Stability and surface errors are investigated numerically for milling operations with a helical tool. A detailed two degree of freedom mechanical model is derived that includes both surface regeneration and the helical teeth of the tool. The governing delay-differential equation is analyzed by the semi-discretization method. The surface errors are predicted based on the (stable) forced motion of the tool. New surface error parameters were introduced to characterize the properties of the spatial machined surface. The errors were calculated numerically for a given machine tool and workpiece for different axial depths of cut and spindle speeds. It is shown that both good surface properties and large material removal rate can be achieved by appropriate axial immersion in case of helical fluted tool. This phenomenon was proved analytically by means of the Fourier transformation of the cutting force.

Keywords appropriate axial immersion, cutting force, helical mills, material removal rate, stability, surface location error

INTRODUCTION

Machine tool vibration is one of the most important problems that face the machinist. Undesired self-excited vibrations during machining, often called machine tool chatter, shorten the lifetime of the tool and result in unacceptable surface quality. Recently, the research interest has focused on the determination of the chatter-free machining process parameters to avoid the above-mentioned effects.

Efficient machining process parameters for stable (chatter-free) cutting can be chosen from the well-known stability charts (Stépán, 1989; Tlusty and Poláček, 1963; Tobias, 1965), however, in most cases, forced vibrations still occur at these parameters. In milling processes, these forced vibrations...
are often significant and they also affect surface quality, which has special importance for finishing.

Most studies in the literature of milling tool chatter consider straight fluted tool models (see Gradišek et al., 2005; Insperger et al., 2006; Mann et al., 2005; Peigné et al., 2004), and few articles use models to describe the effects of the helix angle of the tool (Zatarain et al., 2006). Surface properties are usually determined by a straight fluted tool model (Insperger et al., 2006; Peigné et al., 2004), even though cutting tools typically have helical edges. Several geometric descriptions of the surface can be found for a helical tool model (Montgomery and Altintas, 1991; Shirase and Altintas, 1996; Surmann, 2006; Xu et al., 2003).

The goal of this paper is to predict the machined surface quality in case of helical milling tools. Apart from the existing surface quality parameters (Kline et al., 1982; Mann et al., 2004; Schmitz and Ziegert, 1999), such as the surface location error (SLE) and the total height of the profile (Rt), some modified/extended parameters are introduced that take into account the error variation along the axial immersion. The final goal of the paper is to find the appropriate axial immersions for milling with helical tools where the surface quality is optimal in some sense.

A two-degrees of freedom (DoF) model is used to describe the dynamical behaviour of the machine tool. The modal parameters are taken from the impact tests on an actual existing machine tool (Gradišek et al., 2005), where the chatter-free parameter domains are well estimated by theoretical and experimental methods. In this study, only those machining process parameters are considered and used where the milling process is stable. At these parameters, the motion of the cutting edges is calculated analytically from the governing equation and the corresponding surface quality parameters are evaluated. The surface errors are presented in the plane of the spindle speed and the axial immersion parameters. Optimal machining process parameters are identified where SLE and other quality parameters are minimal. We refer to these parameters as appropriate axial immersions, which are either constants or linear functions of the spindle speed.

This study was partially initiated by our preliminary experimental work (Bachrathy and Stépán, 2008) on an artificially flexible one DoF test rig, where epoxy resin was machined by a helical milling tool. However, the results presented in the current paper have not been tested experimentally in a realistic machine tool environment yet.

The structure of the paper is as follows: we describe the structural modelling and also the applied model of the machining process leading to the governing equations, while the forced stationary vibration is calculated analytically using Fourier transformation of the cutting force. The appropriate segments of the machined surface are generated. We then introduce additional and modified surface quality parameters that
are needed in case of helical milling tools, as opposed to the case of straight fluted tools, and the main results are derived. First, the structural parameters of a case study are defined, and then the effects of the machining process parameters on surface quality parameters are calculated based on the previous analytical solutions. We also present closed form expressions for the appropriate axial immersions and spindle speeds and prove their existence in the chatter-free parameter domain.

