*Nonlinear Dynamics* **26:** 121–142, 2001.


# Subcritical Hopf Bifurcation in the Delay Equation Model for Machine Tool Vibrations

TAMÁS KALMÁR-NAGY
*Department of Theoretical and Applied Mechanics, Cornell University, Ithaca, NY 14853, U.S.A.*

GÁBOR STÉPÁN
*Department of Applied Mechanics, Technical University of Budapest, H-1521 Budapest, Hungary*

FRANCIS C. MOON
*Sibley School of Aerospace & Mechanical Engineering, Cornell University, Ithaca, NY 14853, U.S.A.*

(Received: 9 April 1998; accepted: 17 January 2001)

**Abstract.** We show the existence of a subcritical Hopf bifurcation in the delay-differential equation model of the so-called regenerative machine tool vibration. The calculation is based on the reduction of the infinite-dimensional problem to a two-dimensional center manifold. Due to the special algebraic structure of the delayed terms in the nonlinear part of the equation, the computation results in simple analytical formulas. Numerical simulations gave excellent agreement with the results.

**Keywords:** Hopf bifurcation, delay differential equation, center manifold, chatter.

## 1. Introduction

One of the most important effects causing poor surface quality in a cutting process is vibration arising from delay. Because of some external disturbances the tool starts a damped oscillation relative to the workpiece thus making its surface uneven. After one revolution of the workpiece the chip thickness will vary at the tool. The cutting force thus depends not only on the current position of the tool and the workpiece but also on a delayed value of the displacement. The length of this delay is the time-period $\tau$ of one revolution of the workpiece. This is the so-called regenerative effect (see, for example, [12, 16, 19, 20]). The corresponding mathematical model is a delay-differential equation. In order to study phenomena related to delay effects, a simple 1 DOF model of the tool was considered. Even though the model has only 1 DOF, the delay term makes the phase space infinite-dimensional.

Experimental results of Shi and Tobias [15] and Kalmár-Nagy et al. [9] clearly showed the existence of 'finite amplitude instability', that is unstable periodic motion of the tool around its asymptotically stable position related to the stationary cutting.

Recently, there has been increased interest in the subject. The Ph.D. theses of Johnson [8] and Fofana [4], and the paper of Nayfeh et al. [13] presented the analysis of the Hopf bifurcation in different models using different methods, like the method of multiple scales, harmonic balance, Floquet Theory (see also [14]) and of course, numerical simulations. New models have been proposed to explain chip segmentation by Burns and Davies [1]. This high-frequency process may also affect the tool dynamics.

The aim of this paper is to give a rigorous analytical investigation of the Hopf bifurcation present in the regenerative machine tool vibration model using the theory and tools of the Hopf

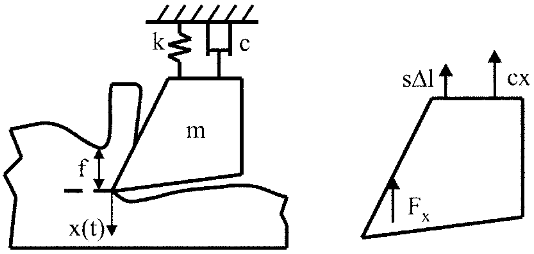


*Figure 1.* 1 DOF mechanical model.

Bifurcation Theorem and the Center Manifold Theorem. Although these have been available for a long time [5, 7] the closed form calculation regarding the existence and the nature of the corresponding Hopf bifurcation in the mathematical model is only feasible by using computer algebra (see also [2]).

## 2. Mechanical Model for Tool Vibrations

Figure 1 shows a 1 DOF mechanical model of the regenerative machine tool vibration in the case of the so-called orthogonal cutting ($f$ denotes chip thickness). The model is the simplest one which still explains the basic stability problems and nonlinear vibrations arising in this system [16–18]. The corresponding Free Body Diagram (ignoring horizontal forces) is also shown in Figure 1. Here $\Delta l = l - l_0 + x(t)$ where $l, l_0$ are the initial spring length and spring length in steady-state cutting, respectively. The zero value of the general coordinate $x(t)$ of the tool edge position is set in a way that the $x$ component $F_x$ of the cutting force $F$ is in balance with the spring force when the chip thickness $f$ is just the prescribed value $f_0$ (steady-state cutting). The equation of motion of the tool is clearly

$$m\ddot{x} = -s\Delta l - F_x - c\dot{x}. \tag{1}$$

In steady-state cutting ($x = \dot{x} = \ddot{x} = 0$)

$$0 = -s(l - l_0) - F_x(f_0) \Rightarrow F_x(f_0) = -s(l - l_0),$$

i.e., there is pre-stress in the spring. If we write $F_x = F_x(f_0) + \Delta F_x$ then Equation (1) becomes

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = -\frac{1}{m}\Delta F_x, \tag{2}$$

where $\omega_n = \sqrt{s/m}$ is the natural angular frequency of the undamped free oscillating system, and $\zeta = c/(2m\omega_n)$ is the so-called relative damping factor.

The calculation of the cutting force variation $\Delta F_x$ requires an expression of the cutting force as a function of the technological parameters, primarily as a function of the chip thickness $f$ which depends on the position $x$ of the tool edge. The traditional models [20, 21] use the cutting coefficient $k_1$ derived from the stationary idea of the cutting force as an empirical function of the technological parameters like the chip width $w$, the chip thickness $f$, and the cutting speed $v$.

A simple but empirical way to calculate the cutting force is using a curve fitted to data obtained from cutting tests. Shi and Tobias [15] gave a third-order polynomial for the cutting force (similar to Figure 2). The coefficient of the second-order term is negative which suggests

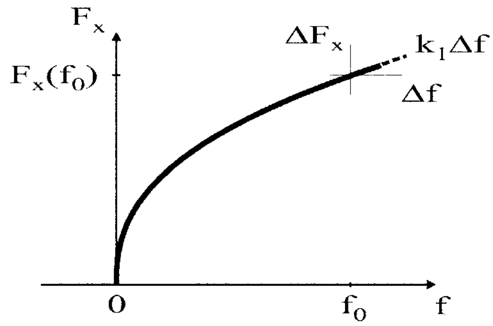


*Figure 2.* Cutting force and chip thickness relation.

a simple power law with exponent less than 1. Here we will use the formula given by Taylor [18]. According to this, the cutting force $F_x$ depends on the chip thickness as

$$F_x(f) = Kwf^{3/4},$$

where the parameter $K$ depends on further technological parameters considered to be constant in the present analysis. Expanding $F_x$ into a power series form around the desired chip thickness $f_0$ and keeping terms up to order 3 yields

$$F_x(f) \approx Kw\left(f_0^{3/4} + \frac{3}{4}(f - f_0)f_0^{-1/4} - \frac{3}{32}(f - f_0)^2 f_0^{-5/4} + \frac{5}{128}(f - f_0)^3 f_0^{-9/4}\right).$$

Or, equivalently, we can express the cutting force variation $\Delta F_x = F_x(f) - F_x(f_0)$ as the function of the chip thickness variation $\Delta f = f - f_0$, like

$$\Delta F_x(\Delta f) \approx Kw\left(\frac{3}{4}f_0^{-1/4}\Delta f - \frac{3}{32}f_0^{-5/4}\Delta f^2 + \frac{5}{128}f_0^{-9/4}\Delta f^3\right). \tag{3}$$

The coefficient of $\Delta f$ on the right-hand side of Equation (3) is usually called the *cutting force coefficient* and denoted by $k_1$ ($k_1 = (3/4)Kwf_0^{-1/4}$). Note, that $k_1$ is linearly proportional to the width $w$ of the chip, so in the upcoming calculations it will serve as a bifurcation parameter. Then Equation (3) can be rewritten as

$$\Delta F_x(\Delta f) \approx k_1\Delta f - \frac{1}{8}\frac{k_1}{f_0}\Delta f^2 + \frac{5}{96}\frac{k_1}{f_0^2}\Delta f^3.$$

Even though only the local bifurcation at $f = f_0 \Leftrightarrow \Delta f = 0$ will be investigated in this study we mention the case when the tool leaves the material, that is $f < 0 \Leftrightarrow \Delta f < -f_0$. In this case

$$\Delta F_x(\Delta f) = -F_x(f_0)$$

and so the regenerative effect is 'switched off' until the tool makes contact with the workpiece again.

