Pitman Research Notes in Mathematics Series

G Stépán

Retarded dynamical systems: stability and characteristic functions

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ABOUT THIS VOLUME

This Research Note gives an analytical stability criterion for linear autonomous retarded functional differential equations. It includes detailed proofs and a wide selection of applications. The applications are mainly from the fields of mechanical engineering, machine dynamics, man-machine systems, robotics and partly from bioecology. These examples lead to mathematical problems containing more discrete delays than one or the integral of the past. In these cases, it is often very difficult or practically too complicated to investigate the stability of trivial solutions or to construct stability charts by means of conventional methods. The role of long delays is analyzed in oscillatory systems. Non-linear vibrations, limit cycles and their stability are also studied using the Hopf bifurcation method.

The book gives a simple introduction to the stability analysis of retarded dynamical systems and supports the direct investigation of the characteristic functions. The great number of stability charts help the researcher to understand the often peculiar behaviour of these systems.

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Retarded Dynamical Systems: Stability and Characteristic Functions

G. Stépán

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Preface

This book is based on the author's research work in the last ten years. In the Research Group on Mechanics of the Hungarian Academy of Sciences, a lot of practical problems have been investigated which required a general and efficient method to analyse the stability of trivial and periodic solutions of dynamical systems with memory. In the literature, the great number of the analytical, numerical and geometrical stability criteria for linear autonomous systems show that none of these criteria can be used generally in a simple way. For example, the Pontryagin criterion is preferred in theoretical mathematics, the Nyquist criterion and D-subdivision method in control theory, the tau-decomposition method in mechanics and there exist a lot of other special methods for particular problems.

In the design work of engineers, stability charts have a significant role. For the investigation of oscillatory systems with delay effects, with more delays than one or with an integral over the past, an alternative stability criterion was needed in order to construct these stability charts. This analytical stability criterion is more general for retarded systems than the existing ones.

The first chapter gives a short survey of the existing methods. The stability criterion for linear autonomous retarded functional differential equations is presented in the second chapter. The detailed proof of the criterion, the manner of its generalization to a class of neutral systems and the basis of the investigation of non-linear systems are also given there. The third chapter contains basic stability charts. They are constructed with the help of a series of theorems deduced from the stability criterion. In the fourth chapter, the bioecological examples are chosen from the specialist literature, but the problems of man-machine systems, robotics and machine tool vibrations have been formed from practical examples. These results were often investigated and verified experimentally by the author at the Department of Engineering Mechanics of the Technical University of Budapest and at the Mechanical Engineering Department of the University of Newcastle upon Tyne. This fact guarantees that the examples are not artificial, although the book has a mathematical character and does not deal with the practical validity of the mathematical models in question.

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1. Introduction and Survey

A lot of analogue expressions are used which cover more or less the same idea: retarded dynamical systems, hereditary systems, systems with delayed feedback, systems with memory, with aftereffect or time lag, etc. The expressions delayed self-excitation, regenerative or reproductive vibrations can also be found in the specialist literature referring to these systems. These ideas are not defined exactly, but experts working on different fields of applications understand them well. However, mathematics has had the exact definitions and terminology since these systems have been described by functional differential equations. It became clear a long time ago, that these systems could be handled as infinite dimensional problems, but only the paper [39] of Myshkis gave the first correct mathematical formulation of the initial value problem in 1949.

A retarded functional differential equation (RFDE) describes a system where the rate of change of state is determined by the present and past states of the system. If the rate of change of state depends on its own past values as well, the system can be governed by a neutral functional differential equation (NFDE). In the special case, when only discrete values of the past have influence on the present rate of change of state, the corresponding mathematical equations can be either retarded or neutral differential-difference equations (RDDE or NDDE respectively). These are special cases of the functional differential equations (FDE), as the ordinary differential equations are the special cases of the DDEs. The basic theory of ordinary differential equations has been generalized for these infinite dimensional problems. The theorems about the local existence and uniqueness of a solution and its continuous dependence on the initial data are not presented in this book. The formulation of the initial value problem for RFDEs and some basic definitions will be given in order to call the reader's attention to the basic differences between ordinary and functional differential equations and to introduce the notation used throughout the book.

The basic theory for DDEs can be found in the book [3] of Bellman and Cooke. The theory of FDEs with bounded delay is presented in the basic work [25] of Hale. The applied mathematical book [35] of Kolmanovskii and Nosov gives a good summary of the most important theorems for FDEs without proofs but with an extensive reference list. To generalize the Hopf bifurcation method for the investigation of non-linear RFDEs, it is necessary to formulate the problem as an operator differential equation. This can, for example, be found in the book [27].

In this chapter, the most important results about the zeros of characteristic functions as well as a brief survey of the basic methods of stability investigations of linear autonomous FDEs will be given.

1.1. Initial value problem and stability definitions

Suppose that $h \geq 0$ is a given number with understanding that h may be $+\infty$, and $\mathbf{R}^n$ is an n-dimensional linear vectorspace over the reals with norm $|\,.\,|$. Let $\mathbf{B}$ denote the vector space of continuous and bounded functions mapping the interval $[-h, 0]$ into $\mathbf{R}^n$. With the norm $||\,.\,||$ given by

$$||\phi|| = \sup_{\theta \in [-h,0]} |\phi(\theta)|, \quad \phi \in \mathbf{B}, \tag{1.1}$$

$\mathbf{B}$ is a Banach space.

