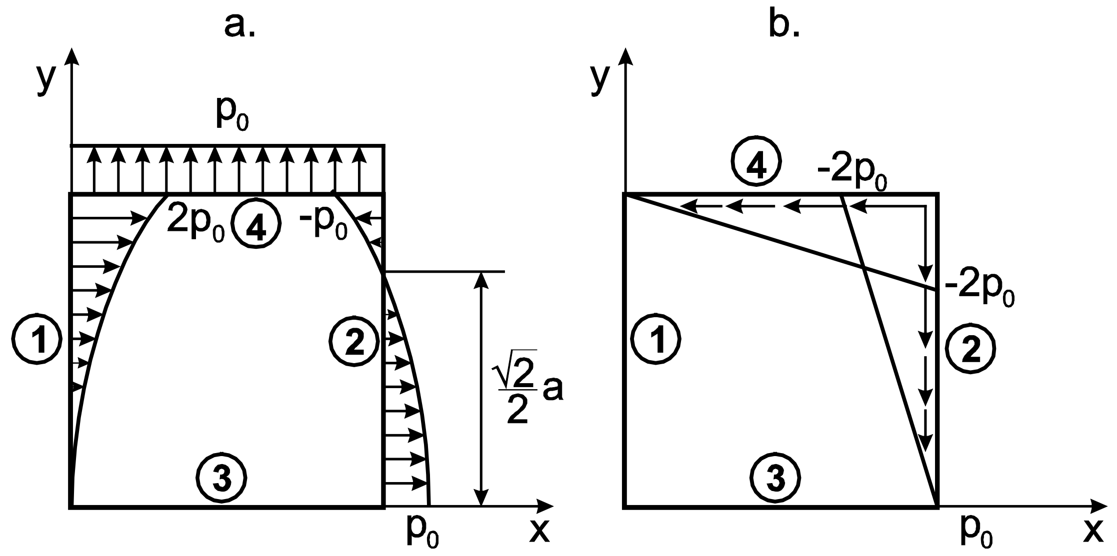

Fig.11.5. Normal (a) and tangential (b) loads on the boundary curves of a square plate under plane stress state.

11.5.2 Analysis of a tangentially loaded plate

For the plate shown in Fig.11.6 with dimensions of $2h{\cdot}L$ the body force is negligible, we can assume plane stress state. The form of the Airy's stress function for the load shown in Fig. 11.6 is [3]:

$$\chi(x,y) = \frac{p_t}{4}\left(xy - \frac{xy^2}{h} - \frac{xy^3}{h^2} + \frac{Ly^2}{h} + \frac{Ly^3}{h^2} \right). \qquad (11.54)$$

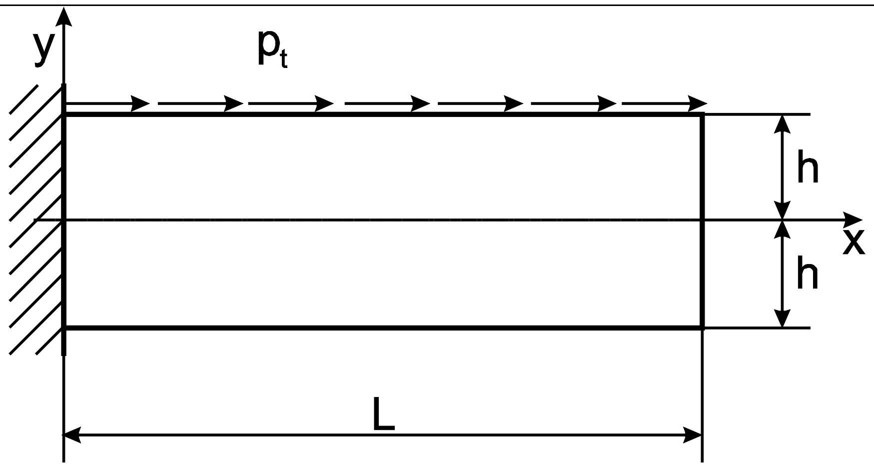

Fig.11.6. Thin plate loaded by tangentially distributed force under plane stress state.

Is the given $\chi(x,y)$ function an exact solution of the problem above?

