1 Introduction

The difficulty of predicting machine tool chatter was already recognized and described by Taylor [1]. Since then, several attempts have been made to explain this phenomenon. The introduction of the so-called regenerative effect resulted in a breakthrough in the modelling of chatter. This effect is related to the cutting force variation due to the wavy workpiece surface cut one revolution ago (in turning) or one tooth ago (in milling). After the extensive work of Tlusty et al. [2], Tobias [3] and Kudinov [4], regenerative effect became the most commonly accepted explanation for machine tool chatter [5–8]. The corresponding mathematical models are delay-differential equations (DDEs) with infinite dimensional state spaces. General stability criteria for these equations were given and applied for advanced cutting models by Stepán [9,10].

The study of nonlinear phenomena during the cutting process showed the existence of unstable periodic motions about the stationary cutting (a so-called subcritical Hopf bifurcation) which was shown experimentally by Shi and Tobias [11] and analytically by Stepán and Kalmar-Nagy [12].

As a consequence, the vibrations usually increase until the tool starts leaving the material either in case of unstable stationary cutting, or in case of linearly stable stationary cutting subjected to perturbations larger than the unstable periodic motion around it. Clearly, these vibrations lead to a different kind of system sometimes called as self-interrupted cutting (as opposed to the parametrically interrupted cutting where the teeth constantly enter and leave the cut as the cutter rotates [13]). In these cases, the system has sudden switches between the infinite dimensional dynamics of regenerative cutting and the finite dimensional one of a damped oscillator as the tool enters and leaves the material, respectively [10]. The resulting vibrations can be calculated by simulation [14–15] but there also exist analytical methods to estimate its amplitude [16]. Interrupted cutting can also be a desired way of machining as shown by Batzer et al. [17] for vibratory drilling. When the ratio of time spent cutting to not cutting is small, highly interrupted cutting models are used (see e.g. Davies and Balachandran [18] for milling of thin-walled structures). For analytical investigation of highly interrupted cutting, Davies et al. [19] developed special discrete map models as opposed to the conventional nonlinear DDE approach.

The modelling of the milling process is somewhere between the two extreme models: the autonomous DDE for turning and the discrete map model of highly interrupted cutting. From the viewpoint of a single tooth of the milling tool, the process may look like an interrupted cutting. However, the resultant cutting force varies with the number of active teeth, and this may show only a small periodic component if this number is great. For these cases, the conventional time-averaging was used in the classical literature [3] and so the stability results were similar to those of turning. From mathematical viewpoint, this averaging can hardly be justified since significant errors (even qualitative ones) can occur due to the time-dependent parameters in the model [20,21]. A more precise model of milling leads to parametric excitation in the regenerative effect. The equation of motion is a DDE with periodic coefficients, and the stability criteria can not be given in closed form even in the linear case. Minis and Yanushevsky [22] used the first harmonics of the time-periodic parameters and showed slight deviations in the stability of milling relative to the results with the time-averaging method. A higher order harmonic balance was used and applied in the milling problem by Altintas and Budak [23–25]. The harmonic balance method with some new numerical techniques was used by Corpus and Endres [26] and by Tian and Hutton [27]. Recently, other methods were developed, like the Fargue-type approximation and the Semi-Discretization method by Insperger and Stepán [28,29], or the Finite Element Analysis in Time by Bayly et al. [30].

Numerical simulation can be used to capture the interrupted nature of the milling process [15,31], but the exploration of parameter space via time domain simulation is inefficient. Unstable machining vibrations can also be detected by the analysis of the chatter signal [32–33]. Trajectory reconstruction methods of stochastic processes can effectively be used for identifying the relative motion of the tool and the workpiece from noisy time series [34].