The chip thickness variation $\Delta f$ can easily be expressed as the difference of the present tool edge position $x(t)$ and the delayed one $x(t - \tau)$ in the form

$$\Delta f = x(t) - x(t - \tau) = x - x_\tau,$$

where the delay $\tau = 2\pi/\Omega$ is the time period of one revolution with $\Omega$ being the constant angular velocity of the rotating workpiece. By bringing the linear terms to the left-hand side, Equation (2) becomes

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \left(\omega_n^2 + \frac{k_1}{m}\right)x - \frac{k_1}{m}x_\tau = \frac{k_1}{8f_0 m}\left((x - x_\tau)^2 - \frac{5}{12f_0}(x - x_\tau)^3\right).$$

Let us introduce the nondimensional time $\tilde{t}$ and displacement $\widetilde{x}$

$$\tilde{t} = \omega_n t, \quad \widetilde{x} = \frac{5}{12f_0}x,$$

and the nondimensional bifurcation parameter $p = k_1/(m\omega_n^2)$ (note that the nondimensional time delay is $\tilde{\tau} = \omega_n\tau$). Dropping the tilde we arrive at

$$\ddot{x} + 2\zeta\dot{x} + (1 + p)x - px_\tau = \frac{3p}{10}((x - x_\tau)^2 - (x - x_\tau)^3).$$

This second-order equation is transformed into a two-dimensional system by introducing

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix}$$

and we obtain the delay-differential equation

$$\dot{\mathbf{x}}(t) = \mathbf{L}(p)\mathbf{x}(t) + \mathbf{R}(p)\mathbf{x}(t - \tau) + \mathbf{f}(\mathbf{x}(t), p), \tag{4}$$

where the dependence on the bifurcation parameter $p$ is also emphasized:

$$\mathbf{L}(p) = \begin{pmatrix} 0 & 1 \\ -(1 + p) & -2\zeta \end{pmatrix},$$

$$\mathbf{R}(p) = \begin{pmatrix} 0 & 0 \\ p & 0 \end{pmatrix},$$

$$\mathbf{f}(\mathbf{x}(t), p) = \frac{3p}{10}\begin{pmatrix} 0 \\ (x_1(t) - x_1(t - \tau))^2 - (x_1(t) - x_1(t - \tau))^3 \end{pmatrix}. \tag{5}$$

## 3. Linear Stability Analysis

The characteristic function of Equation (4) can be obtained by substituting the trial solution $\mathbf{x}(t) = \mathbf{c}\exp(\lambda t)$ into its linear part:

$$\begin{aligned} D(\lambda, p) &= \det(\lambda\mathbf{I} - \mathbf{L}(p) - \mathbf{R}(p)\,\mathrm{e}^{-\lambda\tau}) \\ &= \lambda^2 + 2\zeta\lambda + (1 + p) - p\,\mathrm{e}^{-\lambda\tau}. \end{aligned} \tag{6}$$

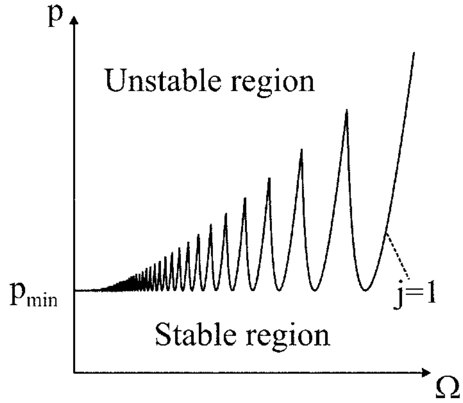


*Figure 3.* Stability chart.

The necessary condition for the existence of a nonzero solution is

$$D(\lambda, p) = 0.$$

On the stability boundary shown in Figure 3 the characteristic equation has one pair of pure imaginary roots (except the intersections of the lobes, where it has two pairs of imaginary roots). To find this curve, we substitute $\lambda = i\omega$, $\omega > 0$ into Equation (6).

$$D(i\omega, p) = 1 + p - \omega^2 - p\cos\omega\tau + i(2\zeta\omega + p\sin\omega\tau) = 0.$$

This complex equation is equivalent to the two real equations

$$1 - \omega^2 + p(1 - \cos\omega\tau) = 0, \tag{7}$$

$$2\zeta\omega + p\sin\omega\tau = 0. \tag{8}$$

The trigonometric terms in Equations (7) and (8) can be eliminated to yield

$$p = \frac{(1-\omega^2)^2 + 4\zeta^2\omega^2}{2(\omega^2 - 1)}.$$

Since $p > 0$ this also implies $\omega > 1$.

With the help of the trigonometric identity

$$\frac{1 - \cos\omega\tau}{\sin\omega\tau} = \tan\frac{\omega\tau}{2}$$

$\tau$ can be expressed from Equations (7) and (8) as

$$\tau = \frac{2}{\omega}\left(j\pi - \arctan\frac{\omega^2 - 1}{2\zeta\omega}\right), \quad j = 1, 2, \ldots,$$

where $j$ corresponds to the $j$th ‘lobe’ (parameterized by $\omega$) from the right in the stability diagram ($j$ must be greater than 0, because $\tau > 0$). And finally

$$\Omega = \frac{2\pi}{\tau} = \frac{\omega\pi}{j\pi - \arctan\frac{\omega^2-1}{2\zeta\omega}}, \quad j = 1, 2, \ldots.$$

At the minima ('notches') of the stability boundary $\omega$, $p$, $\Omega$ assume particularly simple forms. To find these values $\mathrm{d}p/\mathrm{d}\omega = 0$ has to be solved. Then we find

$$\omega = \sqrt{1+2\zeta}, \quad p = 2\zeta(\zeta+1),$$

$$\tau = \frac{2\left(j\pi - \arctan\frac{1}{\sqrt{1+2\zeta}}\right)}{\sqrt{1+2\zeta}}, \quad j = 1, 2, \ldots,$$

$$\Omega = \frac{\sqrt{1+2\zeta}\,\pi}{j\pi - \arctan\frac{1}{\sqrt{1+2\zeta}}}, \quad j = 1, 2, \ldots. \tag{9}$$

To simplify calculations we present results obtained at these parameter values (the calculations can be carried out in general along the stability boundary, see [10]).

The location of the characteristic roots has now to be established. For $p = 0$ Equation (6) has only two roots ($\lambda_{1,2} = -\zeta \pm i\sqrt{1-\zeta^2}$) and these are located in the left half plane. Increasing the value of $p$ results in characteristic roots 'swarming out' from minus complex infinity (the north pole of the Riemann-sphere). So for small $p$ all the characteristic roots are in the left half plane (this can be proved with Rouché's Theorem, see [10]).

The necessary condition for the existence of periodic orbits is that by varying the bifurcation parameter ($p$) the critical characteristic roots cross the imaginary axis with nonzero velocity, that is

$$\mathrm{Re}\,\frac{\mathrm{d}\lambda(p_{\mathrm{cr}})}{\mathrm{d}p} \neq 0.$$

The characteristic function (6) has two zeros $\lambda = \pm i\sqrt{1+2\zeta}$ at the notches.

The change of the real parts of these critical characteristic roots can be determined via implicit differentiation of the characteristic function (6) with respect to the bifurcation parameter $p$

$$\frac{\mathrm{d}D(\lambda(p), p)}{\mathrm{d}p} = \frac{\partial D(\lambda(p), p)}{\partial p} + \frac{\partial D(\lambda(p), p)}{\partial \lambda}\frac{\mathrm{d}\lambda(p)}{\mathrm{d}p} = 0,$$

$$\frac{\mathrm{d}\lambda(p_{\mathrm{cr}})}{\mathrm{d}p} = -\left.\frac{\frac{\partial D(\lambda(p),p)}{\partial p}}{\frac{\partial D(\lambda(p),p)}{\partial \lambda}}\right|_{\lambda = i\sqrt{1+2\zeta}},$$

$$\gamma := \mathrm{Re}\,\frac{\mathrm{d}\lambda(p_{\mathrm{cr}})}{\mathrm{d}p} = \frac{1}{2(1+\zeta)^2(1+\zeta\tau)}. \tag{10}$$

Since $\gamma$ is always positive and all the characteristic roots but the critical ones of (6) are located in the left half complex plane, the conditions of an infinite-dimensional version of the Hopf Bifurcation Theorem given in [7] are satisfied. $\gamma$ will later be used in the estimation of the vibration amplitude.

## 4. Operator Differential Equation Formulation

In order to study the *critical* infinite-dimensional problem on a two-dimensional center manifold we need the operator differential equation representation of Equation (4).

This delay-differential equation can be expressed as the abstract evolution equation [2, 6, 11] on the Banach space $\mathcal{H}$ of continuously differentiable functions $\mathbf{u} : [-\tau, 0] \to \mathbb{R}^2$

$$\dot{\mathbf{x}}_t = \mathcal{A}\mathbf{x}_t + \mathcal{F}(\mathbf{x}_t). \tag{11}$$

Here $\mathbf{x}_t(\varphi) \in \mathcal{H}$ is defined by the shift of time

$$\mathbf{x}_t(\varphi) = \mathbf{x}(t+\varphi), \quad \varphi \in [-\tau, 0]. \tag{12}$$

The linear operator $\mathcal{A}$ *at the critical value of the bifurcation parameter* assumes the form

$$\mathcal{A}\mathbf{u}(\vartheta) = \begin{cases} \frac{\mathrm{d}}{\mathrm{d}\vartheta}\mathbf{u}(\vartheta), & \vartheta \in [-\tau, 0), \\ \mathbf{L}\mathbf{u}(0) + \mathbf{R}\mathbf{u}(-\tau), & \vartheta = 0, \end{cases}$$

while the nonlinear operator $\mathcal{F}$ can be written as

$$\mathcal{F}(\mathbf{u})(\vartheta) = \begin{cases} \mathbf{0}, & \vartheta \in [-\tau, 0), \\ \frac{3p}{10}\begin{pmatrix} 0 \\ (u_1(0) - u_1(-\tau))^2 - (u_1(0) - u_1(-\tau))^3 \end{pmatrix}, & \vartheta = 0, \end{cases}$$

where $\mathbf{u} \in \mathcal{H}$ (cf. Equation (5)).