A function, $\chi(x,y)$ is the exact solution of the problem if it satisfies the governing partial differential equation of plane problems and the dynamic boundary conditions. Based on the given $\chi(x,y)$ function it is seen that Eq.(11.39) is satisfied in this case, since the governing equation is a fourth order partial differential equation, while the functions contains to a maximum the third power of y. Let us investigate the dynamic boundary conditions! Similarly to the former example we calculate the stress field first:

$$\sigma_x = \frac{\partial^2 \chi}{\partial y^2} = \frac{1}{2} p_t \left(\frac{L-x}{h} + \frac{3(L-x)}{h^2} y \right),\ \sigma_y = \frac{\partial^2 \chi}{\partial x^2} = 0, \quad (11.55)$$

$$\tau_{xy} = \tau_{yx} = -\frac{\partial^2 \chi}{\partial x \partial y} = -\frac{1}{4} p_t \left(1 - \frac{2y}{h} - \frac{3y^2}{h^2} \right),\ \sigma_z = 0.$$

Based on the stresses, the loads on the boundary curves are:

$$x = L:\ \sigma_x = 0,\ \tau_{yx} = -\frac{1}{4} p_t \left(1 - \frac{2y}{h} - \frac{3y^2}{h^2} \right), \quad (11.56)$$

$$y = h:\ \sigma_y = 0,\ \tau_{xy} = p_t,$$

$$y = -h:\ \sigma_y = 0,\ \tau_{xy} = 0.$$

Finally, independently of Eq.(11.56) we formulate the dynamic boundary conditions by the help of Fig. 11.6. In accordance with the dynamic boundary condition definition the stress components acting on the actual boundary curve should be equal to the corresponding (normal or tangential) components of the traction vector. That means:

$$x = L:\ \sigma_x = 0,\ \tau_{yx} = 0, \quad (11.57)$$

$$y = h:\ \sigma_y = 0,\ \tau_{xy} = p_t,$$

$$y = -h:\ \sigma_y = 0,\ \tau_{xy} = 0.$$

Comparing the boundary conditions to the boundary loads it is seen, that one condition is not satisfied, namely the shear stress, τ_{yx} on the boundary at $x = L$ is not zero, i.e. one of the conditions is violated. Nevertheless, there are two points, where in accordance with the formula:

$$-\frac{1}{4}p_t\left(1-\frac{2y}{h}-\frac{3y^2}{h^2}\right)=0\Rightarrow 1-\frac{2y}{h}-\frac{3y^2}{h^2}=0\Rightarrow 3y^2+2yh-h^2=0, \tag{11.58}$$

with solutions of $y_1 = 1/3 \cdot h$ and $y_2 = -h$, i.e. at two points the dynamic boundary condition is satisfied. As a final word, the given $\chi(x,y)$ function is not the exact solution of the problem in Fig.11.6, because one of the dynamic boundary conditions is violated. After all, it is acceptable, since together with Eq.(11.39) the given function satisfies eight from the total ten conditions. It should be highlighted, that the boundary at $x = 0$ is a fixed boundary, which involves kinematic boundary condition, that is why we did not investigated this boundary curve in the example.

11.6 The governing equation of plane problems using polar coordinates

The solutions of many elasticity problems are conveniently formulated in terms of cylindrical coordinates. On the base of Fig.11.7 we have the functional relations [1]:

$$x = r\cos\vartheta,\ y = r\sin\vartheta, \tag{11.59}$$

$$\vartheta = \arctan\frac{y}{x},\ r^2 = x^2 + y^2.$$

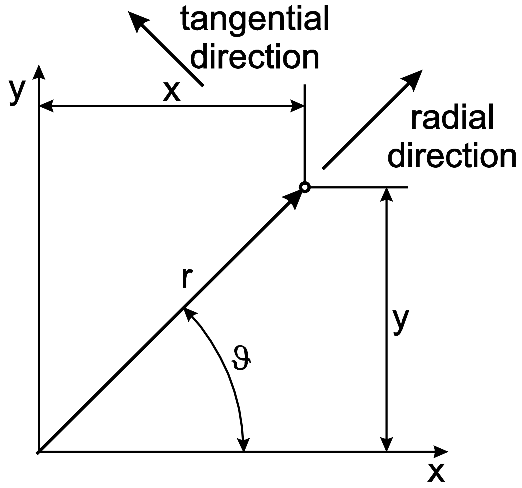

Fig.11.7. Parameters of a polar coordinate system.

The derivatives of the polar coordinates with respect to x and y using the last of Eq.(11.59) are:

$$\frac{\partial r}{\partial x}=\frac{x}{r}=\cos\vartheta,\ \frac{\partial r}{\partial y}=\frac{y}{r}=\sin\vartheta, \tag{11.60}$$

$$\frac{\partial\vartheta}{\partial x}=-\frac{y}{r^2}=-\frac{\sin\vartheta}{r},\ \frac{\partial\vartheta}{\partial y}=\frac{x}{r^2}=\frac{\cos\vartheta}{r}.$$