**MECHANICAL MODELLING**

In our analysis, a two DoF flexible tool and rigid workpiece model is used (Figure 1), because the lowest modal frequencies in the x and y modal directions of the cylindrical milling tool are the relevant ones. In finishing operations, only the stable process is acceptable, which can be determined by a linear stability calculation. Thus, the cutting force, $F_j$, of the $j$th tooth is approximated as a linear function of the chip thickness, $h_j$, at the stationary cutting. The tangential and the radial cutting force components can be given as:

$$F_{jt}(t) = wK_{jt} h_j(t),$$
$$F_{jr}(t) = wK_{jr} h_j(t) = k_r F_{jt}(t),$$

(1)

![FIGURE 1 Two DoF model of the flexible tool and milling process.](image-url)
where \( w \) is the axial immersion, \( K_t \) and \( K_r \) are the specific cutting force coefficients, while \( k_r \) denotes their ratio, \( k_r = K_r / K_t \). These coefficients can be determined through a set of test measurements at different machining process parameters by using force sensors (Budak and Altintas, 1996).

The chip thickness consists of two parts: the stationary chip thickness, and the dynamic one. The stationary part is generated by the constant feed velocity, \( v \), while the dynamic part is generated by the vibration of the tool. If the motion of the tool centre is described by the general coordinate vector, \( \mathbf{q} = [x \ y]^T \), then the chip thickness at the \( j \)th tooth assumes the form

\[
h_j(t) = \left[ \sin(\phi_j(t)) \cos(\phi_j(t)) \right] \left[ \begin{array}{c} \tau v \\ 0 \end{array} \right] + \mathbf{q}(t) - \mathbf{q}(t - \tau).
\]

(2)

Here \( \phi_j(t) \) is the angular position of the \( j \)th tooth, the regenerative time delay is equal to the tooth-pass period

\[
\tau = \frac{2\pi}{\Omega Z}.
\]

(3)

\( Z \) is the number of the teeth and \( \Omega \) is the angular velocity of the tool. The term \( \tau v \) is equal to the feed per tooth.

With the help of the modal matrices \( \mathbf{M} \) (mass), \( \mathbf{C} \) (damping) and \( \mathbf{K} \) (stiffness), the governing equation is given by

\[
\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{C}\dot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = \mathbf{W}(t)\left( \begin{array}{c} \tau v \\ 0 \end{array} \right) + \mathbf{q}(t) - \mathbf{q}(t - \tau),
\]

(4)

where the right-hand side determines the resultant cutting force vector acting on the tool. This linear non-homogeneous delay-differential equation (DDE) can be calculated from Equations (1) and (2). The resulting directional force coefficient matrix \( \mathbf{W}(t) \) is determined next.

**Straight-Edged Tool**

In case of a straight edged tool, it is straightforward to calculate the directional force coefficient matrix \( \mathbf{W}(t) \) from Equations (1) and (2) (see, for example, Bachrathy, 2006; Gradišek et al., 2005; Insperger et al., 2006). The result is

\[
\mathbf{W}(t) = \sum_{j=1}^{Z} \frac{wK_t}{2} g(\phi_j(t)) \mathbf{T}(\phi_j(t)).
\]

(5)

The screen function, \( g \), indicates with a value 1 if the \( j \)th tooth is in contact with the workpiece, that is, if the angle \( \phi_j(t) \) is between the entrance angle \( \phi_{\text{in}} \) and exit angles \( \phi_{\text{out}} \), while it is 0 otherwise. The projection matrix, \( \mathbf{T} \), is used to project the \( j \)th cutting force vector from
the local radial-tangential coordinate system fixed to the jth tooth to the global (x, y) coordinate system fixed to the tool centre:

\[
T_{11}(\phi_j(t)) = -\sin(2\phi_j(t)) - k_r + k_t \cos(2\phi_j(t))
\]
\[
T_{12}(\phi_j(t)) = -1 - \cos(2\phi_j(t)) - k_t \sin(2\phi_j(t))
\]
\[
T_{21}(\phi_j(t)) = 1 - \cos(2\phi_j(t)) - k_t \sin(2\phi_j(t))
\]
\[
T_{22}(\phi_j(t)) = \sin(2\phi_j(t)) - k_t - k_r \cos(2\phi_j(t)).
\]