Definition 1.1. The relation

$$\dot{x}(t) = f(t, x_t) \tag{1.2}$$

is called RFDE where $f : \mathbf{R} \times \mathbf{B} \to \mathbf{R}^n$ is a given function, the overdot represents the right-hand derivative with respect to t and the function $x_t \in \mathbf{B}$ is defined by

$$x_t(\theta) = x(t+\theta), \quad \theta \in [-h, 0]. \tag{1.3}$$

From now on, t is assumed to be the time, and h is the length of retardation. If h is finite, that is the delay is bounded, then the norm in (1.1) is equivalent to $\max_{\theta \in [-h,0]} |\phi(\theta)|$. If $h = +\infty$ then the delay is unbounded. In this case the choice of the phase space has an important role in the qualitative theory of FDEs since x_t contains all the past history, in accordance with (1.3). As a matter of fact, a lot of spaces are presented and used in the literature [10,24,26,35,40,51,66] for systems with infinite delay, which are often more general than the space $\mathbf{B}$ used here. The publications of Schumacher [48] and Naito [41] give some details of the problem of finding weak assumptions on the phase space when $h = +\infty$.

Definition 1.2. The function $x : \mathbf{R} \to \mathbf{R}^n$ is a solution of the RFDE (1.2) with the initial condition

$$x_\sigma = \phi; \quad \sigma \in \mathbf{R}, \quad \phi \in \mathbf{B} \tag{1.4}$$

if there exists a scalar $\delta > 0$ such that $x_t \in \mathbf{B}$ and x satisfies (1.2) and (1.4) for all $t \in [\sigma, \sigma + \delta]$.

The notation $x(t; \sigma, \phi)$ also refers to the solution of (1.2) and (1.4), and the function $x_t(\sigma, \phi)$ is defined according to (1.3):

$$x_t(\sigma, \phi)(\theta) = x(t + \theta; \sigma, \phi), \quad \theta \in [-h, 0].$$

If the function f in (1.2) is continuous and $f(t, \phi)$ satisfies a local Lipschitz condition in ϕ then the local existence and uniqueness of the solution of (1.2) and (1.4) can be proved as well as its continuous dependence on the initial data [25,35].

Definition 1.3. Let us suppose that $f(t, 0) = 0$ for all $t \in \mathbf{R}$ in the RFDE (1.2). The trivial solution $x = 0$ of (1.2) is stable in the Lyapunov sense if for every $\sigma \in \mathbf{R}$ and

$\epsilon > 0$ there exists a $\delta = \delta(\sigma, \epsilon)$ such that $||x_t(\sigma, \phi)|| \leq \epsilon$ for any $t \geq \sigma$ and for any initial function ϕ satisfying $||\phi|| < \delta$. The trivial solution is called asymptotically stable if it is stable and for every $\sigma \in \mathbf{R}$ there exists a $\Delta = \Delta(\sigma)$ such that $\lim_{t\to+\infty} ||x_t(\sigma, \phi)|| = 0$ for any ϕ satisfying $||\phi|| < \Delta$.

In exactly the same way, the standard definitions of uniform and exponential stability can also be extended for infinite dimensional problems. The idea of orbital stability will also be used when periodic solutions of autonomous RFDEs are investigated. For autonomous systems, the stability of a trivial solution yields its uniform stability [35].

The general form of a linear autonomous (and homogeneous) RFDE is

$$\dot{x}(t) = L(x_t) \tag{1.5}$$

where the functional $L : \mathbf{B} \to \mathbf{R}^n$ is continuous and linear. According to the Riesz Representation Theorem, (1.5) can be represented as

$$\dot{x}(t) = \int_{-\infty}^{0} [d\eta(\theta)]x(t+\theta), \tag{1.6}$$

where η is an $n \times n$ matrix of functions of bounded variation on $(-\infty, 0]$ and the integral is a Riemann-Stiltjes one. Let $\mathbf{C}$ denote the set of complex numbers.

Definition 1.4. The function given by

$$D(\lambda) = \det\big(\lambda I - \int_{-\infty}^{0} e^{\lambda\theta} d\eta(\theta)\big), \quad \lambda \in \mathbf{C} \tag{1.7}$$

is called the characteristic function corresponding to the linear autonomous RFDE (1.6) where I is the $n \times n$ unit matrix.

The characteristic function can be obtained either by the substitution of non-trivial solutions $x(t) = Ke^{\lambda t}$, $K \in \mathbf{R}^n$ into (1.6) or by means of the application of a Laplace transformation to (1.6).

The theory of NFDEs is more complicated than that of the RFDEs. However, the linear autonomous NFDEs have also a simple form:

$$\dot{x}(t) = \int_{-\infty}^{0} [d\eta_1(\theta)]\dot{x}(t+\theta) + \int_{-\infty}^{0} [d\eta_0(\theta)]x(t+\theta) \tag{1.8}$$

where η_0 and η_1 are $n \times n$ matrices of functions of bounded variation. Its characteristic function D is defined by

$$D(\lambda) = \det\bigl(\lambda I - \lambda \int_{-\infty}^{0} e^{\lambda\theta} d\eta_1(\theta) - \int_{-\infty}^{0} e^{\lambda\theta} d\eta_0(\theta)\bigr). \tag{1.9}$$

Definition 1.5. The characteristic function D of a RFDE or a NFDE is called stable if

$$\{\lambda \in \mathbf{C} : \operatorname{Re}\lambda \geq 0,\, D(\lambda) = 0\} = \emptyset$$

where D is given by (1.7) or (1.9) respectively.

The linear equations (1.6) or (1.8) can, of course, represent the cases of finite delays as well. The delay has a finite length h when η, or η_0 and η_1, are constants in $(-\infty, -h)$. Thus, all the results obtained for unbounded delays are valid for the special case of finite delays. However, we shall often present theorems for finite delays only (for example in *Theorems 1.7, 1.8* and *1.9* of the following section) since we know much more about them.

In the case of ordinary differential equations, the stability of the characteristic polynomial is equivalent to the exponential stability of the trivial solution. The situation is much more complicated for FDEs. Some results of the literature about the zeros of characteristic functions and about the stability of solutions are presented in the following section.