Again, the derivatives with respect to x and y can be formulated based on the chain rule:

$$\frac{\partial}{\partial x}=\frac{\partial r}{\partial x}\frac{\partial}{\partial r}+\frac{\partial\vartheta}{\partial x}\frac{\partial}{\partial\vartheta}=\cos\vartheta\frac{\partial}{\partial r}-\frac{\sin\vartheta}{r}\frac{\partial}{\partial\vartheta}, \tag{11.61}$$

$$\frac{\partial}{\partial y}=\frac{\partial r}{\partial y}\frac{\partial}{\partial r}+\frac{\partial\vartheta}{\partial y}\frac{\partial}{\partial\vartheta}=\sin\vartheta\frac{\partial}{\partial r}+\frac{\cos\vartheta}{r}\frac{\partial}{\partial\vartheta}$$

To derive the governing equations in terms of polar coordinates we incorporate the stress transformation expressions [1]. The normal and shear stresses are transformed to a coordinate system given by rotation about axis *z* by an angle ϑ:

$$\sigma_n = \underline{n}^T \underline{\underline{\sigma}}\,\underline{n}, \quad \tau_{mn} = \underline{m}^T \underline{\underline{\sigma}}\,\underline{n}, \tag{11.62}$$

where:

$$\underline{n}^T = [\cos\vartheta \quad \sin\vartheta \quad 0], \quad \underline{m}^T = [-\sin\vartheta \quad \cos\vartheta \quad 0], \tag{11.63}$$

which leads to:

$$\begin{aligned} \sigma_x &= \sigma_r \cos^2\vartheta + \sigma_\vartheta \sin^2\vartheta + \tau_{r\vartheta}\sin 2\vartheta, \\ \sigma_y &= \sigma_r \sin^2\vartheta + \sigma_\vartheta \cos^2\vartheta - \tau_{r\vartheta}\sin 2\vartheta, \\ \tau_{xy} &= (\sigma_\vartheta - \sigma_r)\sin\vartheta\cos\vartheta + \tau_{r\vartheta}(\cos^2\vartheta - \sin^2\vartheta), \end{aligned} \tag{11.64}$$

The strain components (ε_x, ε_y, γ_{xy}) can be transformed similarly. Taking Eq.(11.64) back into the equilibrium equations given by Eq.(11.4), moreover by assuming that there are also body forces, we have [1,2]:

$$\begin{aligned} \frac{\partial\sigma_r}{\partial r} + \frac{1}{r}\frac{\partial\tau_{r\vartheta}}{\partial\vartheta} + \frac{\sigma_r - \sigma_\vartheta}{r} + q_r = 0, \\ \frac{1}{r}\frac{\partial\sigma_\vartheta}{\partial\vartheta} + \frac{\partial\tau_{r\vartheta}}{\partial r} + \frac{2\tau_{r\vartheta}}{r} + q_\vartheta = 0, \end{aligned} \tag{11.65}$$

where the former is the equation in the radial, the latter is the equation in the tangential direction. By a similar technique, the strain-displacement equations may be transformed into:

$$\varepsilon_r = \frac{\partial u_r}{\partial r}, \quad \varepsilon_\vartheta = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\vartheta}{\partial\vartheta}, \quad \gamma_{r\vartheta} = \frac{1}{r}\frac{\partial u_r}{\partial\vartheta} + \frac{\partial u_\vartheta}{\partial r} - \frac{u_\vartheta}{r}, \tag{11.66}$$

where u_r and u_ϑ are the radial and tangential displacements. Eliminating the displacement components we obtain the compatibility equation:

$$\frac{\partial^2\varepsilon_\vartheta}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\varepsilon_r}{\partial\vartheta^2} + \frac{2}{r}\frac{\partial\varepsilon_\vartheta}{\partial r} - \frac{1}{r}\frac{\partial\varepsilon_r}{\partial r} = \frac{1}{r}\frac{\partial^2\gamma_{r\vartheta}}{\partial r\partial\vartheta} + \frac{1}{r^2}\frac{\partial\gamma_{r\vartheta}}{\partial\vartheta}. \tag{11.67}$$

In the case of Hooke's law there is no need to perform the transformation, due to the fact that the polar coordinate system is an orthogonal system. Therefore, e.g. in Eq.(11.20) referring to plane stress state, we have to substitute *x* by *r*, and *y* by ϑ:

$$\begin{aligned} \varepsilon_r = \frac{1}{E}(\sigma_r - \nu\sigma_\vartheta), \quad \varepsilon_\vartheta = \frac{1}{E}(\sigma_\vartheta - \nu\sigma_r), \quad \gamma_{r\vartheta} = \frac{2(1+\nu)}{E}\tau_{r\vartheta}, \\ \sigma_r = \frac{E}{1-\nu^2}[\varepsilon_r + \nu\varepsilon_\vartheta], \sigma_\vartheta = \frac{E}{1-\nu^2}[\varepsilon_\vartheta + \nu\varepsilon_r], \tau_{r\vartheta} = \frac{E}{2(1+\nu)}\gamma_{r\vartheta}. \end{aligned} \tag{11.68}$$