The adjoint space $\mathcal{H}^*$ of continuously differentiable functions $\mathbf{v} : [0, \tau] \to \mathbb{R}^2$ with the adjoint operator

$$\mathcal{A}^*\mathbf{v}(\sigma) = \begin{cases} -\frac{\mathrm{d}}{\mathrm{d}\sigma}\mathbf{v}(\sigma), & \sigma \in (0, \tau], \\ \mathbf{L}^*\mathbf{v}(0) + \mathbf{R}^*\mathbf{v}(\tau), & \sigma = 0, \end{cases}$$

is also needed as well as the bilinear form $(\,,\,) : \mathcal{H}^* \times \mathcal{H} \to \mathbb{R}$ defined by

$$(\mathbf{v}, \mathbf{u}) = \mathbf{v}^*(0)\mathbf{u}(0) + \int_{-\tau}^{0} \mathbf{v}^*(\xi + \tau)\mathbf{R}\mathbf{u}(\xi)\,\mathrm{d}\xi. \tag{13}$$

The formal adjoint and the bilinear form provide the basis for a geometry in which it is possible to develop a projection using the basis eigenvectors of the formal adjoint. The significance of this projection is that the critical delay system has a two-dimensional attractive subsystem (the center manifold) and the solutions on this manifold determine the long time behavior of the full system. For a heuristic argument of how these operators and bilinear form arise, see the Appendix. The mathematically inclined can study [6].

Since the critical eigenvalues of the linear operator $\mathcal{A}$ just coincide with the critical characteristic roots of the characteristic function $D(\lambda, p)$, the Hopf bifurcation arising at the degenerate trivial solution can be studied on a two-dimensional center manifold embedded in the infinite-dimensional phase space.

A first-order approximation to this center manifold can be given by the center subspace of the associated linear problem, which is spanned by the real and imaginary parts $\mathbf{s}_1$, $\mathbf{s}_2$ of the *complex* eigenfunction $\mathbf{s}(\vartheta) \in \mathcal{H}$ corresponding to the critical characteristic root $i\omega$. This eigenfunction satisfies

$$\mathcal{A}\mathbf{s}(\vartheta) = i\omega\mathbf{s}(\vartheta),$$

that is

$$\mathcal{A}(\mathbf{s}_1(\vartheta) + i\mathbf{s}_2(\vartheta)) = i\omega(\mathbf{s}_1(\vartheta) + i\mathbf{s}_2(\vartheta)).$$

Separating the real and imaginary parts yields

$$\mathcal{A}\mathbf{s}_1(\vartheta) = -\omega\mathbf{s}_2(\vartheta),$$

$$\mathcal{A}\mathbf{s}_2(\vartheta) = \omega\mathbf{s}_1(\vartheta).$$

Using the definition of $\mathcal{A}$ results the following boundary value problem

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\vartheta}\mathbf{s}_1(\vartheta) &= -\omega\mathbf{s}_2(\vartheta),\\ \frac{\mathrm{d}}{\mathrm{d}\vartheta}\mathbf{s}_2(\vartheta) &= \omega\mathbf{s}_1(\vartheta),\end{aligned} \tag{14}$$

$$\begin{aligned}\mathbf{L}\mathbf{s}_1(0) + \mathbf{R}\mathbf{s}_1(-\tau) &= -\omega\mathbf{s}_2(0),\\ \mathbf{L}\mathbf{s}_2(0) + \mathbf{R}\mathbf{s}_2(-\tau) &= \omega\mathbf{s}_1(0).\end{aligned} \tag{15}$$

The general solution to the differential equation (14) is

$$\begin{aligned}\mathbf{s}_1(\vartheta) &= \cos(\omega\vartheta)\mathbf{c}_1 - \sin(\omega\vartheta)\mathbf{c}_2,\\ \mathbf{s}_2(\vartheta) &= \sin(\omega\vartheta)\mathbf{c}_1 + \cos(\omega\vartheta)\mathbf{c}_2.\end{aligned}$$

The boundary conditions (15) result in a system of linear equations for some of the unknown coefficients:

$$\begin{pmatrix}\mathbf{L} + \cos(\omega\tau)\mathbf{R} & \omega\mathbf{I} + \sin(\omega\tau)\mathbf{R}\end{pmatrix}\begin{pmatrix}\mathbf{c}_1\\ \mathbf{c}_2\end{pmatrix} = \mathbf{0}. \tag{16}$$

The center manifold reduction also requires the calculation of the 'left-hand side' critical real eigenfunctions $\mathbf{n}_{1,2}$ of $\mathcal{A}$ that satisfy the adjoint problem

$$\begin{aligned}\mathcal{A}^*\mathbf{n}_1(\sigma) &= \omega\mathbf{n}_2(\sigma),\\ \mathcal{A}^*\mathbf{n}_2(\sigma) &= -\omega\mathbf{n}_1(\sigma).\end{aligned}$$

This boundary value problem has the general solution

$$\begin{aligned}\mathbf{n}_1(\sigma) &= \cos(\omega\sigma)\mathbf{d}_1 - \sin(\omega\sigma)\mathbf{d}_2,\\ \mathbf{n}_2(\sigma) &= \sin(\omega\sigma)\mathbf{d}_1 + \cos(\omega\sigma)\mathbf{d}_2,\end{aligned}$$

while the boundary conditions simplify to

$$\begin{pmatrix}\mathbf{L}^T + \cos(\omega\tau)\mathbf{R}^T & -\omega\mathbf{I} - \sin(\omega\tau)\mathbf{R}^T\end{pmatrix}\begin{pmatrix}\mathbf{d}_1\\ \mathbf{d}_2\end{pmatrix} = \mathbf{0}. \tag{17}$$

With the help of the bilinear form (13), the 'orthonormality' conditions

$$(\mathbf{n}_1, \mathbf{s}_1) = 1, \quad (\mathbf{n}_1, \mathbf{s}_2) = 0 \tag{18}$$

provide two more equations.

Since Equations (16–18) do not determine the unknown coefficients uniquely (8 unknowns and 6 equations) we can choose two of them freely (so that the others will be of simple form)

$$c_{11} = 1, \quad c_{21} = 0.$$

Then

$$\mathbf{c}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \mathbf{c}_2 = \begin{pmatrix} 0 \\ \omega \end{pmatrix},$$

$$\mathbf{d}_1 = 2\gamma \begin{pmatrix} 2\zeta^2 + 2\zeta + 1 \\ \zeta \end{pmatrix}, \quad \mathbf{d}_2 = 2\gamma\omega \begin{pmatrix} \zeta \\ 1 \end{pmatrix}.$$

Let us decompose the solution $\mathbf{x}_t(\vartheta)$ of Equation (11) into two components $y_{1,2}$ lying in the center subspace and into the infinite-dimensional component $\mathbf{w}$ transverse to the center subspace:

$$\mathbf{x}_t(\vartheta) = y_1(t)\mathbf{s}_1(\vartheta) + y_2(t)\mathbf{s}_2(\vartheta) + \mathbf{w}(t)(\vartheta), \tag{19}$$

where

$$y_1(t) = (\mathbf{n}_1, \mathbf{x}_t)\,|_{\vartheta=0}\,, \quad y_2(t) = (\mathbf{n}_2, \mathbf{x}_t)\,|_{\vartheta=0}\,.$$

With these new coordinates the operator differential equation (11) can be transformed into a 'canonical form' (see the Appendix)

$$\dot{y}_1 = (\mathbf{n}_1, \dot{\mathbf{x}}_t) = \omega y_2 + \mathbf{n}_1^T(0)\mathbf{F}, \tag{20}$$

$$\dot{y}_2 = (\mathbf{n}_2, \dot{\mathbf{x}}_t) = -\omega y_1 + \mathbf{n}_2^T(0)\mathbf{F}, \tag{21}$$

$$\dot{\mathbf{w}} = \mathcal{A}\mathbf{w} + \mathcal{F}(\mathbf{x}_t) - \mathbf{n}_1^T(0)\mathbf{F}\mathbf{s}_1 - \mathbf{n}_2^T(0)\mathbf{F}\mathbf{s}_2, \tag{22}$$

where

$$\mathbf{F} = \mathcal{F}(y_1(t)\mathbf{s}_1(0) + y_2(t)\mathbf{s}_2(0) + \mathbf{w}(t)(0))$$

and in Equation (22) the decomposition (19) should be substituted for $\mathbf{x}_t$. Equations (4) and (5) give rise to the nonlinear operator

$$\mathcal{F}(\mathbf{w} + y_1\mathbf{s}_1 + y_2\mathbf{s}_2)(\vartheta) = \begin{cases} \mathbf{0}, & \vartheta \in [-\tau, 0), \\ \frac{3}{5}\zeta z \begin{pmatrix} 0 \\ \frac{z}{1+\zeta} + 2(w_1(0) - w_1(-\tau)) - \left(\frac{z}{1+\zeta}\right)^2 \end{pmatrix}, & \vartheta = 0, \end{cases} \tag{23}$$

where $z = y_1 - \omega y_2$ and the terms of fourth or higher order were neglected (since $\mathbf{w}(y_1, y_2)$ is second order in Equation (24) and the normal form (35) will only contain terms up to third order).