1.2. Zeros of characteristic functions and stability

The following theorem is given in [24] as a conclusion for linear autonomous RFDEs with unbounded delay, but it can also be deduced from *Theorem 2.3* (p.53) of [35].

Theorem 1.6. Suppose that there exists a scalar $\nu > 0$ such that

$$\int_{-\infty}^{0} e^{-\nu\theta} |d\eta_{jk}(\theta)| < +\infty, \quad j,k = 1,\ldots,n \tag{1.10}$$

in the characteristic function (1.7). If this characteristic function is stable then the trivial solution of the linear RFDE (1.6) is exponentially asymptotically stable. △

As a matter of fact, condition (1.10) guarantees that there does not exist a sequence $\{\lambda_k\}$, $k = 1, 2 \ldots$ of the zeros of (1.7) such that $\operatorname{Re}\lambda_k$ tends to a non-negative number as $k \to +\infty$. When $\eta(\theta)$ is constant for $\theta \in (-\infty, -h]$, that is the length h of the retardation is finite, condition (1.10) automatically holds, and we know even more about the zeros of (1.7) [25]:

Theorem 1.7. Let us suppose that $\eta(\theta)$ is constant for $\theta \in (-\infty, -h]$, i.e. the delay is finite. If there is a sequence $\{\lambda_k\}$ of the zeros of the characteristic function (1.7) such that $|\lambda_k| \to +\infty$ as $k \to +\infty$ then $\operatorname{Re}\lambda_k \to -\infty$ as $k \to +\infty$. △

Reference [17] presents an example where (1.10) is fulfilled, the delay is unbounded, the characteristic function is stable and there does exist a sequence $\{\lambda_k\}$ such that $\operatorname{Re}\lambda_k$ converges to a finite (but negative) number. In [42], an example is investigated where (1.10) is not fulfilled, the characteristic function is stable and the asymptotic stability of the trivial solution of the linear system is not exponential.

Similar, but even more complicated results can be found for NFDEs (see [35] and the references in it). For example, the trivial solution may be stable or unstable if the real parts of a sequence of multiple zeros of (1.9) converge to zero through negative numbers. It is easy to show that the characteristic function (1.9) may have zeros of infinite number on the right half of the complex plane even in the case of a single discrete delay. The following theorems (from [25,35]) are presented for the case when η_1 and η_0 in the NFDE (1.8) are piece-wise constant on $(-\infty, 0]$, i.e. they are step functions with a finite number of discontinuities. This means that (1.8) is a NDDE.

Theorem 1.8. Suppose that there are only discrete time lags of finite number in (1.8), i.e.

$$\dot{x}(t) = \sum_{j=1}^{p} C_j \dot{x}(t - \tau_{1j}) + \sum_{k=0}^{q} B_k x(t - \tau_{0k})$$

where $0 < \tau_{11} < \ldots < \tau_{1p}$, $0 = \tau_{00} < \ldots < \tau_{0q}$, C_j $(j = 1, \ldots, p)$ and B_k $(k = 0, \ldots, q)$ are n-dimensional constant matrices. The trivial solution of the NDDE is exponentially asymptotically stable if there exists a scalar $\epsilon > 0$ such that $\mathrm{Re}\,\lambda_k \leq -\epsilon$ for all the zeros λ_k of the corresponding characteristic function. $\triangle$

Theorem 1.9. Suppose that there is only one discrete time lag in the NDDE, i.e. $p = q = 1$ and $\tau_{11} = \tau_{01} = \tau$. If at least one of the eigenvalues of C_1 is not zero then there exist real scalars α and β such that $\beta < \mathrm{Re}\,\lambda_k < \alpha$ for all the zeros λ_k of the characteristic function. Moreover, if there exists a sequence $\{\lambda_k\}$ of the zeros such that $|\lambda_k| \to +\infty$ as $k \to +\infty$ then there also exists a sequence $\{\lambda'_k\}$ of the zeros of

$$d(\lambda) = \det(I - C_1 e^{-\lambda\tau})$$

such that $(\lambda_k - \lambda'_k) \to 0$ as $k \to +\infty$. If $d(\lambda)$ has a sequence of zeros $\{\lambda'_k\}$ such that $|\lambda'_k| \to +\infty$ as $k \to +\infty$ then the characteristic function of the NDDE also has a sequence $\{\lambda_k\}$ of zeros such that $(\lambda'_k - \lambda_k) \to 0$ as $k \to +\infty$. $\triangle$

As these theorems and the cited examples show, the stability of the characteristic function is only a necessary condition of the exponential asymptotic stability of trivial solutions in linear autonomous FDEs. However, for the class of RFDEs defined by condition (1.10), the stability of the characteristic function is a necessary and sufficient condition of exponential asymptotical stability as this follows from *Theorem 1.6* and from the fact that the trivial solution is not asymptotically stable if $\operatorname{Re}\lambda_k \geq 0$ for at least one zero of the characteristic function.

1.3. Stability criteria

Lyapunov's direct method has been generalized for FDEs. Since the investigation of the transcendental characteristic functions is difficult, the construction of Lyapunov functionals may be a reasonable way of stability investigation even in the case of linear autonomous FDEs [6,7,23,25,35]. However, only those stability criteria are discussed briefly in this section which are based on the stability analysis of characteristic functions. Let us examine, first, the analytical criteria.