The extended investigation of the milling process and the corresponding periodic DDE lead to the realization of a new bifurcation phenomenon. In addition to Hopf bifurcation, period doubling bifurcation is also a typical way of stability loss in milling processes, as it was shown by several analytical approaches.
[19,26,28,30,35], and confirmed by numerical simulation [36] and by experimental verifications [19,30,37,38]. These results were obtained for highly, or at least strongly, interrupted milling processes, like high-speed milling with low number of teeth cutting a thin workpiece. This can be considered as a limit case of general high-speed milling with 2 or 3 active teeth. If the number of active teeth is even larger, then the tooth pass excitation effect becomes more and more negligible, and the cutting process can approximately be modelled by an autonomous DDE. This is equivalent to the autonomous DDE describing turning processes and gives the other limit case. In this paper, the transition between the two stability charts related to the two extreme models, i.e. the discrete map and the autonomous DDE, is presented. This transition contains one critical element, a qualitative difference between the charts, which is posed as a question by Davies et al. [19]: ‘a doubling in the number of stability lobes’. Here a positive answer is given to this question, and the gradual build up of a series of new stability lobes and chatter frequencies are shown and proved as the parameters of the milling process vary from the ones close to turning to the ones close to highly interrupted cutting.

2 Mechanical Model of Milling

A single degree of freedom (SDOF) mechanical model of the milling process can be seen in Fig. 1. The equation of motion can be written in the following form

\[ \ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2x(t) = \frac{1}{m}\Delta F_j, \]  

where \( x(t) \) is the position of the tool edge at the time instant \( t \), \( \omega_n = \sqrt{k/m} \) is the natural angular frequency, \( \zeta = c/(2m\omega_n) \) is the relative damping factor, \( m \) is the modal mass, \( \Delta F_j \) is the \( j \)th component of the cutting force variation, which is time periodic due to the varying number of active teeth.

Let the number of teeth be \( z \) each indexed by \( j = 1, \ldots, z \). The angular position of the tooth \( j \) can be given in the following way

\[ \varphi_j = \hat{\Omega}t + j\vartheta, \]  

where \( \hat{\Omega} \) is the angular velocity of the tool given in [rad/s], and \( \vartheta = 2\pi/z \) is the angle between two edges. The \( j \)th edge works only if its angular position fulfils the condition

\[ \varphi_{j-1} < \varphi_j < \varphi_f, \]

where the angles \( \varphi_{j-1} \) and \( \varphi_f \) (the locations of entering and leaving, respectively) depend on the geometry of the milling operation

\[ \cos \varphi_{j-1} = \frac{B + 2e}{D}, \quad \cos \varphi_f = \frac{B - 2e}{D}, \]  

and \( B \) is the width of the work-piece, \( e \) is the distance between the center lines of the tool and the work-piece, and \( D \) is the diameter of the tool (see Fig. 1).

The \( x \) component of the force acting on tooth \( j \) assumes the form

\[ F_{jx} = F_{jt}\cos \varphi_j + F_{jn}\sin \varphi_j, \]  

where \( F_{jt} \) and \( F_{jn} \) are the tangential and normal components of the cutting force acting on tooth \( j \), respectively (see Fig. 2).

The tangential component of the cutting force acting on tooth \( j \) reads

\[ F_{jt} = \begin{cases} Kwf_j^s g(\varphi_j) & \text{if } \varphi_{j-1} < \varphi_j < \varphi_f \\ 0 & \text{otherwise} \end{cases}, \]  

where \( K \) is the cutting coefficient, \( w \) is the chip width (equal to the depth of cut for zero helix angle), \( f \) is the chip thickness, \( x_f \) is the exponent of the chip thickness (a typical value is \( x_f = 3/4 \)). To compose Eq. (5) in a mathematical form, we introduce the screen function of Laczik [39]

\[ g(\varphi_j) = \frac{1}{2}(1 + \text{sgn}(\sin(\varphi_j - \psi) - p)) = \begin{cases} 1 & \text{if } \varphi_{j-1} < \varphi_j < \varphi_f \\ 0 & \text{otherwise} \end{cases}, \]  

where \( \psi \) and \( p \) are defined as

\[ \tan \psi = \frac{\sin \varphi_{j-1} - \sin \varphi_f}{\cos \varphi_{j-1} - \cos \varphi_f}, \quad p = \sin(\varphi_{j-1} - \psi). \]  