## 5. Two-Dimensional Center Manifold

The center manifold is tangent to the plane $y_1$, $y_2$ at the origin, and it is locally invariant and attractive to the flow of system (11). Since the nonlinearities considered here are nonsymmetric, we have to compute the second-order Taylor-series expansion of the center manifold. Thus, its equation can be assumed in the form of the truncated power series

$$\mathbf{w}(y_1, y_2)(\vartheta) = \frac{1}{2}(\mathbf{h}_1(\vartheta)y_1^2 + 2\mathbf{h}_2(\vartheta)y_1y_2 + \mathbf{h}_3(\vartheta)y_2^2). \tag{24}$$

The time derivative of $\mathbf{w}$ can be expressed both by differentiating the right-hand side of Equation (24) via substituting Equations (20) and (21)

$$\begin{aligned}
\dot{\mathbf{w}} &= \mathbf{h}_1 y_1 \dot{y}_1 + \mathbf{h}_2 y_2 \dot{y}_1 + \mathbf{h}_2 y_1 \dot{y}_2 + \mathbf{h}_3 y_2 \dot{y}_2 \\
&= \dot{y}_1(\mathbf{h}_1 y_1 + \mathbf{h}_2 y_2) + \dot{y}_2(\mathbf{h}_2 y_1 + \mathbf{h}_3 y_2) \\
&= (\omega y_2 + d_{12} f)(\mathbf{h}_1 y_1 + \mathbf{h}_2 y_2) + (-\omega y_1 + d_{22} f)(\mathbf{h}_2 y_1 + \mathbf{h}_3 y_2) \\
&= -\omega \mathbf{h}_2 y_1^2 + \omega(\mathbf{h}_1 - \mathbf{h}_3) y_1 y_2 + \omega \mathbf{h}_2 y_2^2 + o(y^3),
\end{aligned}$$

where $f = (0 \quad 1) \cdot \mathbf{F}$ and also by calculating Equation (22)

$$\frac{\mathrm{d}\mathbf{w}}{\mathrm{d}t} = \mathcal{A}\mathbf{w} + \mathcal{F}(\mathbf{w} + y_1\mathbf{s}_1 + y_2\mathbf{s}_2) - (d_{12}\mathbf{s}_1 + d_{22}\mathbf{s}_2)\, f,$$

where

$$\mathcal{A}\mathbf{w} = \begin{cases} \frac{1}{2}(\dot{\mathbf{h}}_1 y_1^2 + 2\dot{\mathbf{h}}_2 y_1 y_2 + \dot{\mathbf{h}}_3 y_2^2), & \vartheta \in [-\tau, 0), \\ \mathbf{L}\mathbf{w}(0) + \mathbf{R}\mathbf{w}(-\tau), & \vartheta = 0, \end{cases}$$

$$\begin{aligned}
\mathbf{L}\mathbf{w}(0) + \mathbf{R}\mathbf{w}(-\tau) &= \frac{1}{2} y_1^2 (\mathbf{L}\mathbf{h}_1(0) + \mathbf{R}\mathbf{h}_1(-\tau)) \\
&\quad + y_1 y_2 (\mathbf{L}\mathbf{h}_2(0) + \mathbf{R}\mathbf{h}_2(-\tau)) + \frac{1}{2} y_2^2 (\mathbf{L}\mathbf{h}_3(0) + \mathbf{R}\mathbf{h}_3(-\tau)).
\end{aligned}$$

Equating like coefficients of the second degree expressions $y_1^2$, $y_1y_2$, $y_2^2$ we obtain a six-dimensional linear boundary value problem for the unknown coefficients $\mathbf{h}_1, \mathbf{h}_2, \mathbf{h}_3$

$$\begin{aligned}
&\frac{1}{2}\dot{\mathbf{h}}_1 = -\omega \mathbf{h}_2 + f_{11}(d_{12}\mathbf{s}_1(\vartheta) + d_{22}\mathbf{s}_2(\vartheta)), \\
&\dot{\mathbf{h}}_2 = \omega \mathbf{h}_1 - \omega \mathbf{h}_3 + f_{12}(d_{12}\mathbf{s}_1(\vartheta) + d_{22}\mathbf{s}_2(\vartheta)), \\
&\frac{1}{2}\dot{\mathbf{h}}_3 = \omega \mathbf{h}_2 + f_{22}(d_{12}\mathbf{s}_1(\vartheta) + d_{22}\mathbf{s}_2(\vartheta)),
\end{aligned} \tag{25}$$

$$\begin{aligned}
&\frac{1}{2}(\mathbf{L}\mathbf{h}_1(0) + \mathbf{R}\mathbf{h}_1(-\tau)) = -\omega \mathbf{h}_2(0) + f_{11}(d_{12}\mathbf{s}_1(0) + d_{22}\mathbf{s}_2(0)), \\
&\mathbf{L}\mathbf{h}_2(0) + \mathbf{R}\mathbf{h}_2(-\tau) = \omega \mathbf{h}_1(0) - \omega \mathbf{h}_3(0) + f_{12}(d_{12}\mathbf{s}_1(0) + d_{22}\mathbf{s}_2(0)), \\
&\frac{1}{2}(\mathbf{L}\mathbf{h}_3(0) + \mathbf{R}\mathbf{h}_3(-\tau)) = \omega \mathbf{h}_2(0) + f_{22}(d_{12}\mathbf{s}_1(0) + d_{22}\mathbf{s}_2(0)),
\end{aligned} \tag{26}$$

where the $f_{ij}$'s denote the partial derivatives of $f$ (with the appropriate multiplier) evaluated at $y_1 = y_2 = 0$ (thus giving the coefficient of the corresponding quadratic term)

$$f_{11} = \frac{1}{2}\frac{\partial^2 f}{\partial y_1^2}\bigg|_{\mathbf{0}}, \quad f_{12} = \frac{\partial^2 f}{\partial y_1 y_2}\bigg|_{\mathbf{0}}, \quad f_{22} = \frac{1}{2}\frac{\partial^2 f}{\partial y_2^2}\bigg|_{\mathbf{0}}.$$

Introducing the following notation

$$\mathbf{h} := \begin{pmatrix} \mathbf{h}_1 \\ \mathbf{h}_2 \\ \mathbf{h}_3 \end{pmatrix}, \quad \mathbf{C}_{6\times 6} = \omega \begin{pmatrix} \mathbf{0} & -2\mathbf{I} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} & -\mathbf{I} \\ \mathbf{0} & 2\mathbf{I} & \mathbf{0} \end{pmatrix},$$

$$\mathbf{p} = \begin{pmatrix} f_{11}\mathbf{p}_0 \\ f_{12}\mathbf{p}_0 \\ f_{22}\mathbf{p}_0 \end{pmatrix}, \quad \mathbf{q} = \begin{pmatrix} f_{11}\mathbf{q}_0 \\ f_{12}\mathbf{q}_0 \\ f_{22}\mathbf{q}_0 \end{pmatrix},$$

$$\mathbf{p}_0 = \begin{pmatrix} d_{12} \\ c_{22}d_{22} \end{pmatrix}, \quad \mathbf{q}_0 = \begin{pmatrix} d_{22} \\ -c_{22}d_{12} \end{pmatrix}.$$

Equation (25) can be written as the inhomogeneous differential equation

$$\frac{\mathrm{d}}{\mathrm{d}\vartheta}\mathbf{h} = \mathbf{C}\mathbf{h} + \mathbf{p}\cos(\omega\vartheta) + \mathbf{q}\sin(\omega\vartheta). \tag{27}$$

The general solution of Equation (27) assumes the usual form

$$\mathbf{h}(\vartheta) = \mathrm{e}^{\mathbf{C}\vartheta}\mathbf{K} + \mathbf{M}\cos(\omega\vartheta) + \mathbf{N}\sin(\omega\vartheta). \tag{28}$$

The coefficients $\mathbf{M}$, $\mathbf{N}$ of the nonhomogeneous part are obtained after substituting this solution back to Equation (27) resulting in a 12-dimensional inhomogeneous linear algebraic system

$$\begin{pmatrix} \mathbf{C}_{6\times 6} & -\omega\mathbf{I}_{6\times 6} \\ \omega\mathbf{I}_{6\times 6} & \mathbf{C}_{6\times 6} \end{pmatrix}\begin{pmatrix} \mathbf{M} \\ \mathbf{N} \end{pmatrix} = -\begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix}. \tag{29}$$

Since we will only need the first component $w_1$ of $\mathbf{w}(y_1, y_2)(\vartheta)$ (see Equation (23)) we have to calculate only the first, third and fifth component of $\mathbf{M}$, $\mathbf{N}$, $\mathbf{K}$. From Equation (29)

$$\begin{pmatrix} M_1 \\ M_3 \\ M_5 \end{pmatrix} = \frac{-5(1+\zeta)\omega}{4\zeta\gamma}\begin{pmatrix} \omega(3+2\zeta \\ 2\zeta^2 + 2\zeta + 1 \\ \omega(3+4\zeta) \end{pmatrix}, \tag{30}$$

$$\begin{pmatrix} N_1 \\ N_3 \\ N_5 \end{pmatrix} = \frac{5(1+\zeta)\omega}{4\zeta\gamma}\begin{pmatrix} 2 + 7\zeta + 4\zeta^2 \\ -\zeta\omega \\ 2\zeta^2 - \zeta - 2 \end{pmatrix}. \tag{31}$$

The boundary condition for $\mathbf{h}$ associated with Equation (27) comes from those parts of Equation (22) where $\mathcal{A}$, $\mathcal{F}$ are defined at $\vartheta = 0$. It is

$$\mathbf{P}\mathbf{h}(0) + \mathbf{Q}\mathbf{h}(-\tau) = \mathbf{p} + \mathbf{r} \tag{32}$$

with

$$\mathbf{P}_{6\times 6} = \begin{pmatrix} \mathbf{L} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{L} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{L} \end{pmatrix} - \mathbf{C}_{6\times 6}, \quad \mathbf{Q}_{6\times 6} = \begin{pmatrix} \mathbf{R} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{R} \end{pmatrix},$$

$$\mathbf{r} = -\begin{pmatrix} 0 & f_{11} & 0 & f_{12} & 0 & f_{22} \end{pmatrix}^T .$$