The *Pontryagin criterion* investigates the quasi-polynomials

$$P(\lambda, e^{\lambda}) = \sum_{j=0}^{p} \sum_{k=0}^{q} a_{jk} \lambda^j e^{k\lambda} . \qquad (1.11)$$

All the characteristic equations of RDDEs and NDDEs can be transformed into the form $P(\lambda, e^{\lambda}) = 0$ if there is a single discrete delay or all the ratios of the discrete delays of finite number are rational numbers. Pontryagin's paper [44], published first in 1942, gave necessary and sufficient conditions for the stability of (1.11). Although this method has strong limitations and it may become very complicated for systems with more delays than one, the Pontryagin criterion is widely used in mathematics, vibration and control theory to construct analytical stability conditions and stability charts [3,5,25,34]. As a matter of fact, this is one of the most general analytical criteria.

The *Yesipovich-Svirskii criterion* [65] can be used for quasi-polynomials (1.11) in the special form

$$P(\lambda, e^{\lambda}) = \sum_{j=0}^{p} a_j \lambda^j e^{\lambda} + \sum_{k=0}^{q} b_k \lambda^k . \tag{1.12}$$

As (1.12) shows, DDEs with one discrete delay can be investigated in this way. The method needs a further transformation $\lambda = iz$ ($i = \sqrt{-1}$, $z \in \mathbf{C}$) of $P(\lambda, e^{\lambda}) = 0$ into $F(z) = 0$ where

$$F(z) = \frac{1}{\tan(\frac{1}{2}z)} + \phi(z) . \tag{1.13}$$

The function $\phi(z)$ does not contain transcendental terms. The necessary and sufficient condition of the stability of (1.12) is given by means of the expression

$$\sum_k \operatorname{sgn}\left(\frac{1}{\operatorname{Re} F(u_k)} \frac{d \operatorname{Im} F(u_k)}{du}\right) \tag{1.14}$$

where the values u_k are the real roots of $\operatorname{Im} \phi(z) = 0$. In spite of the fact that this method seems to be effective in special applications, it is not used in the literature.

The *τ-decomposition method* [32] requires the transformation of the characteristic equation into the form

$$e^{\tau\lambda} = D_0(\lambda) \tag{1.15}$$

where $D_0(\lambda)$ is a ratio of two polynomials. Thus, this method is to be used for RDDEs with one discrete delay. The criterion is based on the analysis of the contour $D_0(iu)$, ($u \in \mathbf{R}_+$) around the unit circle in the complex plane. It can be used as an analytical criterion but its long algorithm is preferably used as a numerical method. By means of the computer program presented in [32], multi-degree of freedom retarded oscillatory systems were investigated.

The *Chebotarev criterion* is the most direct generalization of the Routh-Hurwitz criterion for quasi-polynomials (1.11). However, its application as an analytical criterion is not effective practically, since an infinite number of Hurwitz determinants must be calculated. If there are long delays in the system, determinants of high dimension may be responsible for stability.

The *D-subdivision method* helps to construct stability charts in the parameter space of the FDE. This method is always combined with some other analytical or numerical stability criteria. It becomes very effective when the stability chart of a retarded system can be presented by means of the combination of this method and the simple Routh-Hurwitz criterion. However, this trick can be used for special RDDEs only [64].

In control theory, the *Nyquist criterion* is used most frequently. It is usually applied as a numerical or geometrical criterion for DDEs. The equation

$$-1 = D_l(\lambda) \tag{1.16}$$

is investigated, which can always be rearranged to get back the characteristic function of the system, but the actual form of $D_l(\lambda)$ depends on the structure of the feed-back in the control scheme. For analytical investigation, the mathematical form of $D_l(\lambda)$ may be uncomfortable, but this open-loop transfer function is determined easily in control theory. This has advantages when the system is described by transfer functions in the "frequency domain" instead of differential equations in the "time domain". On the basis of the relative positions of the point $-1 + 0i$ and the contour $D_l(iu)$, $(u \in \mathbf{R}_+)$ on the complex plane, the stability properties of the systems can be determined.

As it was noted by Satche [47], a great number of other criteria can be produced if the characteristic equations are rearranged in the form

$$D_1(\lambda) = D_2(\lambda)$$

(like (1.15) or (1.16)) and the contours $D_1(iu)$ and $D_2(iu)$ $(u \in \mathbf{R}_+)$ are plotted in the complex plane.

The *Bode criterion*, the *Nichols criterion* and some others in control theory are the results of further transformations of the Nyquist criterion and they may be convenient in some numerical applications.

In the literature, the *Michailov criterion* [20] is usually mentioned and used as a geometrical method. Since it investigates the characteristic function in its original form

$$D(\lambda) = 0\,,$$

the contour $D(iu)$ $(u \in \mathbf{R}_+)$ has to be plotted and an analysis of how it envelops the origin of the complex plane carried out. This contour is often mentioned as Michailov's hodograph or Satche's diagram [45]. The method is a consequence of the Cauchy Residue Theorem in complex analysis, and almost all of the other stability criteria discussed originate in it. It may generally be used for RFDEs if there exists a scalar $\nu > 0$ such that the characteristic function is bounded and analytical in any closed domain in $\{\lambda : \operatorname{Re}\lambda > -\nu\}$. For example, (1.10) yields the satisfaction of this condition. Although the method may be extended for great classes of NFDEs as well, its application needs special attention since the Michailov hodograph may become very complicated in these cases. We shall discuss this problem briefly in *Section 2.2.*

There are a lot of important but less general criteria for the stability of characteristic functions like those of *Eller* [13], *Barsscz* and *Olbrot* [2], *Tsypkin* [62], *Terjéki* [59], etc. As a matter of fact, there is a wide selection of stability criteria in the literature. However, none of these methods can be used generally for FDEs. For actual problems, it is difficult to find the best one, and the application of any method may need tedious work.

The following chapter presents the direct stability investigation of the characteristic function of FDEs. This leads to an analytical stability criterion.