Thus, the tangential component of the cutting force acting at the tooth \( j \) is the following

\[ F_{jt} = Kwf_j^s g(\varphi_j), \]  

while the normal component can be expressed as

\[ F_{jn} = F_{jt} \tan \gamma, \]  

where \( \gamma = 15^\circ \) in general (see Fig. 2). From Eqs. (4), (8) and (9) we get the \( x \) component of the force acting on edge \( j \)

\[ F_{jx} = -F_{jt}(\cos \varphi_j + \sin \varphi_j \tan \gamma) = -Kwf_j^s g(\varphi_j)(\cos \varphi_j + \sin \varphi_j \tan \gamma). \]  

The \( x \) component of the force acting on the tool is the sum of \( F_{jx} \) for all \( j \)

\[ F_x = \sum_{j=1}^{z} -Kwf_j^s g(\varphi_j)(\cos \varphi_j + \sin \varphi_j \tan \gamma). \]  

Because of the vibrations of the tool, the feed \( s \) per tooth has a deviation from the prescribed value \( s_0 \) in the following way

\[ s = s_0 + x(t) - x(t - \tau), \]  

\[ \tau = \frac{T}{n}, \quad n = 1, 2, 3, \ldots. \]
where the delay $\tau$ is one tooth pass period, that is $\tau = 60/(z\Omega)$ [s], and $\Omega=(30/\pi)\Omega$ is supposed to be given in [rpm]. Consequently, the chip thickness cut by tooth $j$ can be written as

$$f_j=(s_0+x(t)-x(t-\tau))\sin \varphi_j,$$

(13)

The ideal chip thickness reads

$$f_{j0}=s_0 \sin \varphi_j.$$  

(14)

The difference between the ideal and real chip thicknesses assumes the form

$$\Delta f_j=(x(t)-x(t-\tau))\sin \varphi_j.$$  

(15)

Substituting Eq. (13) into Eq. (11) we get the value of $F_x$ as the function of the actual and delayed positions of the tool, $x(t)$ and $x(t-\tau)$, respectively

$$F_x = \sum_{j=1}^{\xi} -Kw (s_0+x(t)-x(t-\tau))\psi (\sin \varphi_j)\psi (\varphi_j)(\cos \varphi_j + \sin \varphi_j\tan \gamma).$$

(16)

The linearization of Eq. (16) around the prescribed feed $s_0$ per tooth yields

$$\Delta F_x = \sum_{j=1}^{\xi} -Kw (s_0+x_0)\psi (\sin \varphi_j)\psi (\varphi_j)(\cos \varphi_j + \sin \varphi_j\tan \gamma) \times (x(t)-x(t-\tau)).$$

(17)

Equations (2), (6) and (17) result in the equation of motion in the form

$$\ddot{x}(t) + 2\zeta \omega_n \dot{x}(t) + \omega_n^2 x(t) = \frac{wKsF_{x0}}{m}h(t)(x(t-\tau)-x(t)),$$

(18)

where $h(t)$ is a time dependent coefficient including system parameters in the form

$$h(t) = \sum_{j=1}^{\xi} \frac{1}{2} (1 + \text{sgn} (\sin (\Omega t + j \vartheta - \psi) - \rho)) (\sin (\Omega t + j \vartheta)) \psi \times (\cos (\Omega t + j \vartheta) + \sin (\Omega t + j \vartheta)\tan \gamma).$$

(19)

Function $h(t)$ is built up from periodic functions of period $2\pi/\Omega$. Still, the period of $h(t)$ is $2\pi/(z\Omega)=60/(z\Omega)$ due to the periodic entering and leaving of the tool edges. This tooth passing period is just equal to the time-delay $\tau$.