The formulation incorporating plane strain state based on Eq.(11.26) leads to:

$$\begin{aligned} \varepsilon_r = \frac{1-\nu^2}{E}(\sigma_r - \frac{\nu}{1-\nu}\sigma_\vartheta), \varepsilon_\vartheta = \frac{1-\nu^2}{E}(\sigma_\vartheta - \frac{\nu}{1-\nu}\sigma_r), \quad \gamma_{r\vartheta} = \frac{2(1+\nu)}{E}\tau_{r\vartheta}, \\ \sigma_r = \frac{E}{1+\nu}\left[\varepsilon_r + \frac{\nu}{1-2\nu}(\varepsilon_r + \varepsilon_\vartheta)\right], \sigma_\vartheta = \frac{E}{1+\nu}\left[\varepsilon_\vartheta + \frac{\nu}{1-2\nu}(\varepsilon_r + \varepsilon_\vartheta)\right], \\ \tau_{r\vartheta} = \frac{E}{2(1+\nu)}\gamma_{r\vartheta}. \end{aligned} \tag{11.69}$$

The first scalar invariant of the strain tensor (plane dilatation) under plane strain state is:

$$\varepsilon_I = \varepsilon_r + \varepsilon_\vartheta = \frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\vartheta}{\partial \vartheta}. \qquad (11.70)$$

Substituting the stress and strain components into the equilibrium equation given by Eq.(11.65) (plane strain) and incorporating the first scalar invariant we obtain the Navier's equation in terms of polar coordinates [1,2]:

$$(\lambda + 2G)\frac{\partial \varepsilon_I}{\partial r} - \frac{2G}{r}\frac{\partial \omega}{\partial \vartheta} + q_r = 0, \qquad (11.71)$$

$$(\lambda + 2G)\frac{1}{r}\frac{\partial \varepsilon_I}{\partial \vartheta} + 2G\frac{\partial \omega}{\partial r} + q_\vartheta = 0,$$

where

$$\omega = \frac{1}{2r}\left(\frac{\partial(ru_\vartheta)}{\partial r} - \frac{\partial u_r}{\partial \vartheta}\right) \qquad (11.72)$$

is the rotation about axis z, λ is the Lamé-constant:

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}. \qquad (11.73)$$

The governing equation of plane problems in terms of polar coordinates can be formulated by using the Hamilton operator. Based on Eqs.(11.48) and (11.61) we get:

$$\nabla^4 \chi = \nabla^2 \nabla^2 \chi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \vartheta^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \vartheta^2}\right)\chi = 0. \qquad (11.74)$$

The stresses may be obtained by using the differential quotients given by Eq.(11.61) and the transformation expressions given by Eq.(11.64):

$$\sigma_r = \frac{1}{r}\frac{\partial \chi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \chi}{\partial \vartheta^2}, \sigma_\vartheta = \frac{\partial^2 \chi}{\partial r^2}, \ \tau_{r\vartheta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \chi}{\partial \vartheta}\right). \qquad (11.75)$$

The last three formulae are equally valid under plane stress and plane strain states. The equilibrium equations, strain-displacement relationship can also be formulated by using infinitesimal elements in polar coordinate system [1].

11.7 Axisymmetric plane problems

The use of polar coordinates is particularly convenient in the solution of revolution symmetric or in other words axisymmetric problems. In this case displacement field, stresses are independent of the angle coordinate (ϑ), consequently the derivates with respect to ϑ vanish everywhere. In accordance with Eq.(11.74) the governing equation of plane problems becomes:

$$\left(\frac{d^4}{dr^4} + \frac{2}{r}\frac{d^3}{dr^3} - \frac{1}{r^2}\frac{d^2}{dr^2} + \frac{1}{r^3}\frac{d}{dr}\right)\chi = 0. \qquad (11.76)$$

By introducing a new independent variable, ξ, this equation can be reduced to a differential equation with constant coefficients:

$$r = e^\xi. \qquad (11.77)$$

As a result, Eq.(11.76) becomes:

$$\left(\frac{d^4}{d\xi^4} - 4\frac{d^3}{d\xi^3} + 4\frac{d^2}{d\xi^2}\right)\chi = 0, \qquad (11.78)$$

for which the general solution is:

$$\chi = A\xi e^{2\xi} + Be^{2\xi} + C\xi + D. \tag{11.79}$$

Taking back e^{ξ} we have:

$$\chi = Ar^2 \ln r + Br^2 + C\ln r + D, \tag{11.80}$$

where *A*, *B*, *C* and *D* are constants. The stresses based on Eq.(11.75) are:

$$\sigma_r = \frac{1}{r}\frac{\partial \chi}{\partial r}, \sigma_\vartheta = \frac{\partial^2 \chi}{\partial r^2}, \tau_{r\vartheta} = 0. \tag{11.81}$$