**K** is found by substituting the general solution (28) into Equation (32)

$$(\mathbf{P} + \mathbf{Q}\,\mathrm{e}^{-\tau\mathbf{C}})\mathbf{K} = \mathbf{r} + \mathbf{p} - \mathbf{PM} - \mathbf{Q}(\cos(\omega\tau)\mathbf{M} - \sin(\omega\tau)\mathbf{N}). \tag{33}$$

Despite its hideous look Equation (33) simplifies, because

$$\mathbf{p} - \mathbf{PM} - \mathbf{Q}(\cos(\omega\tau)\mathbf{M} - \sin(\omega\tau)\mathbf{N}) = \mathbf{0}.$$

For our system

$$\begin{pmatrix} K_1 \\ K_3 \\ K_5 \end{pmatrix} = \frac{6\zeta}{5(9 + 33\zeta + 32\zeta^2)} \begin{pmatrix} 9 + 32\zeta + 32\zeta^2 \\ \omega(3 + 4\zeta) \\ 9 + 34\zeta + 32\zeta^2 \end{pmatrix}. \tag{34}$$

Finally, Equations (30, 31, 34) are substituted into Equation (28) resulting in the second-order approximation of the center manifold (24). It is not necessary to express the center manifold approximation in its full form, since we only need the values of its components at $\vartheta = 0$ and $-\tau$ in the transformed operator equation (20, 21, 22). For example,

$$w_1(0) = \frac{1}{2}((M_1 + K_1)y_1^2 + 2(M_3 + K_3)y_1 y_2 + (M_5 + K_5)y_2^2),$$

while the expression for $w_1(-\tau)$ is somewhat more lengthy.

## 6. The Hopf Bifurcation

In order to restrict a third-order approximation of system (20–22) to the two-dimensional center manifold calculated in the previous section, the second-order approximation $\mathbf{w}(y_1, y_2)$ of the center manifold has to be substituted into the two scalar equations (20) and (21). Then these equations will assume the form

$$\begin{aligned} \dot{y}_1 &= \omega y_2 + a_{20}y_1^2 + a_{11}y_1y_2 + a_{02}y_2^2 + a_{30}y_1^3 + a_{21}y_1^2y_2 + a_{12}y_1y_2^2 + a_{03}y_2^3, \\ \dot{y}_2 &= -\omega y_1 + b_{20}y_1^2 + b_{11}y_1y_2 + b_{02}y_2^2 + b_{30}y_1^3 + b_{21}y_1^2y_2 + b_{12}y_1y_2^2 + b_{03}y_2^3. \end{aligned} \tag{35}$$

Using 10 out of these 14 coefficients $a_{jk}, b_{jk}$, the so-called Poincaré–Lyapunov constant $\Delta$ can be calculated as shown in [5] or [7]

$$\begin{aligned} \Delta &= \frac{1}{8\omega}[(a_{20} + a_{02})(-a_{11} + b_{20} - b_{02}) + (b_{20} + b_{02})(a_{20} - a_{02} + b_{11})] \\ &\quad + \frac{1}{8}(3a_{30} + a_{12} + b_{21} + 3b_{03}). \end{aligned}$$

The negative/positive sign of $\Delta$ determines if the Hopf bifurcation is supercritical or subcritical. Despite the above described tedious calculations $\Delta$ is quite simple:

$$\Delta = \frac{9\zeta\gamma}{50}\frac{45 + 177\zeta + 196\zeta^2 + 24\zeta^3}{9 + 33\zeta + 32\zeta^2} > 0.$$

This means that the Hopf bifurcation is *subcritical*, that is *unstable periodic motion* exists around the stable steady state cutting for cutting coefficients $p$ which are somewhat smaller than the critical value $p_{\mathrm{cr}}$. This unstable limit cycle determines the domain of attraction of the asymptotically stable stationary cutting.

The estimation of the vibration amplitude has the simple form

$$r = \sqrt{-\frac{\gamma}{\Delta}(p - p_{\mathrm{cr}})} = \sqrt{\frac{\gamma p_{\mathrm{cr}}}{\Delta}}\sqrt{1 - \frac{p}{p_{\mathrm{cr}}}}.$$

The approximation of the corresponding periodic solution of the original operator differential equation (11) can be obtained from the definition (12) of $\mathbf{x}_t$ as

$$\begin{aligned}\mathbf{x}_t(\vartheta) &= \mathbf{x}(t+\vartheta) = y_1(t)\mathbf{s}_1(\vartheta) + y_2(t)\mathbf{s}_2(\vartheta) + \mathbf{w}(t)(\vartheta)\\ &\approx r(\cos(\omega t)\mathbf{s}_1(\vartheta) - \omega\sin(\omega t)\mathbf{s}_2(\vartheta)).\end{aligned}$$

The periodic solution of the delay-differential equation (4) can be obtained in the form

$$\begin{aligned}\mathbf{x}(t) &= \mathbf{x}_t(0) \approx y_1(t)\mathbf{s}_1(0) + y_2(t)\mathbf{s}_2(0)\\ &= r\begin{pmatrix}\cos(\omega t)\\ -\omega\sin(\omega t)\end{pmatrix}.\end{aligned} \tag{36}$$

Since the relative damping factor $\zeta$ is usually far less than 1 in realistic machine tool structures, the first-order Taylor expansion of the amplitude $r$ with respect to $\zeta$ is also a good approximation

$$r \approx \frac{30 + 11\zeta}{9\sqrt{5}}\sqrt{1 - \frac{p}{p_{\mathrm{cr}}}}.$$

Let us transform the nondimensonal time and displacement back to the original ones. Selecting the first coordinate of Equation (36), we obtain the approximate form of the unstable periodic motion embedded in the regenerative machine tool vibrations for $p < p_{\mathrm{cr}}$ (see also [17])

$$x(t) \approx \frac{4}{15\sqrt{5}}\frac{30 + 11\zeta}{f_0}\sqrt{1 - \frac{p}{p_{\mathrm{cr}}}}\cos(\omega_n\sqrt{1 + 2\zeta}\, t).$$

## 7. Numerical Results

The results of the above sections were confirmed numerically. Simulations with

$$\zeta = 0.1, \quad j = 1$$

were carried out (see Equations (9)). The full delay equation (4, 5) was integrated in Mathematica. To find the amplitude of the unstable limit cycle this equation was integrated with

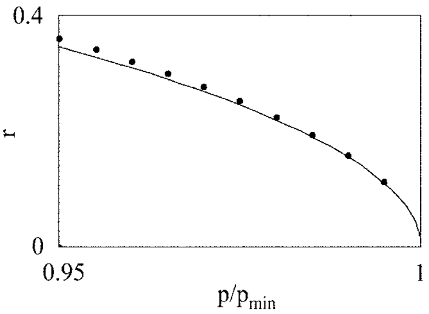


*Figure 4.* Bifurcation diagram.

sinusoidal initial functions of different amplitudes. The growth or decay of the solution (after some transient) decides whether the solution is 'outside' or 'inside' of the unstable limit cycle. Using a bisection routine allows the computation of the location of the unstable limit cycle with good accuracy. The bifurcation diagram (presenting the amplitude of the unstable limit cycle vs the normalized bifurcation parameter) is shown in Figure 4, together with the analytical approximation (solid line). The agreement is excellent.

## 8. Conclusions

The existence and nature of a Hopf bifurcation in the delay-differential equation for self-excited tool vibration is presented and proved analytically with the help of the Center Manifold and Hopf Bifurcation Theory. The simple results are due to the special structure of the nonlinearities considered in the cutting force dependence on the chip thickness. On the other hand this analysis is local in the sense that it does not account for nonlinear phenomena as the tool leaves the material. In this case the regenerative effect disappears, and the result of the local analysis is not valid anymore [3, 9].

Finally, the semi-analytical and numerical results of Nayfeh et al. [13] show some cases where a slight supercritical bifurcation appears before the birth of the unstable limit cycle, and they present also some robust supercritical Hopf bifurcations. These results were calculated at critical parameter values somewhat away from the 'notches' of the stability chart chosen in this study. The model considered there also contained structural nonlinearities.

## Appendix

### Canonical Form for Ordinary and Delay Differential Equations

In this Appendix we will show an analogy between ordinary and delay differential equations thus motivating the definitions of the differential operator, its adjoint and the bilinear form used to investigate Hopf bifurcation in delay equations.

In particular it is shown that the time-delay problem leads to an operator that is the generalization of the defining matrix in a system of ODEs with constant coefficients.