**Helical-Edged Tool**

If helical tools are considered in the model (see Figure 2) then the current angular position of the cutting edge also depends on the axial coordinate, z, of the tool (see Zatarain et al., 2006). Thus,

\[
\phi_j(z, t) = t\Omega + j\frac{2\pi}{Z} - \frac{2\pi z}{p},
\]

where \(p\) is the helix pitch. The tool is divided into elementary disc components of width \(dz\). In a cross section of the tool, the elementary directional force coefficient matrix, \(dW\), is calculated in the same way as in the straight edged model with an infinitesimally small chip width \((w \sim dz)\):

\[
dW(z, t) = \sum_{j=1}^{Z} \frac{K_t}{2} g(\phi_j(z, t))T(\phi_j(z, t))dz.
\]

**FIGURE 2** Geometry of a helical edged tool, including the helix pitch, \(p\).
Equation (8) is then integrated along the $z$ axis from 0 to the axial immersion, $w$, in order to obtain the resultant directional force coefficient matrix:

$$W(t) = \sum_{j=1}^{Z} \int_{z=0}^{w} \frac{K_i}{2} g(\phi_j(z,t))T(\phi_j(z,t))dz.$$  

(9)

The limit of the integration can be changed simply to the angular coordinate because it is a linear function of the $z$ coordinate, as given in Equation (7), where $d\phi/dz = -2\pi/p$. The integration by substitution leads to

$$W(t) = \sum_{j=1}^{Z} \int_{\phi_j(w,t)}^{\phi_j(0,t)} \frac{K_i p}{4\pi} g(\phi_j)T(\phi_j)d\phi_j.$$  

(10)

Figure 3 shows that at a $p/Z$ height, the cutting edges' angles cover a full period $[0,2\pi]$. If the axial immersion $w$ is larger than $p/Z$ then the directional force coefficient matrix in Equation (10) can be separated into two parts:

$$W(t) = N_{\text{full}}\overline{W} + \sum_{j=1}^{Z} \tilde{W}_j(t),$$  

(11)
where $N_{\text{full}} = \text{int}(wZ/p)$, with int() denoting the integer part function. The first term contains those sections where the integration is carried out along full periods ($p/Z$). This part is constant in time:

$$\bar{W} = \int_{0}^{2\pi} \frac{K_p}{4\pi} g(\phi) T(\phi) d\phi = \int_{\phi_{\text{in}}}^{\phi_{\text{out}}} \frac{K_p}{4\pi} T(\phi) d\phi. \tag{12}$$

The second part contains the remainder of the integration, which is time dependent:

$$\tilde{W}_j(t) = \int_{a_j(t)}^{b_j(t)} \frac{K_p}{4\pi} T(\phi) d\phi. \tag{13}$$

The limits, $a_j(t)$ and $b_j(t)$, of the integration can be determined from the systematic scheduling of entry and exit angles, so the directional force coefficient matrix can be calculated analytically (see Bachrathy, 2006). The details of this calculation are not presented here since only the structure of the directional force coefficient matrix is important here: it is a periodic function and its period equals to the time delay ($\tau$). However, it can be time independent if the periodic part, $\tilde{W}_j(t)$, is identically zero. This is the case if we have to integrate along full periods only, that is, when the axial immersion is just

$$w = k \frac{p}{Z}, \quad k = 0, 1, 2, 3, \ldots \tag{14}$$

This case is represented in Figure 4 by straight horizontal lines.

**FIGURE 4** $W_{11}$ component of the directional force coefficient matrix for some axial immersion values ($w$), with $p/Z = 10$ mm and $a_e = 5\%$ radial immersion.
STATIONARY AND TRANSIENT MOTION OF THE MILLING TOOL