Taking the solution function back we see that:

$$\sigma_r = 2A\ln r + \frac{C}{r^2} + A + 2B,\ \sigma_\vartheta = 2A\ln r - \frac{C}{r^2} + 3A + 2B, \tau_{r\vartheta} = 0. \tag{11.82}$$

11.7.1 Solid circular cylinder and thick-walled tube

Let us see some examples for the application of the equations and formulae above [1]! For a solid circular cylinder the stresses at $r = 0$ can not be infinitely high, therefore:

$$A = C = 0. \tag{11.83}$$

The stresses in a solid circular cylinder are:

$$\sigma_r = \sigma_\vartheta = 2B, \tau_{r\vartheta} = 0. \tag{11.84}$$

This is the solution of a circular cylinder loaded by external pressure with magnitude of 2*B* on the outer surface. In the case of a hollow circular cylinder or a thick-walled tube (Fig.11.8a) it is not sufficient to investigate only the dynamic boundary conditions, we need to impose also kinematic boundary conditions.

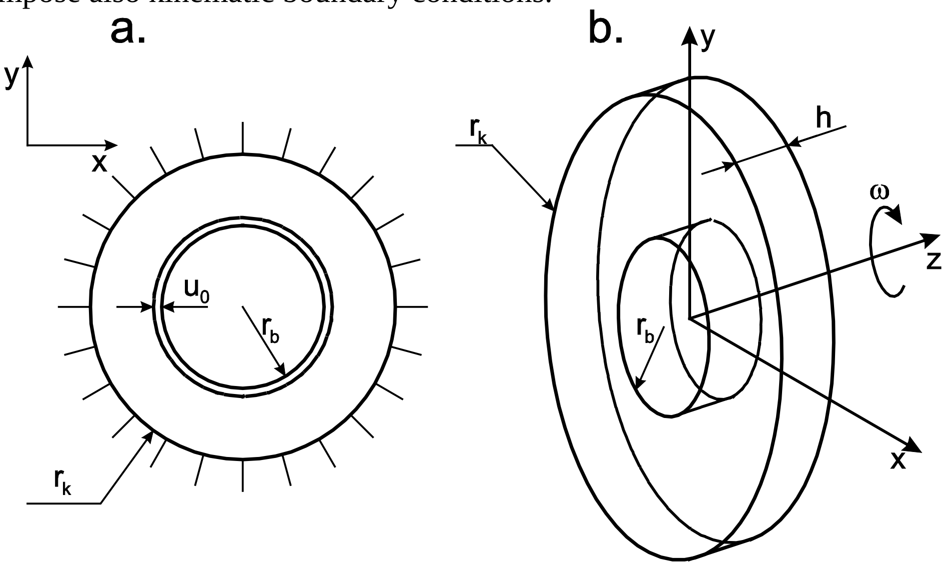

Fig.11.8. Hollow circular cylinder with imposed displacement at the inner boundary (a), thick-walled rotating disk (b).

The strain components by using Eq.(11.66) become:

$$\varepsilon_r = \frac{du_r}{dr}, \varepsilon_\vartheta = \frac{u_r}{r}, \gamma_{r\vartheta} = 0. \tag{11.85}$$

Using the stress-strain relationship given by Eq.(11.68) we obtain the equations below:

$$\frac{du_r}{dr} = K_1(\sigma_r - K_2\sigma_\vartheta)\,, \frac{u_r}{r} = K_1(\sigma_\vartheta - K_2\sigma_r)\,, \qquad (11.86)$$

where:

$$K_1 = \frac{1}{E},\ K_2 = \frac{\nu}{1-\nu}\,, \qquad (11.87)$$

for plane stress, and

$$K_1 = \frac{1-\nu^2}{E},\ K_2 = \nu\,, \qquad (11.88)$$

for plane strain. Next, we express the strain components:

$$\frac{du_r}{dr} = K_1(2A\ln r + \frac{C}{r^2} + A + 2B - K_2(2A\ln r - \frac{C}{r^2} + 3A + 2B))\,,$$

$$(11.89)$$

$$\frac{u_r}{r} = K_1(2A\ln r - \frac{C}{r^2} + 3A + 2B - K_2(2A\ln r + \frac{C}{r^2} + A + 2B))\,.$$