CANONICAL FORM FOR ODES

Here we assume that $\mathbf{x} = \mathbf{0}$ is a fixed point (this can always be achieved with a translation) of the system of nonlinear ordinary differential equations

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{f}(\mathbf{x}(t)) \tag{A.1}$$

and the matrix $\mathbf{A}$ has a pair of pure imaginary eigenvalues $\pm i\omega$ while all its other eigenvalues are 'stable' (have real part less than zero). The linear part $\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t)$ of Equation (A.1) can be transformed into a form that makes the behavior of the solutions more transparent. This is achieved by using the transformation

$$\mathbf{y}(t) = \mathbf{B}\mathbf{x}(t). \tag{A.2}$$

Then Equation (A.1) can be rewritten in terms of the new coordinates as (the BABy formula)

$$\begin{aligned}\dot{\mathbf{y}}(t) &= \mathbf{B}\mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{f}(\mathbf{x}(t)) = \mathbf{B}\mathbf{A}\mathbf{B}^{-1}\mathbf{y}(t) + \mathbf{B}\mathbf{f}(\mathbf{B}^{-1}\mathbf{y}(t)) \\ &= \underbrace{\begin{pmatrix} \begin{matrix} 0 & \omega \\ -\omega & 0 \end{matrix} & \mathbf{0} \\ \mathbf{0} & \Lambda \end{pmatrix}}_{\mathbf{J}} \mathbf{y}(t) + \mathbf{g}(\mathbf{y}(t)),\end{aligned} \tag{A.3}$$

with $\Lambda$ containing the Jordan blocks corresponding to the stable eigenvalues. Or with the decompositions

$$\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \\ \mathbf{w} \end{pmatrix}, \quad \mathbf{g} = \begin{pmatrix} g_1 \\ g_2 \\ \mathbf{g_w} \end{pmatrix},$$

we can write

$$\dot{y}_1 = \omega y_2 + g_1(\mathbf{y}),$$

$$\dot{y}_2 = -\omega y_1 + g_2(\mathbf{y}),$$

$$\dot{\mathbf{w}} = \Lambda \mathbf{w} + \mathbf{g_w}(\mathbf{y}).$$

The geometric meaning of Equation (A.2) is to define the new coordinates as projections of $\mathbf{x}$ onto the real and imaginary part of the critical eigenvector of $\mathbf{A}^*$. This eigenvector satisfies

$$\mathbf{A}^*\mathbf{n} = -i\omega\mathbf{n}, \quad \mathbf{n} = \mathbf{n}_1 + i\mathbf{n}_2. \tag{A.4}$$

The projection is achieved with the help of the usual scalar product $(\mathbf{u}, \mathbf{v}) = \mathbf{u}^*\mathbf{v}$ (where $\mathbf{u}^*$ is the complex conjugate transpose (adjoint) of $\mathbf{u}$) as

$$y_i(t) = (\mathbf{n}_i, \mathbf{x}(t)) = \mathbf{n}_i^*\mathbf{x}(t). \tag{A.5}$$

Then the time evolution of a new coordinate can be expressed as

$$\begin{aligned}\dot{y}_i &= (\mathbf{n}_i, \dot{\mathbf{x}}(t)) = (\mathbf{s}_i, \mathbf{A}\mathbf{x}(t) + \mathbf{f}(\mathbf{x}(t))) = (\mathbf{n}_i, \mathbf{A}\mathbf{x}(t)) + (\mathbf{n}_i, \mathbf{f}(\mathbf{x}(t))) \\ &= (\mathbf{A}^*\mathbf{n}_i, \mathbf{x}(t)) + (\mathbf{n}_i, \mathbf{f}(\mathbf{x}(t))) = \mathbf{n}_i^*\mathbf{A}\mathbf{x}(t) + \mathbf{n}_i^*\mathbf{f}(\mathbf{x}(t)),\end{aligned}$$

where we used the linearity of the scalar product and the identity

$$(\mathbf{u}, \mathbf{A}\mathbf{v}) = (\mathbf{A}^*\mathbf{u}, \mathbf{v}).$$

This result is equivalent with Equation (A.3).

The coordinates $y_1(t)$, $y_2(t)$ of the linear system $\dot{\mathbf{y}}(t) = \mathbf{J}\mathbf{y}(t)$ describe stable (but not asymptotically stable) solutions, while the other coordinates represent exponentially decaying ones. In other words the linear equation

$$\dot{\mathbf{y}}(t) = \mathbf{J}\mathbf{y}(t)$$

has a two-dimensional attractive invariant subspace (a plane). To obtain the two real vectors that span this plane we first find the two complex conjugate eigenvectors that satisfy

$$\mathbf{J}\mathbf{s} = i\omega\mathbf{s}, \tag{A.6}$$

$$\mathbf{J}\bar{\mathbf{s}}^* = -i\omega\bar{\mathbf{s}}^*. \tag{A.7}$$

These are

$$\mathbf{s} = \begin{pmatrix} 1 \\ i \\ \mathbf{0} \end{pmatrix}, \quad \mathbf{s}^* = \begin{pmatrix} 1 \\ -i \\ \mathbf{0} \end{pmatrix}.$$

The two real vectors

$$\mathbf{s}_1 = \operatorname{Re}\mathbf{s} = \begin{pmatrix} 1 \\ 0 \\ \mathbf{0} \end{pmatrix}, \quad \mathbf{s}_2 = \operatorname{Im}\mathbf{s} = \begin{pmatrix} 0 \\ 1 \\ \mathbf{0} \end{pmatrix}$$

span the plane in question. $\mathbf{n}$, $\mathbf{s}$ satisfy the orthonormality conditions

$$(\mathbf{n}_1, \mathbf{s}_1) = (\mathbf{n}_2, \mathbf{s}_2) = 1, \tag{A.8}$$

$$(\mathbf{n}_1, \mathbf{s}_2) = (\mathbf{n}_2, \mathbf{s}_1) = 0. \tag{A.9}$$

Note that since $(\mathbf{n}, \mathbf{s}) = (\mathbf{n}_1, \mathbf{s}_1) + (\mathbf{n}_2, \mathbf{s}_2) + i((\mathbf{n}_2, \mathbf{s}_1) - (\mathbf{n}_1, \mathbf{s}_2))$ Equations (A.8) and (A.9) are equivalent to

$$(\mathbf{n}, \mathbf{s}) = 2.$$

Equations (A.6) and (A.7) can also be written as

$$\begin{aligned} \mathbf{J}\mathbf{s}_1 &= -\omega\mathbf{s}_2, \\ \mathbf{J}\mathbf{s}_2 &= \omega\mathbf{s}_1, \end{aligned}$$

which can then be solved for the real vectors.

CANONICAL FORM FOR DDES

Let us first consider a linear scalar autonomous delay differential equation (and the corresponding initial function) of the form

$$\begin{aligned} \dot{x}(t) &= Lx(t) + Rx(t-\tau) + f(x(t), x(t-\tau)), \\ x(t) &= \phi(t) \quad t \in [-\tau, 0). \end{aligned} \tag{A.10}$$

The first goal is to put Equation (A.10) into a form similar to Equation (A.1). Here an attempted numerical solution could give a clue. Discretizing Equation (A.10) leads to

$$\begin{aligned} \frac{\mathrm{d}x_i}{\mathrm{d}\vartheta} &\approx \frac{1}{\mathrm{d}\vartheta}(x_{i-1} - x_i), \quad i = 1, \ldots, N, \\ \left.\frac{\mathrm{d}x_i}{\mathrm{d}\vartheta}\right|_{\vartheta=0} &= Lx_0 + Rx_N + f(x_0, x_N). \end{aligned}$$

Taking the limit $d\vartheta \to 0$, and defining the 'shift of time' (a chunk of a function)

$$x_t(\vartheta) = x(t + \vartheta),$$

we have the following

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\vartheta}x_t(\vartheta) &= \frac{\mathrm{d}}{\mathrm{d}\vartheta}x_t(\vartheta), \quad \vartheta \in [-\tau, 0), \\ \left.\frac{\mathrm{d}}{\mathrm{d}\vartheta}x_t(\vartheta)\right|_{\vartheta=0} &= Lx_t(0) + Rx_t(-\tau) + f(x_t(0), x_t(-\tau)). \end{aligned}$$

This motivates our definition of the linear differential operator $\mathcal{A}$

$$\mathcal{A}u(\vartheta) = \begin{cases} \frac{\mathrm{d}}{\mathrm{d}\vartheta}u(\vartheta), & \vartheta \in [-\tau, 0), \\ Lu(0) + Ru(-\tau), & \vartheta = 0, \end{cases} \tag{A.11}$$

and the nonlinear operator $\mathcal{F}$

$$\mathcal{F}(u)(\vartheta) = \begin{cases} 0, & \vartheta \in [-\tau, 0), \\ f(u(0)), & \vartheta = 0, \end{cases} \tag{A.12}$$

Since $\mathrm{d}/\mathrm{d}\vartheta = \mathrm{d}/\mathrm{d}t$, we can rewrite Equation (A.10) as

$$\frac{\mathrm{d}}{\mathrm{d}t}x_t(\vartheta) = \dot{x}_t(\vartheta) = \mathcal{A}x_t(\vartheta) + \mathcal{F}(x_t)(\vartheta).$$

This operator formulation can be extended to multidimensional systems as well

$$\dot{\mathbf{x}}_t(\vartheta) = \mathcal{A}\mathbf{x}_t(\vartheta) + \mathcal{F}(\mathbf{x}_t)(\vartheta). \tag{A.13}$$

This form is very similar to the system of ODEs (A.1). There is one very important difference, though. Equation (A.13) represents an infinite-dimensional system. However, the infinite-dimensional phase space of its linear part can also be split into stable, unstable and center subspaces corresponding to eigenvalues having positive, negative and zero real part. If we focus our attention to the case where the operator has imaginary eigenvalues, it means that $\mathcal{A}$ has an pair of complex conjugate *eigenfunctions* corresponding to $\pm i\omega$ satisfying

$$\mathcal{A}\mathbf{s}(\vartheta) = i\omega\mathcal{I}\mathbf{s}(\vartheta), \tag{A.14}$$

$$\mathcal{A}\bar{\mathbf{s}}^*(\vartheta) = -i\omega\mathcal{I}\bar{\mathbf{s}}^*(\vartheta), \tag{A.15}$$

where the identity operator $\mathcal{I}$ (this seemingly superfluous definition is intended to make our life easier as a bookkeping device) is defined as

$$\mathcal{I}\mathbf{u}(\theta) = \begin{cases} \mathbf{u}(\theta), & \theta \neq 0, \\ \mathbf{u}(0), & \theta = 0. \end{cases}$$

Note that Equations (A.14) and (A.15) represent a boundary value problem (because of the definition of $\mathcal{A}$).