2. Direct Stability Investigation

The stability criterion established for RFDEs in *Section 2.1* is based directly on the investigation of the characteristic function in its original form. This can be generalized for a class of NFDEs as well. *Sections 2.3* and *2.4* will present some special necessary or sufficient criteria.

2.1. Retarded functional differential equations

In this section, the characteristic function of the linear and autonomous RFDE (1.6) is investigated when condition (1.10) holds. The main result is given in *Theorem 2.19.*

Lemma 2.1. Consider the characteristic function D of (1.6) given by

$$D(\lambda) = \det\big(\lambda I - \int_{-\infty}^{0} e^{\lambda\theta} d\eta(\theta)\big) = \sum_{k=0}^{n} (-1)^k a_k(\lambda)\lambda^{n-k}, \tag{2.1}$$

where

$$\int_{-\infty}^{0} e^{-\nu\theta} |d\eta_{jk}(\theta)| < +\infty, \quad j,k = 1,\ldots,n \tag{2.2}$$

for some scalar $\nu > 0$, and let $\mathbf{D} \subset \mathbf{C}$ be given as

$$\mathbf{D} = \{\, \lambda \in \mathbf{C} \,:\, \mathrm{Re}\,\lambda \geq -\nu + \epsilon,\ |\lambda| \leq H \,\}$$

where ϵ and H are arbitrarily small and large positive scalars respectively. Then the functions a_k $(k = 1,\ldots,n)$ and D are analytic and regular in $\mathbf{D}$.

Proof. For any $j,k = 1,\ldots,n$ and for every $\lambda \in \mathbf{D}$, the functions

$$f_{jk}(\lambda) = \int_{-\infty}^{0} e^{\lambda\theta} d\eta_{jk}(\theta)$$

have no singularities and they are differentiable since

$$|f_{jk}(\lambda)| \leq \int_{-\infty}^{0} e^{-\nu\theta} |d\eta_{jk}(\theta)| < +\infty$$

follows from $\operatorname{Re}\lambda > -\nu$ (see [9]). The functions a_k are formed from the elements of the matrix $[f_{jk}]$; for example

$$a_1(\lambda) = \operatorname{Tr} \int_{-\infty}^{0} e^{\lambda\theta} d\eta(\theta), \ldots, a_n(\lambda) = \det \int_{-\infty}^{0} e^{\lambda\theta} d\eta(\theta)$$

(Tr denotes the trace of a matrix). This implies that a_k $(k = 1, \ldots, n)$ are analytic and regular functions in **D** and this property is inherited by the characteristic function D according to (2.1). $\triangle$

Remark 2.2. Condition (2.2) is fulfilled, for example, when η has discontinuities of finite number at

$$-\tau_q < \ldots < -\tau_1 < -\tau_0 = 0,$$

it is differentiable elsewhere and there exist $\nu > 0$ and $K > 0$ scalars such that $|\eta'(\theta)| < Ke^{\nu\theta}$. In this case, the corresponding RFDE has the special form

$$\dot{x}(t) = \sum_{k=0}^{q} B_k x(t - \tau_k) + \int_{-\infty}^{0} w(\theta) x(t + \theta) d\theta$$

where the $n \times n$ constant matrices B_k $(k = 0, \ldots, q)$ and the function $w : (-\infty, 0] \to \mathbf{R}^n$ are uniquely determined by the function η. The function $|w|$ decreases exponentially as $\theta \to -\infty$. $\triangle$

The following definition of a contour in the complex plane will be used in the calculation of the number of the characteristic roots with positive real parts.

Definition 2.3. The curve (g) is called a Bromwich contour if

$$(g) = \bigcup_{k=1}^{3} (g_k) \tag{2.3}$$

where (g_k) $(k = 1, 2, 3)$ are given in the complex plane as follows:

(g_1) : $\lambda = He^{i\phi}$, $i = \sqrt{-1}$, $H \in \mathbf{R}_+$, ϕ ranges from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$;

(g_2) : $\lambda = i\omega$, where ω ranges from H to 0;

(g_3) : $\lambda = i\omega$, where ω ranges from 0 to $-H$.

Theorem 2.4. Let the characteristic function (2.1) have no zeros on the imaginary axis, and let (2.2) hold. Then the number N of its zeros in the right half of the complex plane is determined by

$$N = \frac{1}{2\pi i} \lim_{H \to +\infty} \oint_{(g)} \frac{1}{D(\lambda)} \frac{dD(\lambda)}{d\lambda} d\lambda\,, \tag{2.4}$$

where (g) stands for the Bromwich contour.

Proof. According to the Cauchy Residue Theorem, or the principle of arguments [9], the integral in (2.4) gives $(N - P)2\pi i$ where N and P are the number of zeros and poles (including their multiplicities) respectively within the closed contour (g). The contour $\lim_{H \to +\infty}(g)$ encircles the whole right half plane and the characteristic function has no pole there as it was shown in *Lemma 2.1*. Formula (2.4) follows from $P = 0$. $\triangle$

Corollary 2.5. The characteristic function (2.1) with condition (2.2) is stable if and only if

$$D(i\omega) \neq 0\,, \quad \omega \in [0, \infty) \tag{2.5a}$$

and

$$\lim_{H \to +\infty} \oint_{(g)} \frac{1}{D(\lambda)} \frac{dD(\lambda)}{d\lambda} d\lambda = 0\,, \tag{2.5b}$$

where (g) is the Bromwich contour given in *Definition 2.3*.