Insperger et al. (2006) showed that the motion of the tool can be decomposed in the form

\[ q(t) = q_p(t) + q_h(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \begin{bmatrix} \xi(t) \\ \eta(t) \end{bmatrix}, \quad (15) \]

where \( q_p(t) = q_p(t - \tau) \) is the forced (chatter-free) stationary motion of the tool that corresponds to the particular solution of the linear forced DDE (Equation (4)), and \( q_h(t) \) is a perturbation that corresponds to the diminishing transient vibration in case of chatter-free parameters – i.e., it describes the exponentially increasing machine tool chatter in case of unstable machining process parameters, where subscript \( h \) refers to the homogeneous solution of Equation (4). The forced motion \( q_p(t) \) satisfies

\[ M\ddot{q}_p(t) + C\dot{q}_p(t) + Kq_p(t) = W(t) \begin{bmatrix} \tau v \\ 0 \end{bmatrix}, \quad (16) \]

which is an ordinary differential equation (ODE). Substituting Equations (15) and (16) into Equation (4), we obtain the linear parametrically excited DDE

\[ M\ddot{q}_h(t) + C\dot{q}_h(t) + (K - W(t))q_h(t) = -W(t)q_h(t - \tau), \quad (17) \]

which describes the stability of the stationary tool motion \( q_p(t) \). Equation (17) is a linear parametrically forced DDE. If the \( q_h(t) \equiv 0 \) trivial solution of this equation is unstable, then chatter occurs and there is no need to calculate the surface errors since unstable machining is unacceptable. When the trivial solution \( q_h(t) \equiv 0 \) of Equation (17) is stable, the stationary motion of the tool centre is given by the solution of Equation (16). This (slightly) damped periodically forced oscillator always has a periodic solution. By using this solution, we can compute the stationary motion of the teeth, which generates the machined surface.

The parameter domains of stable cutting can be calculated from Equation (17) using the so-called semi-discretization method based on the classical Nyquist-like criteria. This efficient method was developed, described and tested in detail by Insperger and Stépán (2002).
FORCED VIBRATION AND MACHINED SURFACE

In order to calculate a particular solution analytically, we approximate the cutting force by its truncated Fourier series

\[ W(t) \begin{bmatrix} \tau v \\ 0 \end{bmatrix} \approx a_0 + \sum_{j=1}^{N_F} a_j \cos \left( \frac{2\pi j t}{\tau} \right) + b_j \sin \left( \frac{2\pi j t}{\tau} \right). \]  

(18)

During the calculations, the first \( N_F = 25 \) Fourier terms are used. Similarly, the particular solution is considered in the form

\[ q_p(t) \approx c_0 + \sum_{j=1}^{N_F} c_j \cos \left( \frac{2\pi j t}{\tau} \right) + d_j \sin \left( \frac{2\pi j t}{\tau} \right). \]  

(19)

The coefficients \( c_j \) and \( d_j \) can be determined after the substitution of Equations (18) and (19) into the governing Equation (4):

\[
\begin{bmatrix}
  c_j \\
  d_j 
\end{bmatrix} = \begin{bmatrix}
  K - \left( \frac{2\pi j}{\tau} \right)^2 M & C \left( \frac{2\pi j}{\tau} \right) \\
  -C \left( \frac{2\pi j}{\tau} \right) & K - \left( \frac{2\pi j}{\tau} \right)^2 M
\end{bmatrix}^{-1} \begin{bmatrix}
  a_j \\
  b_j
\end{bmatrix}, \quad j = 1, 2, 3, \ldots, N_F. \tag{20}
\]

Using the solution \( q_p(t) \), the position vector \( e_j(z, t) \) of the points at height \( z \) of the \( j \)th cutting edge can be described by superposing the forced stationary vibration onto the cylindrical rotation of the ideally rigid (non-vibratory) tool (see Figure 5). Thus,

\[ e_j(z, t) = \begin{bmatrix}
  x(t) \\
  y(t)
\end{bmatrix} + \begin{bmatrix}
  R \cos(\phi_j(z, t)) \\
  R \sin(\phi_j(z, t))
\end{bmatrix}, \tag{21}
\]

where \( R \) is the radius of the tool and \( \phi_j \) comes from Equation (7).