Function $h(t)$ depends on the geometry of milling through the parameters $D, B, e, z, s_x, \gamma$. For highly interrupted cutting ($D>B, z=1$ or 2), $h(t)$ can well be approximated as a piecewise constant $\tau$-periodic function

$$h(t) \approx \begin{cases} 
 1 & \text{if } 0<t<\rho \tau, \\
 0 & \text{if } \rho \tau < t < \tau, 
\end{cases}$$

(20)

where $\rho$ is the fraction of time spent cutting to tooth pass period. Consequently, the tool has no contact with the workpiece for the time length $(1-\rho)r$. If $\rho$ is small, then the milling process can be modelled as the tool were hit by the workpiece in each tooth pass period. This is the basic point of the kicked harmonic oscillator model of Davies et al. [19] mentioned in the Introduction.

For the case of turning, the governing equation of motion is Eq. (18) with $h(t)=1$. This provides the other limit case: if the number of teeth is large ($z\approx8$) and the radial depth of cut is also large ($D>B$ with $e=0$), then $h(t)=1$. The transition of the function $h(t)$ from the case of turning to the case of highly interrupted milling can be seen later, in Fig. 4.

### 3 Stability Analysis

Equation (18) is a time periodic DDE, and the stability criteria can not be given in closed form. For autonomous DDEs, the stability criteria are given by a characteristic equation. For time periodic ordinary differential equations, the stability properties can be determined via the Floquet transition matrix. This matrix usually does not have a closed form, it is still a finite dimensional matrix, though. For time periodic DDEs, the stability properties are determined by the infinite dimensional version of the Floquet transition matrix, by the so-called monodromy or Floquet operator. The non-zero eigenvalues of the monodromy operator are called characteristic multipliers. If all the characteristic multipliers are in modulus less than one, then the corresponding DDE is asymptotically stable [20,21].

To achieve some stability prediction of the milling process, approximation methods, like the Fargue-type approximation [28], the Semi-Discretization method [29] or the Finite Element Analysis in Time [30], should be used. All of these methods result in a finite dimensional approximation of the infinite dimensional monodromy operator. Here, the Semi-Discretization method is used. The detailed description of the method including the proof of convergence was given by Insperger and Stépán [29], while its application for the milling stability problem (together with a comparison to the Finite Element Analysis in Time of Bayly et al. [30]) can be seen in the paper of Insperger et al. [35].

Here, the Semi-Discretization technique is used to construct stability charts. Its first step is the construction of the time interval division $[t_i,t_{i+1}]$ of length $\Delta t$, $i=0,1,\ldots$ so that $\tau=k\Delta t$, where $k$ is an integer that can be considered as an approximation parameter. The main point of the method is that the time periodic terms and the delayed term in Eq. (18) is approximated by piecewise constant values in each discretization interval $[t_i,t_{i+1}]$:

$$\ddot{x}(t) + 2\zeta \omega_n \dot{x}(t) + \omega_n^2 x(t) = \frac{wKsF_{x0}}{m}h_i x(t)$$  

(21)

where

$$h_i = \frac{1}{\Delta t} \int_{t_i}^{t_{i+1}} h(t)dt,$$  

(22)

and

$$x_{t_i} = x(t-i), \quad x_{t_i+(i+1)} = x(t-i+1).$$

Now, Eq. (21) can be solved for each discretization interval as a linear ordinary differential equation (ODE):

$$x_{i+1}=x(t_{i+1})=a_0 x_i + a_1 x_i + a_2 x_{i-1} + a_3 x_{i-2},$$  

(24)

$$\dot{x}_{i+1} = \dot{x}(t_{i+1}) = b_0 x_i + b_1 x_i + b_2 x_{i-1} + b_3 x_{i-2},$$  