Proof. Condition $(2.5a)$ means that the characteristic function has no zero in the imaginary axis. Condition $(2.5b)$ is equivalent to (2.4) in *Theorem 2.4* when $N = 0$

which yields that (2.1) has no zero in the right half plane either. Thus, the characteristic function is stable according to *Definition 1.5.* $\triangle$

Remark 2.6. The formula (2.5*b*) has a clear geometrical meaning originated in the argument principle. Let us stick a pin into the origin of the complex plane and lay a closed loop of thread along the contour $D(\lambda)|_{\lambda\in(g)}$. This closed contour may be very complicated and it may encircle the pin to and fro several times. If we want to know whether the linear system is stable or not, then we have to pull the thread and to watch whether the thread can leave the pin or it is obstructed by it. If the thread comes off from the pin, there is stability. $\triangle$

Let us consider the continuous functions

$$f:\Omega_2\to\Omega_3\,,\quad \Omega_2,\Omega_3\subset \mathbf{C}$$

and

$$z:[a,b]\to\Omega_1\,,\quad \Omega_1\subset\Omega_2$$

where $[a,b]$ is a compact interval. The following notation is used throughout this chapter:

$$R(\omega)=\operatorname{Re} f(z(\omega)),\quad S(\omega)=\operatorname{Im} f(z(\omega)),\quad \omega\in[a,b]. \tag{2.6}$$

If R has real zeros of finite number in $[a,b]$ then they are denoted by

$$\rho_1\ge\rho_2\ge\ldots\ge\rho_r\,;\quad R(\rho_k)=0\,,\quad k=1,\ldots,r\,. \tag{2.7a}$$

If there are zeros with multiplicity two or more, they get two or more subscripts respectively. If $r=0$ then R has no zero in $[a,b]$. Similarly, when S has real zeros of finite number in $[a,b]$ then these zeros are denoted by

$$\sigma_1\ge\sigma_2\ge\ldots\ge\sigma_s\,;\quad S(\sigma_k)=0\,,\quad k=1,\ldots,s\,. \tag{2.7b}$$

The argument of the function f is

$$\varsigma=\arg f=\arctan\frac{S}{R}\,. \tag{2.8}$$

$\Delta_{[a,b]}\varsigma$ denotes the change of $\varsigma(\omega) = \arg f(z(\omega))$ as ω decreases from b to a.

The following two lemmas are independent from the characteristic functions, but they will be applied to the characteristic functions in order to calculate the integrals in (2.4) or (2.5b).

Lemma 2.7. Let the functions R and S given by (2.6) have zeros of finite number in an interval $[a,b]$ and let $S(a) = S(b) = 0$. If

$$f(z(\omega)) \neq 0, \quad \omega \in [a,b] \tag{2.9}$$

then

$$\Delta_{[a,b]}\varsigma = \pi \operatorname{sgn} R(\rho_1 + 0) \sum_{k=1}^{r} (-1)^{k+1} \operatorname{sgn} S(\rho_k), \tag{2.10}$$

where the real zeros ρ_k, $(k = 1, \dots, r)$ of R in $[a,b]$ are defined in (2.7a).

Proof. Let us take the zeros σ_j, $(j = 1, \dots, s)$ of S in $[a,b]$ as defined in (2.7b). Obviously, $\sigma_s = a$ and $\sigma_1 = b$. Condition (2.9) means that $\rho_k \neq \sigma_j$ $(k = 1, \dots, r,\ j = 1, \dots, s)$. The change of ς will be determined in the following steps:

1. Let $R(\rho_1 + 0) > 0$.

1.1. Let R have only one zero ρ_{2k+1} in an interval $I_j = (\sigma_{j+1}, \sigma_j)$. It has an odd subscript, thus $R(\rho_{2k+1} + 0) > 0$ and $R(\rho_{2k+1} - 0) < 0$.

1.1.1. If $S(\omega) > 0$, $\omega \in I_j$ then (2.8) gives the following values of the argument of f:

$$\varsigma(\sigma_j) = 0; \quad \varsigma(\rho_{2k+1} + 0) = \frac{\pi}{2} - 0;$$

$$\varsigma(\rho_{2k+1} - 0) = \frac{\pi}{2} + 0; \quad \phi(\sigma_{j+1}) = \pi.$$

This means that

$$\Delta_{I_j}\varsigma = \varsigma(\sigma_{j+1}) - \varsigma(\sigma_j) = \pi. \tag{2.11}$$

1.1.2. If $S(\omega) < 0$ in I_j then the result is

$$\Delta_{I_j}\varsigma = -\pi\,.$$

1.1.3. The expression

$$\Delta_{I_j}\varsigma = \pi \operatorname{sgn} S(\rho_{2k+1})$$

takes into account the sign of S in I_j.

1.2. Let the only zero of R lying in I_j have an even subscript: $\rho_{2k} \in I_j$. In this case we get everything in the same way as in 1.1 but with opposite signs:

$$\Delta_{I_j}\varsigma = -\pi \operatorname{sgn} S(\rho_{2k})\,.$$

1.3. If R has only one zero ρ_k in I_j then the change of the argument is given by

$$\Delta_{I_j}\varsigma = (-1)^{k+1}\pi \operatorname{sgn} S(\rho_k)\,.$$

1.4. If R has zeros $\rho_k, \rho_{k+1}, \ldots, \rho_{k+l}$ in I_j then

$$\Delta_{I_j}\varsigma = \pi \sum_{m=0}^{l} (-1)^{k+m+1} \operatorname{sgn} S(\rho_{k+m})\,,$$

that is $\Delta_{I_j}\varsigma$ is $+\pi$ or $-\pi$ depending on the fact whether the number of the odd or even subscripts is more in the zeros of R in I_j.