In the milling process, each tooth creates a single surface segment \( s(z, t) \) during one revolution. The finished surface of the workpiece is formed by the succeeding surface segments (see Figure 6). The surface properties can be described by a single surface segment, which can be determined by the motion \( e(z, t) \) of any of the cutting edges relative to the workpiece

\[ s(z, t) = e(z, t(z)) + \begin{bmatrix}
  vt \\
  0
\end{bmatrix}, \quad t(z) \in [t_{\text{enter}}(z), t_{\text{exit}}(z)], \tag{22}
\]

where the redundant subscript \( j \) is dropped accordingly. We have to determine the entering and exiting time instants \( t_{\text{enter}}(z), t_{\text{exit}}(z) \) at a certain height, \( z \), where the cutting edge just touches the boundary of the surface.
segment of the workpiece. These can be calculated from

\[ \mathbf{e}(z, t_{\text{enter}}) + \begin{bmatrix} \nu \lambda_{\text{enter}}(z) \\ 0 \end{bmatrix} + \begin{bmatrix} \tau v \\ 0 \end{bmatrix} = \mathbf{e}(z, t_{\text{exit}}) + \begin{bmatrix} \nu \lambda_{\text{exit}}(z) \\ 0 \end{bmatrix} \]  

(23)

by using a simple gradient based numerical method.
SURFACE QUALITY PARAMETERS

In case of the straight fluted tool, two surface parameters are used commonly. The first is the total height of the profile, $R_t$, which describes the surface roughness. It is defined by the difference of the maximum peak height and the maximum valley depth of the surface (Kline et al., 1982; Montgomery and Altintas, 1991; Peigné et al., 2004; Shirase and Altintas, 1996; Surmann, 2006; Xu et al., 2003):

$$R_t = \max(s_y(t)) - \min(s_y(t)),$$  \hfill (24)

where the dependence on the coordinate, $z$, is not denoted for the straight fluted tool. The second parameter is the surface location error, $SLE$ (Insperger et al., 2006; Mann et al., 2005; Schmitz and Ziegert, 1999), which is an offset error defined as the maximal distance between the desired surface and the machined surface. $SLE$ can be calculated by

$$SLE = \max(s_y(t)) - R.$$  \hfill (25)

In case of the helical-edged tool, the total height of the profile and the surface location error are different at each height, $z$. Consequently, the corresponding parameters, $R_t(z)$ and $SLE(z)$, are functions of the axial coordinate, $z$, but the use of scalar surface quality parameters are preferred in practice. In case of the total height of the profile obtained with a helical tool, there are two different types of errors. One is measured along the feed direction, $x$, which is similar to the total height of the profile at the straight fluted tool model. The second is measured along the $z$ direction, and this is formed by the varying $SLE(z)$ (see Figure 6). This variation is caused by the helical edge only, and it is always zero for a straight fluted tool.

It can be seen from Equations (21) and (7) that the surface is changing periodically along the $z$ coordinate, too. The variation of $SLE$ has a wavelength $p/Z$ in the direction $z$. For this reason, the variation of $SLE$ is not really a roughness but rather a waviness. Consequently, the surface error is separated into two parts: the first is the total height of the profile in feed direction

$$R_{tx} = \max(R_t(z)),$$  \hfill (26)

while the second part is the total height of the profile in axial direction

$$R_{tz} = \max(SLE(z)) - \min(SLE(z)).$$  \hfill (27)
An overall scalar characteristic value of the surface could be the maximum surface location error

$$MSLE = \max(s_j(z,t)) - R = \max(SLE(z)).$$  \hspace{1cm} (28)

If we use straight fluted tool, $SLE$ and $MSLE$ are the same. The three types of surface errors, $R_{tx}$, $R_{tz}$ and $MSLE$, are represented graphically in Figure 6.

**EFFECT OF MACHINING PROCESS PARAMETERS ON THE PROFILE**

**Real-Case Mechanical Parameters**

To describe the effect of the machining process parameters on the surface quality, we choose a particular machine tool and corresponding workpiece parameters. The machine tool is described by the same modal parameters as measured by Gradišek et al. (2005). They determined the frequency response function of the machine tool by a standard impact test procedure. The machine tool had a flexible tool with small diameter and large overhang, so the system has one dominant mode in each direction. These modes correspond to the first bending mode of the tool, so the two natural frequencies of these modes are very close to each other ($\omega_{nx} = 4537.3 \text{[rad/s]}$ and $\omega_{ny} = 4537.6 \text{[rad/s]}$) due to the cylindrical symmetry. The mode coupling was also very weak. The modal parameters, such as the mass ($M$), the damping ($C$) and the stiffness ($K$) matrices, were identified by curve fitting using a commercial modal analysis software