(25)

where the coefficients $a_0$, $a_1$, $a_2$, $b_0$, $b_1$ and $b_2$ are obtained with the help of the well-known closed-form solution of Eq. (21) with initial conditions $x(t_i)=x_i$ and $\dot{x}(t_i)=\dot{x}_i$. This leads to the $k+2$ dimensional discrete map

$$z_{i+1} = D z_i,$$  

(26)

where
Finally, by coupling Eq. (26) for \( i = 0, 1, \ldots, k-1 \), the finite dimensional approximate monodromy matrix (or transition matrix) \( \Phi \) over the tooth pass period \( \tau = k \Delta t \) is obtained:

\[
\Phi = D_{k-1}D_{k-2}\cdots D_1D_0. \tag{29}
\]

For fixed machining parameters, \( \Phi \) can be determined numerically. The relevant characteristic multiplier \( \mu \) is the eigenvalue of \( \Phi \) that is largest in modulus. If \( |\mu| < 1 \), then the system is asymptotically stable, the milling operation is chatter-free.

For the case \( |\mu| = 1 \), the arising chatter frequencies are determined by \( \mu \) [37]. If the relevant characteristic multiplier is a complex pair in the form \( \mu = e^{\pm \omega t} \), then the chatter frequencies are given by

\[
f_H = \pm \omega + n \frac{2 \pi}{\tau}, \quad [\text{rad/s}] = \pm \frac{\omega}{2\pi} + \frac{n \Omega}{60}, \quad [\text{Hz}],
\]

where the angular chatter frequency is \( \omega = \text{Im}(\ln \mu)/(\omega_0 \tau) \) with the restriction \( \omega \in [0, 2\pi] \). This is the case of secondary Hopf bifurcation.

If \( \omega \tau = \pi \), then \( \mu = e^{\pm i\pi} = -1 \) is real. In this case, the chatter frequencies are

\[
f_{PD} = \pm \frac{\pi}{\tau} + \frac{2 \pi}{\tau}, \quad [\text{rad/s}] = \pm \frac{\Omega}{30} + \frac{n \Omega}{60}, \quad [\text{Hz}],
\]

where real characteristic multiplier occurs with multiplicity 2. This is the case of period doubling. The reason of this expression is that the frequency \( 1/(2\pi) \) [Hz] is just the half of the tooth pass frequency \( 1/\tau \) [Hz], that is the period of chatter is just double of the tooth pass period.

### 4 Stability Charts

Stability charts are presented in the plane of dimensionless spindle speed \( \Omega/(60f_n) \) and the dimensionless depth of cut \( \bar{w} = (wKx_{s_0}x_{s_0}^{-1})/(m_0a) \). Here, \( f_n = \omega_0/(2\pi) \) is the natural frequency of the tool in [Hz]. The chatter frequencies are presented as the ratio \( \Omega/(60f_n) \), where \( \Omega \) denotes either \( \Omega^H \) or \( \Omega^{PD} \) frequencies.

Stability charts were determined for up-milling operations for various radial depth of cut ratio \( a/D \) and number of teeth \( z \) (see Fig. 3). For calculations, \( B = a \) and \( e = (D - a)/2 \) can be used in Eq. (3). All the other parameters are fixed: \( \xi = 0.01, \gamma = 15^\circ, x_F = 0.75 \).

Various machining operations including the special cases of turning and highly interrupted cutting were investigated. The resulted stability charts and the chatter frequencies are shown in Fig. 4, together with the technological parameters and the function \( h(t) \).

As it can be seen, a new series of extra stability lobes arise in addition to the Hopf lobes of turning (see chart (C) in Fig. 4). The numerical calculation of the relevant characteristic multipliers shows a new kind of bifurcation phenomenon: these extra lobes are related to period two or flip bifurcation. A schematic picture of the framed part of chart (C) in Fig. 4 and the wandering of the relevant characteristic multipliers can be seen in Fig. 5. Through the parameter points 1-2-3, the critical pair of characteristic multipliers moves into the unit circle presenting a Hopf bifurcation at point 2. Through points 3-4-5, the complex pair of characteristic multipliers becomes real, then through 5-6-7, one of them moves out of the unit circle at \(-1 \) presenting a flip bifurcation at point 6. Through 7-8-9, the relevant real characteristic multiplier moves back into the unit circle presenting another flip bifurcation at point 8. Through 9-10-11, the two relevant real characteristic multipliers become a pair of complex conjugate roots again, and they decrease in modulus, while through 11-12-13, another complex pair of characteristic multipliers moves out of the unit circle in the positive half of the complex plane presenting another Hopf bifurcation at point 12.