1.5. For the interval

$$[\sigma_s, \sigma_1] = \bigcup_{j=1}^{s-1} [\sigma_{j+1}, \sigma_j]\,,$$

the results obtained in 1.4 can be summed:

$$\Delta_{[\sigma_s,\sigma_1]}\varsigma = \pi \sum_{k=1}^{r} (-1)^{k+1} \operatorname{sgn} S(\rho_k)\,.$$

2. If $R(\rho_1 + 0) < 0$ then we get this result with opposite sign.

3. The formula (2.10) involves the sign of $R(\rho_1 + 0)$ and it serves the variation of the argument generally. $\triangle$

Lemma 2.8. Let the functions R and S given by (2.6) have zeros of finite number in the interval $[a, b]$ and let $R(a) = R(b) = 0$. If

$$f(z(\omega)) \neq 0, \quad \omega \in [a, b]$$

then

$$\Delta_{[a,b]}\varsigma = \pi \operatorname{sgn} S(\sigma_1 + 0) \sum_{k=1}^{s} (-1)^k \operatorname{sgn} R(\sigma_k), \tag{2.12}$$

where the real zeros σ_k, $(k = 1, \ldots, s)$ of S in $[a, b]$ are defined in (2.7*b*).

Proof. The proof of *Lemma 2.7* must be repeated. The basic step (which resulted in (2.11)) is as follows. Let us take the zeros ρ_j, $(j = 1, \ldots, r)$ of R in $[a, b]$ as defined in (2.7*a*), $\rho_r = a$, $\rho_1 = b$. If $S(\sigma_1 + 0) > 0$ and S has only one zero σ_{2k+1} in an interval $I_j = (\rho_{j+1}, \rho_j)$ where R is positive then

$$\varsigma(\rho_j - 0) = \frac{\pi}{2} - 0; \quad \varsigma(\sigma_{2k+1}) = 0;$$

$$\varsigma(\rho_{j+1} + 0) = -\frac{\pi}{2} + 0$$

as it is calculated from (2.8). Hence,

$$\Delta_{I_j}\varsigma = \varsigma(\rho_{j+1}) - \varsigma(\rho_j) = -\pi.$$

It is similar to (2.11), but it has a negative sign. All the results of the further steps of the proof are similar to those of the proof in *Lemma 2.7*, but with opposite signs. This can be seen in the final formulae (2.10) and (2.12) as well. $\triangle$

Let us return, now, to the investigation of the characteristic function (2.1), more exactly, to the calculation of the integral in the statement of *Theorem 2.4*.

Lemma 2.9. If (g_1) is the part of the Bromwich contour as it is in *Definition 2.3*, and D is the characteristic function (2.1) where (2.2) holds, then

$$\lim_{H\to+\infty}\int_{(g_1)}\frac{1}{D(\lambda)}\frac{dD(\lambda)}{d\lambda}d\lambda = n\pi i\,. \tag{2.13}$$

Proof. The calculation of the integral gives

$$\int_{(g_1)}\frac{1}{D(\lambda)}\frac{dD(\lambda)}{d\lambda}d\lambda = \Delta_{(g_1)}\ln(|D(\lambda)|e^{i\varsigma(\lambda)}) = i\Delta_{(g_1)}\varsigma$$

where

$$\varsigma(\lambda) = \arg D(\lambda)\,.$$

On the curve (g_1), $\lambda = He^{i\phi}$ holds. If

$$R(\phi) = \operatorname{Re} D(He^{i\phi}) \quad \text{and} \quad S(\phi) = \operatorname{Im} D(He^{i\phi})$$

then

$$\Delta_{(g_1)}\varsigma = \arctan\frac{S(\phi)}{R(\phi)}\bigg|_{\phi=-\pi/2}^{\pi/2}.$$

Lemma 2.1 implies that the functions R and S can be given as

$$R(\phi) = H^n\cos(n\phi) + o(H^n)\,,$$

$$S(\phi) = H^n\sin(n\phi) + o(H^n)\,.$$

This means that

$$\lim_{H\to+\infty} i\arctan\frac{S(\phi)}{R(\phi)}\bigg|_{\phi=-\pi/2}^{+\pi/2} = in\pi$$

and formula (2.13) is proved. △

Remark 2.10. *Lemma 2.9* can also be proved by the direct integration of the left-hand side of (2.13):

$$\lim_{H\to+\infty}\int_{-\pi/2}^{+\pi/2}\frac{e^{i\phi n}(n - \operatorname{Tr}\int_{-\infty}^{0}\theta e^{H\theta e^{i\phi}}d\eta(\theta)) + \frac{o(H^n)}{H^n}}{e^{i\phi n} + \frac{o(H^n)}{H^n}}id\phi = in\pi\,,$$

where we recall that the integrand is continuous for $\phi \in (-\frac{\pi}{2}, +\frac{\pi}{2})$ and bounded for $\phi \in [-\frac{\pi}{2}, +\frac{\pi}{2}]$ and $\cos\phi \geq 0$ in this interval. The application of *Lemmas 2.7* and *2.8* also leads to this result since the zeros ρ_k and σ_j of R and S can easily be determined as $H \to +\infty$. They are just at $(p\pi + \pi/2)/n$ and $(p\pi/n)$ respectively (p is integer). $\triangle$

Lemma 2.11. Consider the characteristic function (2.1) with condition (2.2), let D have no pure imaginary zero, and take the contours (g_2) and (g_3) given in *Definition 2.3*. Then

$$\int_{(g_2)\cup(g_3)} \frac{1}{D(\lambda)}\frac{dD(\lambda)}{d\lambda}d\lambda = 2i\Delta_{[0,H]} \arg D(i\omega)\,. \tag{2.14}$$

Proof. In the case of the contours (g_2) and (g_3), the functions R and S have the actual forms

$$R(\omega) = \operatorname{Re} D(i\omega)\,; \quad S(\omega) = \operatorname{Im} D(i\omega)\,. \tag{2.15}$$