Integrating the former equation we get:

$$u_r = K_1(2Ar\ln r - Ar + 2Br - \frac{C}{r} - K_2(2Ar\ln r + Ar + 2Br + \frac{C}{r}) + H)\,,$$

$$(11.90)$$

where H is an integration constant. Dividing the formulae above by r and equating it to the second of Eq.(11.89) gives the following:

$$4Ar - H = 0\,. \qquad (11.91)$$

Since the equation must be satisfied for all values of r in the region, we must consider the trivial solution:

$$A = H = 0\,. \qquad (11.92)$$

The remaining constants, B and C, are to be determined from the boundary conditions imposed on the inner and outer boundary surfaces. Therefore, the general solution is:

$$u_r(r) = K_1(2Br(1-K_2) - \frac{C}{r}(1+K_2))\,. \qquad (11.93)$$

The problem of hollow circular cylinder can also be solved by Navier’s equation. If the displacement field is independent of coordinate ϑ, then ω = 0, i.e. from Eqs.(11.70)-(11.71) we obtain:

$$\frac{d^2u_r}{dr^2} + \frac{1}{r}\frac{du_r}{dr} - \frac{u_r}{r^2} = 0\,, \qquad (11.94)$$

for which the general solution is:

$$u_r(r) = c_1 r + \frac{c_2}{r}\,. \qquad (11.95)$$

It is seen that it is mathematically identical to (11.93). For a circular cylinder with fixed outer surface and with internal pressure the kinematic boundary conditions are:

$$u_r(r_b) = u_0\,,\ u_r(r_k) = 0\,. \qquad (11.96)$$

Based on the solution function the constants are:

$$c_1 = \frac{r_b}{r_b^2 - r_k^2}u_0\,,\ c_2 = \frac{-r_b r_k^2}{r_b^2 - r_k^2}u_0\,, \qquad (11.97)$$

and the solution is:

$$u_r(r) = \frac{r_b u_0}{r_b^2 - r_k^2}\left(r - \frac{r_k^2}{r}\right). \qquad (11.98)$$

The strain components are to be determined by Eq.(11.85), the stresses by Eq.(11.68).

11.7.2 Rotating disks

If the thickness of the circular cylinder is small, then it is said to be a disk (Fig.11.8b). If the disk rotates, then there is a body force in the reference coordinate system. The equilibrium equation in the radial direction (see Eq.(11.65)) becomes [2]:

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\vartheta}{r} + q_r = 0 \text{ and } q_r = \rho r \omega^2, \qquad (11.99)$$

where ω is the angular velocity of the disk, ρ is the density of the disk material. Rearranging the equation we obtain:

$$\frac{d}{dr}(r\sigma_r) - \sigma_\vartheta + \rho r^2\omega^2 = 0. \qquad (11.100)$$

This equation can be satisfied by introducing the stress function, F, in accordance with the following:

$$r\sigma_r = F, \; \sigma_\vartheta = \frac{dF}{dr} + \rho r^2\omega^2. \qquad (11.101)$$

The strain components have already been derived for a hollow circular cylinder, eliminating u_r from Eq.(11.85) we obtain:

$$\varepsilon_\vartheta - \varepsilon_r + r\frac{d\varepsilon_\vartheta}{dr} = 0. \qquad (11.102)$$

Assuming plane stress state and utilizing Eq.(11.68) we have:

$$\varepsilon_r = \frac{1}{E}(\sigma_r - \nu\sigma_\vartheta) = \frac{1}{E}\left(\frac{F}{r} - \nu\left(\frac{dF}{dr} + \rho r^2\omega^2\right)\right), \qquad (11.103)$$

$$\varepsilon_\vartheta = \frac{1}{E}(\sigma_\vartheta - \nu\sigma_r) = \frac{1}{E}\left(\frac{dF}{dr} + \rho r^2\omega^2 - \nu\frac{F}{r}\right).$$

Taking it back into Eq.(11.101) yields the following:

$$r^2\frac{d^2F}{dr^2} + r\frac{dF}{dr} - F + (3+\nu)\rho r^3\omega^2 = 0, \qquad (11.104)$$

i.e. we have a second order differential equation for the stress function, which involves the following solution:

$$F = Ar + B\frac{1}{r} - \frac{3+\nu}{8}\rho r^3\omega^2. \qquad (11.105)$$

The stress components based on Eq.(11.101) are:

$$\sigma_r(r) = A + B\frac{1}{r^2} - \frac{3+\nu}{8}\rho r^2\omega^2, \; \sigma_\vartheta(r) = A - B\frac{1}{r^2} - \frac{1+3\nu}{8}\rho r^2\omega^2, \qquad (11.106)$$

where A and B are integration constants, which can be determined by the boundary conditions. To calculate the displacement field we incorporate Eq.(11.85), from which we have:

$$\frac{du_r}{dr} = A\frac{(1-\nu)}{E} + B\frac{(1+\nu)}{Er^2} - \frac{3(1-\nu^2)}{8E}\rho r^2\omega^2, \qquad (11.107)$$

and the integration of it yields:

$$u_r(r) = A\frac{(1-\nu)}{E}r - B\frac{(1+\nu)}{Er} - \frac{(1-\nu^2)}{8E}\rho r^3\omega^2 \qquad (11.108)$$