$$M = \begin{bmatrix} m_x & 0 \\ 0 & m_y \end{bmatrix} = \begin{bmatrix} 0.01986 & 0 \\ 0 & 0.02008 \end{bmatrix} [\text{kg}], \hspace{1cm} (29)$$

$$C = \begin{bmatrix} c_x & 0 \\ 0 & c_y \end{bmatrix} = \begin{bmatrix} 1.60312 & 0 \\ 0 & 1.155697 \end{bmatrix} \left[\frac{\text{Ns}}{\text{m}}\right], \hspace{1cm} (30)$$

$$K = \begin{bmatrix} k_x & 0 \\ 0 & k_y \end{bmatrix} = \begin{bmatrix} 408,866 & 0 \\ 0 & 413,445 \end{bmatrix} \left[\frac{\text{N}}{\text{m}}\right]. \hspace{1cm} (31)$$

During the calculations, we use tools with a standard radius $R = 4\text{mm}$. The material of the workpiece is AlMgSi0.5 aluminium alloy, the corresponding tangential cutting coefficient $K_t$ and the non-dimensional force ratio $k_t$ are (Gradišek et al., 2004):

$$K_t = 644 \times 10^6 \left[\frac{\text{N}}{\text{m}^2}\right], \hspace{0.5cm} k_t = 0.368. \hspace{1cm} (32)$$
Effect of Machining Process Parameters

The machined surface parameters are calculated in those machining process parameter regions where the cutting process is stable. The surface quality is especially important in finishing operations, where typically, a small radial immersion is used. Calculations using the above structural model and cutting parameters were carried out to predict surface properties at a fixed 5% radial immersion, \( \alpha_r \), for down milling. Figure 7(a) shows the surface location error (\( SLE \)) and Figure 7(b) shows the total height of the profile (\( R_t \)) for down-milling over the stability chart for a straight fluted tool. Note that the total height of the profile in axial direction is always zero for the straight fluted tool.

We can see that there are large offset errors if we are in the vicinity of those angular velocities where the ratio of the natural frequency (\( \omega_n/2\pi \)) of the system and the tooth-passing frequency (\( 1/\tau = \Omega Z/2\pi \)) is close to an integer. In these cases, the tool vibration is close to resonance because one of the frequencies of higher harmonics of the cutting force is close to the natural frequency.

In Figure 8, the calculated the surface parameters (\( MSLE, R_{ts} \) and \( R_{tz} \)) for the same machining conditions as in Figure 7, but with a helix pitch of 10 mm, are presented. It is observed that the helix pitch has a great influence on the surface quality. Figure 8 shows that both good surface properties and large material removal rate (that is, large axial immersion) can be achieved by the appropriate axial immersion with a helical fluted tool. We found that the total height of the profile in axial direction (\( R_{tz} \)) is on the same order of magnitude as the maximum surface location error (\( MSLE \)) (see Figures 8(a) and (c)).

A strange pattern of the surface properties can be viewed by plotting the surface properties above the unstable regions, too (see Figure 9). Along the dashed line at \( p/Z \), when the directional force coefficient matrix

![FIGURE 7](attachment:image.png)  
(a) Surface location error, \( SLE \), and (b) total height of the profile, \( R_t \), for down-milling with straight fluted tool in case of stable cutting parameters (\( Z = 4, \tau v = 0.1 \text{ mm}, p \to \infty, \alpha_r = 5\% \)).
is constant (see Equation (14)), there is a small but non-zero MSLE. Here the cutting force is constant, hence, there is no vibration and the total height of the profile in axial direction is zero: $R_t = 0$. Along the dotted slanting lines, the surface properties are also very good, in spite of the fact that the cutting force itself is not constant there. This phenomenon is explained below by the Fourier transformation of the cutting force signal.