Introducing the real functions

$$\begin{aligned} \mathbf{s}_1(\vartheta) &= \operatorname{Re}\mathbf{s}(\vartheta), \\ \mathbf{s}_2(\vartheta) &= \operatorname{Im}\mathbf{s}(\vartheta). \end{aligned}$$

Equations (A.14) and (A.15) can be rewritten as

$$\begin{aligned} \mathcal{A}\mathbf{s}_1(\vartheta) &= -\omega\mathcal{I}\mathbf{s}_2(\vartheta), \\ \mathcal{A}\mathbf{s}_2(\vartheta) &= \omega\mathcal{I}\mathbf{s}_1(\vartheta). \end{aligned}$$

The two functions $\mathbf{s}_1(\vartheta)$, $\mathbf{s}_2(\vartheta)$ ‘span’ the center subspace which is tangent to the two-dimensional center manifold embedded in the infinite-dimensional phase space. With the help of these functions we can *try* to define the new coordinates (similarly to Equation (A.5))

$$\begin{aligned} y_1(t) &= (\mathbf{n}_1(\vartheta), \mathbf{x}_t(\vartheta)), \\ y_2(t) &= (\mathbf{n}_2(\vartheta), \mathbf{x}_t(\vartheta)). \end{aligned} \tag{A.16}$$

where $\mathbf{n}$ satisfies $\mathcal{A}^*\mathbf{n} = -i\omega\mathbf{n}$. The evolution of the new coordinate $y_1$ would then be given by

$$\begin{aligned} \dot{y}_1(t) &= (\mathbf{n}_1(\vartheta), \dot{\mathbf{x}}_t(\vartheta)) = (\mathbf{n}_1(\vartheta), \mathcal{A}\mathbf{x}_t(\vartheta) + \mathcal{F}(\mathbf{x}_t)(\vartheta)) \\ &= (\mathcal{A}^*\mathbf{n}_1(\vartheta), \mathbf{x}_t(\vartheta)) + (\mathbf{n}_1(\vartheta), \mathcal{F}(\mathbf{x}_t)(\vartheta)). \end{aligned}$$

Two questions arise immediately: how shall we define the pairing $(.,.)$ and what is the adjoint operator $\mathcal{A}^*$? To answer the first question we can first try to use the usual inner product in the space of continously differentiable functions on $[-\tau, 0)$

$$(\mathbf{u}(\vartheta), \mathbf{v}(\vartheta)) = \int_{-\tau}^{0} \mathbf{u}^*(\vartheta)\mathbf{v}(\vartheta)\,\mathrm{d}\vartheta. \tag{A.17}$$

However, we also want to include the ‘boundary terms’ at $\vartheta = 0$ from Equations (A.11) and (A.12) so we can try to modify Equation (A.17) as

$$(\mathbf{u}(\vartheta), \mathbf{v}t(\vartheta)) = \int_{-\tau}^{0} \mathbf{u}^*(\vartheta)\mathbf{v}(\vartheta)\,\mathrm{d}\vartheta + \mathbf{u}^*(0)\mathbf{v}(0). \tag{A.18}$$

Now we try to find the adjoint operator from the definition

$$(\mathcal{A}^*\mathbf{u}, \mathbf{v}) = (\mathbf{u}, \mathcal{A}\mathbf{v})$$

$$(\mathcal{A}^*\mathbf{u}, \mathbf{v}) = \int_{-\tau}^{0} \mathcal{A}^*\mathbf{u}\mathbf{v}\, \mathrm{d}\vartheta + [\mathcal{A}^*\mathbf{u}(0)]^*\mathbf{v}(0), \tag{A.19}$$

$$\begin{aligned}
(\mathbf{u}, \mathcal{A}\mathbf{v}) &= \int_{-\tau}^{0} \mathbf{u}^*\mathcal{A}\mathbf{v}\, \mathrm{d}\vartheta + \mathbf{u}^*(0)\mathcal{A}\mathbf{v}(0) \\
&\overset{\text{by parts}}{=} -\int_{-\tau}^{0} \frac{\mathrm{d}}{\mathrm{d}\vartheta}\mathbf{u}^*\mathbf{v}\, \mathrm{d}\vartheta + \mathbf{u}^*(\vartheta)\mathbf{v}(\vartheta)|_{-\tau}^{0} + \mathbf{u}^*(0)[\mathbf{L}\mathbf{v}(0) + \mathbf{R}\mathbf{v}(-\tau)] \\
&= -\int_{-\tau}^{0} \frac{\mathrm{d}}{\mathrm{d}\vartheta}\mathbf{u}^*\mathbf{v}\, \mathrm{d}\vartheta + \mathbf{u}^*(0)\mathbf{v}(0) - \mathbf{u}^*(\vartheta)\mathbf{v}(\vartheta)|_{-\tau} \\
&\quad + \mathbf{u}^*(0)\mathbf{R}\mathbf{v}(-\tau) + \mathbf{u}^*(0)\mathbf{L}\mathbf{v}(0) \\
&= \int_{-\tau}^{0} \left(-\frac{\mathrm{d}}{\mathrm{d}\vartheta}\right)\mathbf{u}^*\mathbf{v} d\vartheta + [(\mathbf{I} + \mathbf{L})^*\mathbf{u}(0)]^*\mathbf{v}(0) \\
&\quad - \mathbf{u}^*(-\tau)\mathbf{v}(-\tau) + \mathbf{u}^*(0)\mathbf{R}\mathbf{v}(-\tau).
\end{aligned} \tag{A.20}$$

The first two terms of Equation (A.20) are similar to those in Equation (A.19) so we seek to eliminate $\mathbf{u}^*(0)\mathbf{R}\mathbf{v}(-\tau) - \mathbf{u}^*(-\tau)\mathbf{v}(-\tau)$. Since this term is usually nonzero, we can try to modify Equation (A.18) to get $\mathbf{u}^*(0)\mathbf{R}\mathbf{v}(-\tau) - \mathbf{u}^*(\tau - \tau)\mathbf{R}\mathbf{v}(-\tau) = 0$ instead. This can be achieved with the modification

$$(\mathbf{u}(\vartheta), \mathbf{v}(\vartheta)) = \int_{-\tau}^{0} \mathbf{u}^*(\vartheta + \tau)\mathbf{R}\mathbf{v}(\vartheta)\, \mathrm{d}\vartheta + \mathbf{u}^*(0)\mathbf{v}(0). \tag{A.21}$$

Using Equation (A.21)

$$(\mathbf{u}, \mathcal{A}\mathbf{v}) = -\int_{-\tau}^{0} \frac{\mathrm{d}}{\mathrm{d}\vartheta}\mathbf{u}^*(\vartheta + \tau)\mathbf{R}\mathbf{v}(\vartheta)\, \mathrm{d}\vartheta + [\mathbf{L}^*\mathbf{u}(0) + \mathbf{R}^*\mathbf{u}(\tau)]^*\mathbf{v}(0).$$

Defining $\gamma = \vartheta + \tau$

$$\begin{aligned}
(\mathbf{u}, \mathcal{A}\mathbf{v}) &= (\mathcal{A}^*\mathbf{u}, \mathbf{v}) \\
&= \int_{0}^{\tau} \left(-\frac{\mathrm{d}}{\mathrm{d}\gamma}\right)\mathbf{u}^*(\gamma)\mathbf{R}\mathbf{v}(\gamma - \tau)\, \mathrm{d}\gamma + [\mathbf{L}^*\mathbf{u}(0) + \mathbf{R}^*\mathbf{u}(\tau)]^*\mathbf{v}(0)
\end{aligned}$$

gives the definition of the adjoint operator

$$\mathcal{A}^*\mathbf{u}(\gamma) = \begin{cases} -\frac{\mathrm{d}}{\mathrm{d}\gamma}\mathbf{u}(\gamma), & \gamma \in (0, \tau], \\ \mathbf{L}^*\mathbf{u}(0) + \mathbf{R}^*\mathbf{u}(\tau), & \gamma = 0, \end{cases}$$

with the two complex conjugate eigenfunctions

$$\mathcal{A}^*\mathbf{n}(\gamma) = -i\omega\mathcal{I}\mathbf{n}(\gamma), \tag{A.22}$$

$$\mathcal{A}^*\bar{\mathbf{n}}^*(\gamma) = i\omega\mathcal{I}\bar{\mathbf{n}}^*(\gamma). \tag{A.23}$$

Introducing the real functions

$$\begin{aligned} \mathbf{n}_1(\gamma) &= \operatorname{Re}\mathbf{n}(\gamma), \\ \mathbf{n}_2(\gamma) &= \operatorname{Im}\mathbf{n}(\gamma). \end{aligned}$$