The basic equations of the rotating disk are then:

$$\sigma_r(r) = A + B\frac{1}{r^2} + C_1 r^2, \qquad (11.109)$$

$$\sigma_\vartheta(r) = A - B\frac{1}{r^2} + C_2 r^2,$$

$$u_r(r) = ar - b\frac{1}{r} + cr^3,$$

where:

$$C_1 = -\frac{3+\nu}{8}\rho\omega^2, C_2 = -\frac{1+3\nu}{8}\rho\omega^2, \qquad (11.110)$$

$$a = A\frac{(1-\nu)}{E}, b = B\frac{(1+\nu)}{E}, c = -\frac{(1-\nu^2)}{8E}\rho\omega^2. \qquad (11.111)$$

Let us solve an example using the equations above! The elastic disk shown in Fig.11.9 is fixed to the shaft with an overlap of δ [3].

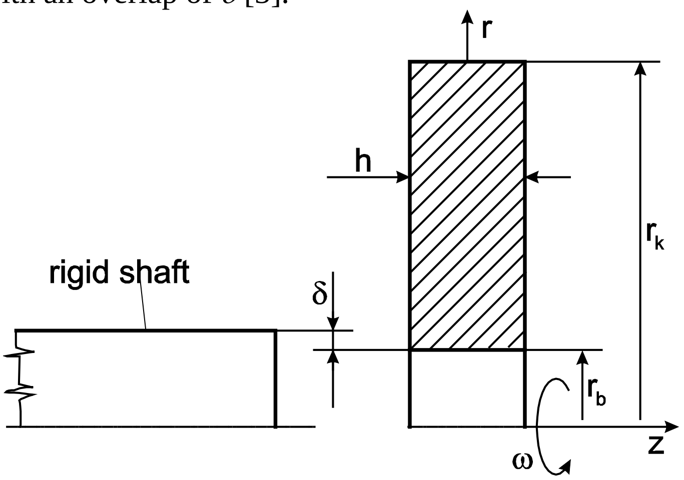

Fig.11.9. Rotating disk on a rigid shaft.

Given:

r_b = 0,02 m, r_k = 0,2 m, h = 0,04 m, δ = 0,02·10^{-3} m, ρ = 7800 kg/m^3, E = 200 GPa, ν = 0,3.

a. *How large can be the maximum angular velocity if we want the disk not to get loose?*
b. *Calculate the contact pressure between the shaft and disk, when the structure does not rotate!*

For point *a*. first we formulate the boundary conditions. A kinematic boundary condition is, that he radial displacement on the inner surface of the disk must be equal to the value of overlap:

$$u_r(r_b) = \delta \Rightarrow ar_b - b\frac{1}{r_b} + cr_b^3 = \delta. \qquad (11.112)$$

The outer surface of the disk is free to load, therefore in accordance with the dynamic boundary condition, the radial stress perpendicular to the outer surface is zero:

$$\sigma_r(r_k) = 0 \Rightarrow A + B\frac{1}{r_k^2} + C_1 r_k^2 = 0 . \qquad (11.113)$$

If the disk gets loose, then a free surface is created, that is why the radial stress should be equal to zero, i.e.:

$$\sigma_r(r_b) = 0 \Rightarrow A + B\frac{1}{r_b^2} + C_1 r_b^2 = 0 . \qquad (11.114)$$

The system of equations contains three unknowns: A, B and ω, since a and b are not independent of A and B. We now subtract Eq.(11.113) from Eq.(11.114) and we obtain:

$$B\left(\frac{1}{r_k^2} - \frac{1}{r_b^2}\right) + C_1(r_k^2 - r_b^2) = 0 \Rightarrow B = C_1 r_b^2 r_k^2 . \qquad (11.115)$$

The back substitution into Eq.(11.114) gives:

$$A = -C_1(r_b^2 + r_k^2) , \qquad (11.116)$$

consequently:

$$a = -\frac{(1-\nu)}{E} C_1(r_b^2 + r_k^2) ,\ b = \frac{(1+\nu)}{E} C_1 r_b^2 r_k^2 . \qquad (11.117)$$

Taking the constants back into the kinematic boundary condition equation yields:

$$-\frac{(1-\nu)}{E} C_1(r_b^2 + r_k^2) r_b - \frac{(1+\nu)}{E} C_1 r_b^2 r_k^2 \frac{1}{r_b} - \frac{(1-\nu^2)}{8E}\rho\omega^2 r_b^3 = \delta . \qquad (11.118)$$