### Fourier Transformation of the Cutting Force

The specific cutting force is distributed along the helical cutting edge and can be calculated from the right-hand side of Equation (4), with Equation (8) for the forced (chatter-free) periodic solution:

$$f(\phi(z, t)) = \sum_{j=1}^{Z} K_j \frac{g(\phi_j(z, t))T(\phi_j(z, t))}{\tau v} \begin{bmatrix} \tau v \\ 0 \end{bmatrix},$$

(33)

where the angular position $\phi_j(z, t)$ of the $j$th edge is given by Equation (7). Function $f(\phi(z, t))$ is a periodic function in time with the tooth-pass period $\tau$ due to the trigonometric functions in the expression of $T$ in
Equation (6). Consequently, \( f(\phi) \) is also periodic in \( \phi \), with period \( 2\pi/Z \). By using Equations (4) and (9), the Fourier transformation of the cutting force is given by

\[
\mathcal{F}_{\text{cut}}(\omega) = \frac{1}{\tau} \int_0^\tau \left( \int_0^\omega f\left( t\Omega - \frac{2\pi z}{p}\right) dz \right) e^{-i\omega t} dt.
\]  

(34)

The axial space coordinate \( z \) is substituted by the new time variable \( \theta \) according to the formula

\[
z = (t - \theta) \frac{\Omega p}{2\pi},
\]

(35)
and the integration by substitution leads to the convolution integral

\[
\mathcal{F}_{\text{cut}}(\omega) = -\frac{1}{\tau} \int_0^\tau \left( \int_{-\frac{2\pi p}{\Omega Z}}^{\frac{2\pi p}{\Omega Z}} f(\theta \Omega) d\theta \right) e^{-i\omega t} dt
\]

\[
= -\frac{1}{\tau} \int_0^\tau \left( \int_{-\infty}^{\infty} f(\theta \Omega) u(t - \theta) d\theta \right) e^{-i\omega t} dt, \quad (36)
\]

with

\[
u(t) = \begin{cases} 
1 & \text{if } 0 < t < \frac{2\pi p}{\Omega Z} \\
0 & \text{otherwise} \end{cases} \quad (37)
\]

The Convolution Theorem states that the Fourier transform transforms a convolution into a multiplication (Randal, 1987), thus Equation (36) can be written as

\[
\mathcal{F}_{\text{cut}}(\omega) = -\frac{1}{\tau} \int_0^\tau f(\theta \Omega) e^{-i\omega t} dt \cdot \int_{-\infty}^{\infty} u(t) e^{-i\omega t} dt
\]

\[
= -\frac{1}{\tau} \int_0^\tau f(\theta \Omega) e^{-i\omega t} dt \cdot i \frac{e^{i2\pi Z \omega}}{\omega} (1 - e^{i\frac{2\pi p}{\Omega Z} \omega}). \quad (38)
\]

Since \( f(\phi) \) is a periodic function, its Fourier transform is a discrete function. It can be seen from Equation (38), and also in Figure 10, that the resonant Fourier component of the cutting force \( F(\omega_n) \) is zero if either the first part of the product in Equation (38) is zero:

\[
\omega_n \neq \frac{2\pi}{\tau} k = k\Omega Z \Rightarrow \Omega \neq \frac{\omega_n}{kZ}, \quad k = 1, 2, 3 \ldots , \quad (39)
\]

or the second part of the product is zero:

\[
\frac{\omega_n w}{\Omega p} = k \Rightarrow w = \frac{\Omega p}{\omega_n} k, \quad k = 1, 2, 3 \ldots . \quad (40)
\]

**FIGURE 10** Representation of the functions \( f(\theta \Omega) \) and \( u(t) \) and their Fourier transforms.
The straight lines, \( w(\Omega) \), described by Equation (40) are the same as the dotted slanting ones in Figure 9. The other condition, Equation (39), determines the values of the well-known resonant angular velocities. It is also important that the appropriate axial immersion does not depend on the force characteristics. The dashed lines at \( kp/Z \ (k = 1, 2, 3, \ldots) \) represent minimal surface quality parameters, as already explained by Equation (14).

**CONCLUSION**

In this paper, we investigated the effects of the parameters of a helical tool model to predict the surface properties of the machined surface in high-speed milling. First, the governing equation of the tool motion was determined. Then the workpiece surface for the parameter region of stable cutting was generated. New surface parameters, such as the maximum surface location error, \( MSLE \), the total height of the profile in feed direction, \( R_{tx} \), and the total height of the profile in axial direction, \( R_{tz} \), were introduced to characterize the properties of the surface. The influence of the helix pitch on surface quality was the focus of our analyses.