Equations (A.22) and (A.23) can be rewritten as

$$\mathcal{A}^*\mathbf{n}_1(\gamma) = \omega\mathcal{I}\mathbf{n}_2(\gamma),$$

$$\mathcal{A}^*\mathbf{n}_2(\gamma) = -\omega\mathcal{I}\mathbf{n}_1(\gamma).$$

Since Equation (A.21) requires functions from two different spaces ($C^1_{[-1,0]}$ and $C^1_{[0,1]}$) it is a *bilinear form* instead of an inner product. The 'orthonormality' conditions (see Equations (A.8) and (A.9)) are

$$(\mathbf{n}_1, \mathbf{s}_1) = (\mathbf{n}_2, \mathbf{s}_2) = 1,$$

$$(\mathbf{n}_2, \mathbf{s}_1) = (\mathbf{n}_1, \mathbf{s}_2) = 0.$$

The new coordinates $y_1$, $y_2$ can be found by the projections (instead of Equation (A.16))

$$\begin{aligned} y_1(t) &= y_{1t}(0) = (\mathbf{n}_1, \mathbf{x}_t)|_{\vartheta=0}, \\ y_2(t) &= y_{2t}(0) = (\mathbf{n}_2, \mathbf{x}_t)|_{\vartheta=0}. \end{aligned}$$

Now $\mathbf{x}_t(\vartheta)$ can be decomposed as

$$\mathbf{x}_t(\vartheta) = y_1(t)\mathbf{s}_1(\vartheta) + y_2(t)\mathbf{s}_2(\vartheta) + \mathbf{w}(t)(\vartheta) \tag{A.24}$$

and the operator differential equation (A.13) can be transformed into the 'canonical form'

$$\begin{aligned} \dot{y}_1 &= (\mathbf{n}_1, \dot{\mathbf{x}}_t)|_{\vartheta=0} = (\mathbf{n}_1, \mathcal{A}\mathbf{x}_t + \mathcal{F}(\mathbf{x}_t))|_{\vartheta=0} \\ &= (\mathbf{n}_1, \mathcal{A}\mathbf{x}_t)|_{\vartheta=0} + (\mathbf{n}_1, \mathcal{F}(\mathbf{x}_t))|_{\vartheta=0} \\ &= (\mathcal{A}^*\mathbf{n}_1, \mathbf{x}_t)|_{\vartheta=0} + (\mathbf{n}_1, \mathcal{F}(\mathbf{x}_t))|_{\vartheta=0} \\ &= \omega(\mathbf{n}_2, \mathbf{x}_t)|_{\vartheta=0} + (\mathbf{n}_1, \mathcal{F}(\mathbf{x}_t))|_{\vartheta=0} = \omega y_2 + \mathbf{n}_1^T(0)\mathbf{F}, \end{aligned}$$

$$\dot{y}_2 = (\mathbf{n}_2, \dot{\mathbf{x}}_t)|_{\vartheta=0} = -\omega y_1 + \mathbf{n}_2^T(0), \mathbf{F}$$

where $\mathbf{F} = \mathcal{F}(y_1(t)\mathbf{s}_1(0) + y_2(t)\mathbf{s}_2(0) + \mathbf{w}(t)(0))$ was used and

$$\begin{aligned}
\dot{\mathbf{w}} &= \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{x}_t - y_1\mathbf{s}_1 - y_2\mathbf{s}_2) = \mathcal{A}\mathbf{x}_t + \mathcal{F}(\mathbf{x}_t) - \dot{y}_1\mathcal{I}\mathbf{s}_1 - \dot{y}_2\mathcal{I}\mathbf{s}_2 \\
&= \mathcal{A}(y_1\mathbf{s}_1 + y_2\mathbf{s}_2 + \mathbf{w}) + \mathcal{F}(\mathbf{x}_t) - \dot{y}_1\mathcal{I}\mathbf{s}_1 - \dot{y}_2\mathcal{I}\mathbf{s}_2 \\
&= y_1(-\omega\mathcal{I}\mathbf{s}_2) + y_2(\omega\mathcal{I}\mathbf{s}_1) + \mathcal{A}\mathbf{w} + \mathcal{F}(\mathbf{x}_t) \\
&\quad - (\omega y_2 + \mathbf{n}_1^T(0)\mathbf{F})\mathcal{I}\mathbf{s}_1 - (-\omega y_1 + \mathbf{n}_2^T(0)\mathbf{F})\mathcal{I}\mathbf{s}_2 \\
&= \mathcal{A}\mathbf{w} + \mathcal{F}(\mathbf{x}_t) - \mathbf{n}_1^T(0)\mathbf{F}\mathcal{I}\mathbf{s}_1 - \mathbf{n}_2^T(0)\mathbf{F}\mathcal{I}\mathbf{s}_2 \\
&= \begin{cases} \frac{\mathrm{d}}{\mathrm{d}\vartheta}\mathbf{w} - \mathbf{n}_1^T(0)\mathbf{F}\mathbf{s}_1 - \mathbf{n}_2^T(0)\mathbf{F}\mathbf{s}_2, & \vartheta \in [-\tau, 0), \\ \mathbf{L}\mathbf{w}(0) + \mathbf{R}\mathbf{w}(-\tau) + \mathbf{F} - \mathbf{n}_1^T(0)\mathbf{F}\mathbf{s}_1(0) - \mathbf{n}_2^T(0)\mathbf{F}\mathbf{s}_2(0), & \vartheta = 0. \end{cases}
\end{aligned}$$

## Acknowledgements

The authors would like to thank Drs Dave Gilsinn, Jon Pratt, Richard Rand and Steven Yeung for their valuable comments.

## References

1. Burns, T. J. and Davies, M. A., ‘Nonlinear dynamics model for chip segmentation in machining’, *Physical Review Letters* **79**(3), 1997, 447–450.
2. Campbell, S. A., Bélair, J., Ohira, T., and Milton, J., ‘Complex dynamics and multistability in a damped oscillator with delayed negative feedback’, *Journal of Dynamics and Differential Equations* **7**, 1995, 213–236.
3. Doi, S. and Kato, S., ‘Chatter vibration of lathe tools’, *Transactions of the American Society of Mechanical Engineers* **78**(5), 1956, 1127–1134.
4. Fofana, M. S., ‘Delay dynamical systems with applications to machine-tool chatter’, Ph.D. Thesis, University of Waterloo, Department of Civil Engineering, 1993.
5. Guckenheimer, J. and Holmes, P. J., *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields*, Applied Mathematical Sciences, Vol. 42, Springer-Verlag, New York, 1986.
6. Hale, J. K., *Theory of Functional Differential Equations*, Applied Mathematical Sciences, Vol. 3, Springer-Verlag, New York, 1977.
7. Hassard, B. D., Kazarinoff, N. D., and Wan, Y. H., *Theory and Applications of Hopf Bifurcations*, London Mathematical Society Lecture Note Series, Vol. 41, Cambridge University Press, Cambridge.
8. Johnson, M. A., ‘Nonlinear differential equations with delay as models for vibrations in the machining of metals’, Ph.D. Thesis, Cornell University, 1996.
9. Kalmár-Nagy, T., Pratt, J. R., Davies, M. A., and Kennedy, M. D., ‘Experimental and analytical investigation of the subcritical instability in turning’, in *Proceedings of the 1999 ASME Design Engineering Technical Conferences*, ASME, New York, 1999, DETC99/VIB80-10, pp. 1–9.
10. Kalmár-Nagy, T., Stépán, G., and Moon, F. C., ‘Regenerative machine tool vibrations’, in *Dynamics of Continuous, Discrete and Impulsive Systems*, to appear.
11. Kuang, Y., *Delay Differential Equations with Applications in Population Dynamics*, Mathematics in Science and Engineering, No. 191, Academic Press, New York, 1993.
12. Moon, F. C., ‘Chaotic dynamics and fractals in material removal processes’, in *Nonlinearity and Chaos in Engineering Dynamics*, J. Thompson and S. Bishop (eds.), Wiley, New York, 1994, pp. 25–37.
13. Nayfeh, A., Chin, C., and Pratt, J., ‘Applications of perturbation methods to tool chatter dynamics’, in *Dynamics and Chaos in Manufacturing Processes*, F. C. Moon (ed.), Wiley, New York, 1997, pp. 193–213.
14. Nayfeh, A. H. and Balachandran, B., *Applied Nonlinear Dynamics*, Wiley, New York, 1995.

15. Shi, H. M. and Tobias, S. A., 'Theory of finite amplitude machine tool instability', *International Journal of Machine Tool Design and Research* **24**(1), 1984, 45–69.
16. Stépán, G., *Retarded Dynamical Systems: Stability and Characteristic Functions*, Pitman Research Notes in Mathematics, Vol. 210, Longman Scientific and Technical, Harlow, 1989.
17. Stépán, G., 'Delay-differential equation models for machine tool chatter', in *Dynamics and Chaos in Manufacturing Processes*, F. C. Moon (ed.), Wiley, New York, 1997, pp. 165–191.
18. Taylor, F. W., 'On the art of cutting metals', *Transactions of the American Society of Mechanical Engineers* **28**, 1907, 31–350.
19. Tlusty, J. and Spacek, L., *Self-Excited Vibrations on Machine Tools*, Nakl CSAV, Prague, 1954 [in Czech].
20. Tobias, S. A., *Machine Tool Vibration*, Blackie and Son, Glasgow, 1965.