Incorporating the constant C_1, and rearranging the resulting equation the maximum angular velocity becomes:

$$\omega = 880{,}5 \text{ rad/s} = \omega_{\max} . \qquad (11.119)$$

In terms of the angular velocity the constants can be determined:

$$A = 1{,}008 \cdot 10^8 \text{ Pa} ,\ B = -39915 \text{ N} ,\ C_1 = -2{,}495 \cdot 10^9 \text{ N/m}^4 , \qquad (11.120)$$

$$C_2 = -1{,}436 \cdot 10^9 \text{ N/m}^4 ,\ a = 3{,}53 \cdot 10^{-4} ,\ b = 2{,}59 \cdot 10^{-7} \text{ m}^2 ,\ c = -3{,}439 \cdot 10^{-3} \text{ 1/m}^2 .$$

For point *b*. we find out that if the disk does not rotate then $\omega = 0$ and this way: $C_1 = C_2 = c = 0$. Under these circumstances the radial displacement on the inner surface must be equal to the value of overlap:

$$u_r(r_b) = \delta \Rightarrow a r_b - b\frac{1}{r_b} + c r_b^3 = \delta . \qquad (11.121)$$

The outer surface of the disk is still free to load, i.e.:

$$\sigma_r(r_k) = 0 \Rightarrow A + B\frac{1}{r_k^2} + C_1 r_k^2 = 0 . \qquad (11.122)$$

The solution is:

$$A = 1{,}530 \cdot 10^6 \text{ Pa} ,\ B = -61208{,}9 \text{ N} , \qquad (11.123)$$

$$a = 5{,}356 \cdot 10^{-6} ,\ b = 3{,}978 \cdot 10^{-7} \text{ m}^2 .$$

The distribution of the radial and tangential stresses under two different conditions are demonstrated in Fig.11.10.

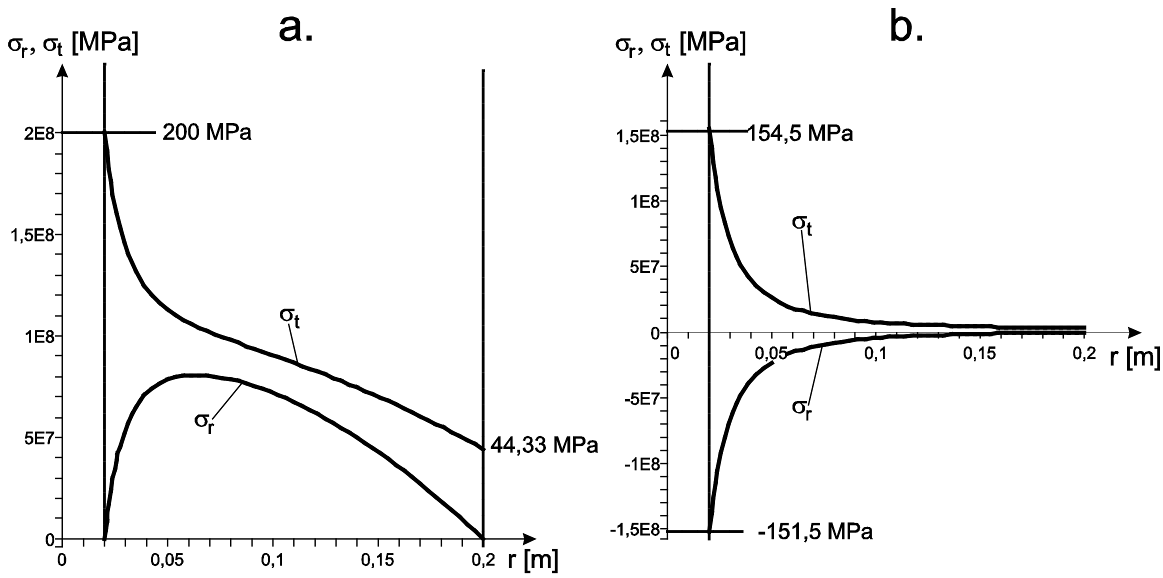

Fig.11.10. Distribution of the radial and tangential stresses in the disk structure when the structure rotates (a) and when there is no rotation (b).

11.8 Bibliography

[1] Pei Chi Chou, Nicholas J. Pagano, *Elasticity – Tensor, dyadic and engineering approaches*, D. Van Nostrand Company, Inc., 1967, Princeton, New Jersey, Toronto, London.

[2] S. Timoshenko, J. N. Godier. *Theory of elasticity*. McGraw-Hill Book Company, Inc., 1951, New York, Toronto, London.

[3] József Uj, *Lectures and practices of the subject Elasticity and FEM*, Budapest University of Technology and Economics, Faculty of Mechanical Engineering, Department of Applied Mechanics, 1998/1999 autumn semester, Budapest (in Hungarian).