The most important observation is that good surface properties and small vibration level can be achieved even for resonant angular velocities \( (\Omega = \omega_n/kZ, k = 1, 2, 3, \ldots) \) if certain appropriate axial immersion values are applied. Some of these values are the trivial ones, where the directional force coefficient matrix is time-independent, i.e., where the axial immersions are equal to an integer multiple of the helix pitch, \( p \), over the number of cutting teeth, \( Z \). Non-trivial appropriate axial immersions are also found, which depend linearly on the cutting speed. At the critical resonant cutting speeds, these are equally spaced between the trivial appropriate axial immersions.

The identification of the non-trivial appropriate axial immersions is of great practical importance because just the resonant cutting speeds are preferred to achieve high material removal rates under stable (chatter-free) machining conditions. If non-appropriate axial immersions are used at these cutting speeds, the machined surface quality parameters might be extremely poor.

Although the theory as well as the analytical and the numerical calculations were implemented on a real and dynamically well-identified machine tool using realistic cutting parameters (Gradišek et al., 2004, 2005), the resulting surface quality parameter diagrams above the plane of the (stable) machining process parameters have not been checked experimentally yet.
ACKNOWLEDGMENTS

This research was supported in part by the János Bolyai Research Scholarship of the Hungarian Academy of Sciences, and by the Hungarian National Sciences Foundation under grant no. OTKA K72911 and OTKA K68910. The authors also acknowledge with thanks the discussions and the useful comments of Dr. Martin Homer (University of Bristol).

NOMENCLATURE

\( a_r \) ratio of the radial immersion and the tool diameter, [%]

\( e_j(z, t) \) the path of the cutting edge as a function of height, \( z \), and time, \( t \), [m]

\( f \) specific cutting force vector, [N/m]

\( F_j \) cutting force vector acting on the \( j \)th tooth, [N]

\( F_r^j, F_t^j \) radial and tangential cutting force component acting on the \( j \)th tooth, [N]

\( \mathcal{F}_{\text{cut}} \) Fourier transform of the cutting force, [N]

\( g \) the screen function of the tooth, 1 if the tooth is in contact, 0 otherwise, [-]

\( h_j \) chip thickness at the \( j \)th tooth, [m]

\( K_r, K_t \) the radial and the tangential cutting coefficient, [N/m²]

\( k_r \) ratio of the radial and the tangential cutting coefficient, [-]

\( \text{MSLE} \) maximum surface location error, [m]

\( \mathbf{M}, \mathbf{C}, \mathbf{K} \) the mass, the damping and the stiffness modal matrices, [kg], [Ns/m], [N/m]

\( N_F \) number of the Fourier terms, [-]

\( p \) helix pitch, [m]

\( \mathbf{q} = [x, y]^T \) position vector of the tool centre, [m]

\( \mathbf{q}_p(t), \mathbf{q}_h(t) \) the forced motion and the transient motion of the tool centre, [m]

\( R_t \) total height of the profile in case of straight fluted tool, [m]

\( R_{tx} \) total height of the profile in feed direction (“waviness”), [m]

\( R_{tz} \) total height of the profile in axial direction, [m]

\( s(z, t) \) single surface segment of the machined workpiece, [m]

\( \text{SLE} \) Surface Location Error in case of straight fluted tool, [m]

\( \mathbf{T} \) projection matrix, [-]

\( u \) screen function in the convolution integral [-]

\( w \) axial immersion, [m]

\( \mathbf{W} \) the directional force coefficient matrix, [N/m]

\( \bar{\mathbf{W}}, \tilde{\mathbf{W}} \) average and remainder of the directional force coefficient matrix, [N/m]

\( Z \) number of cutting teeth of milling tool, [-]
\[ \phi_j \] the angular position of the jth tooth, [rad]

\[ \tau \] the tooth-pass period, [s]

\[ \omega_n \] natural frequency of the system, [rad/s]

\[ \Omega \] the angular velocity of the tool, spindle speed, [rad/s]

REFERENCES