The dashed curve in the chart of Fig. 5 presents the parameters where real characteristic multiplier occurs with multiplicity 2. This curve crosses the intersection of the two kinds of stability limits at \( 2^r \) presenting a degenerate (co-dimension 2) flip-Hopf bifurcation (see points \( 1'-2'-3' \)), and proceeds in the unstable domain (see points \( 1''-2''-3''-4''-5'' \)).

The arising chatter frequencies presented above the stability charts in Fig. 4 show an essential difference between turning and milling. While chatter in turning is characterized by a single frequency that is usually 0–15% above the first natural frequency \( f_n \) of the tool [9], chatter in milling is characterized by an infinite series of vibration frequencies [37] given either by Eq. (30) or by Eq. (31). This is due to the fact that turning is described by an autonomous DDE, while milling is described by a non-autonomous DDE.

In spite of this sharp qualitative difference, the transition between the turning and milling is still smooth. Special boundary points of the stability charts denoted by A, B, C, D, E and F in Fig. 4 were investigated. The dimensionless depths of cut are \( \bar{w} = 0.0205, 0.0205, 0.041, 0.0626, 0.138, 0.4505 \), respectively, while the dimensionless spindle speed is \( \Omega/(60f_n) = 1.3 \) for all these points. The vibration signals for these cases were determined by the Semi-Discretization method [29], and power spectra were calculated. The results are shown in Fig. 6.

The diagram (A) in Fig. 6 shows the well defined single chatter frequency of turning process at \( f_{f_n} = f_n \). If the number of active teeth is large, then the dynamics of the milling process is close to that of turning (since \( h(t) = 1 \)). This is valid for the case (B). In this case, there is a dominant chatter frequency at \( f_{f_n} = 1 \), and the
role of all the other frequencies are vanishing. As the number of active teeth decreases and the cutting becomes more and more interrupted, as it is for the cases (C), (D), (E) and (F), the amplitudes belonging to the lower and higher frequency peaks get larger. This explains that the transition between the frequency diagrams of turning and milling is continuous.

5 Discussion

For several decades, machine tool chatter research has had only very limited influence on manufacturing industry, it often had an academic nature. Vibration monitoring systems or adaptive control on machine tools were predicted to have much more industrial success. This was partly caused by the complexity of the models involved in the description of chatter, partly by the unreliable parameter identification of the machine tool structure and the cutting force itself. The improved experimental modal testing, the more sophisticated mechanical models, the latest mathematical results in nonlinear dynamics, and the use of computer algebra have made the latest research efforts more accessible for industrial applications during the last decade. This is true for industrial applications from the conventional turning [40], to the advanced high-speed milling [41].

The stability investigation of the milling process is extremely difficult due to the infinite dimensional phase space caused by the regenerative effect, and due to the parametric excitation caused by the time-varying number of active teeth. Via Semi-Discretization, the stability chart in the plane of the dimensionless technological parameters, and also the chatter frequencies were determined. The analysis showed that new unstable domains arise in case of high-speed milling associated with period doubling vibrations. The wandering of the relevant characteristic multipliers was illustrated. A series of stability charts and frequency diagrams were constructed presenting the transition from turning through general

![Fig. 4 Stability charts and chatter frequencies for various machining processes](image-url)
milling processes to highly interrupted milling. The continuity of this transition was explained by showing power spectra diagrams. The investigations confirm the results of Davies et al. [19] about the doubling in the number of stability lobes for highly interrupted cutting.

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References