It is easy to check that R is an even function and S is an odd one, that is

$$R(-\omega) = R(\omega)\,; \quad S(-\omega) = -S(\omega)\,.$$

This fact implies that

$$|D(i\omega)| = \sqrt{R^2(\omega) + S^2(\omega)}$$

is an even function, and

$$\arg D(i\omega) = \arctan\frac{S(\omega)}{R(\omega)}$$

is an odd function. Thus,

$$\int_{(g_2)\cup(g_3)} \frac{1}{D(\lambda)}\frac{dD(\lambda)}{d\lambda}d\lambda =$$

$$\Delta_{[-H,+H]} \ln|D(i\omega)| + i\Delta_{[-H,+H]} \arg D(i\omega) = 2i\Delta_{[0,H]} \arg D(i\omega)\,,$$

where ω ranges from H to 0 in $[0, H]$. The lemma is proved. $\triangle$

Lemma 2.12. Consider the characteristic function (2.1) where the dimension n of the corresponding RFDE (1.6) is even and (2.2) holds. Then the number of the real zeros of R in (2.15) is finite. If n is odd then the function S in (2.15) has real zeros of finite number.

Proof. For the pure imaginary values $\lambda = i\omega$, the separation of the real and imaginary parts of the characteristic function (2.1) results in

$$R(\omega) = (-1)^m \omega^n + o(\omega^n)\,; \quad S(\omega) = o(\omega^n) \quad \text{if } n = 2m\,; \tag{2.16a}$$

$$R(\omega) = o(\omega^n)\,; \quad S(\omega) = (-1)^m \omega^n + o(\omega^n) \quad \text{if } n = 2m+1\,, \tag{2.16b}$$

where m is integer. These formulae can be deduced from *Lemma 2.1* and they imply the statement of the lemma. $\triangle$

Remark 2.13. It is possible that the function R has no real zero at all. The function S always has at least one zero since it is an odd function, i.e. $S(0) = 0$.

Lemma 2.14. The characteristic function (2.1) is not stable if

$$D(0) \leq 0\,.$$

Proof. If $D(0) = 0$ then zero is a root of the characteristic equation and D is not stable. If $D(0) < 0$ then there exists at least one positive real root of the characteristic equation since

$$\lim_{H\to+\infty} D(H) = +\infty\,, \quad H \in \mathbf{R}$$

and D is continuous in $[0,\infty)$. In this case D is not stable either. $\triangle$

The following two theorems give useful formulae for the calculation of the number of characteristic roots with positive real parts according to *Theorem 2.4*.

Theorem 2.15. Let the dimension n of the RFDE (1.6) be even, that is $n = 2m$. If the corresponding characteristic function (2.1) has no zero on the imaginary axis, and (2.2) holds, then the number N of its zeros with positive real parts is given by

$$N = m + (-1)^m \sum_{k=1}^{r} (-1)^{k+1} \operatorname{sgn} S(\rho_k) \tag{2.17}$$

where $S(\omega) = \operatorname{Im} D(i\omega)$ and $\rho_1 \geq \ldots \geq \rho_r > 0$ are the real positive zeros of $R(\omega) = \operatorname{Re} D(i\omega)$.

Proof. *Lemma 2.12* implies that the number r of the positive zeros of R is finite (it may also be 0 as it was noted in *Remark 2.13*). The following calculation of N is a consequence of *Theorem 2.4, Lemmas 2.9* and *2.11*:

$$N = \frac{1}{2\pi i} \lim_{H \to +\infty} \sum_{k=1}^{3} \int_{(g_k)} \frac{1}{D(\lambda)} \frac{dD(\lambda)}{d\lambda} d\lambda = \frac{n}{2} + \frac{1}{\pi} \Delta_{[0,\infty)} \varsigma \tag{2.18}$$

where

$$\varsigma(\omega) = \arg D(i\omega) = \arctan \frac{S(\omega)}{R(\omega)}.$$

Since the characteristic function has no pure imaginary zero, the functions R and S have no common real zero. In the interval $[0, \rho_1]$, let the real zeros of S be denoted by $\sigma_1 \geq \ldots \geq \sigma_{s-1} \geq \sigma_s = 0$, where $\rho_1 > \sigma_1$. Let l be the number of the zeros of R which are greater than σ_1, that is $\rho_l > \sigma_1$ but $\rho_{l+1} < \sigma_1$. If $z(\omega) = i\omega$, $f = D$ and $a = \sigma_s$, $b = \sigma_1$ in *Lemma 2.7* then

$$\Delta_{[\sigma_s, \sigma_1]} \varsigma = \Delta_{[0,\sigma_1]} \varsigma = \pi \operatorname{sgn} R(\rho_1 + 0) \sum_{k=l+1}^{r} (-1)^{k+1} \operatorname{sgn} S(\rho_k).$$

In order to determine the variation of ς in the interval $[\sigma_1, +\infty)$, the same method can be used as in the proof of *Lemma 2.7*. Formula (2.16*a*) of *Lemma 2.12* explains that

$$\lim_{\omega \to +\infty} \varsigma(\omega) = \lim_{\omega \to +\infty} \arctan \frac{S(\omega)}{R(\omega)} = 0.$$

Repeating the steps which lead to (2.11), for the interval $[\sigma_1, \infty)$, we obtain the formula

$$\Delta_{[\sigma_1,\infty)} \varsigma = \pi \operatorname{sgn} R(\rho_1 + 0) \sum_{k=1}^{l} (-1)^{k+1} \operatorname{sgn} S(\rho_k).$$

Obviously,

$$\Delta_{[0,\infty)} \varsigma = \Delta_{[\sigma_s, \sigma_1)} \varsigma + \Delta_{[\sigma_1,\infty)} \varsigma.$$

The expression (2.16a) of the function R also shows that

$$\operatorname{sgn} R(\rho_1 + 0) = \operatorname{sgn} \lim_{\omega \to +\infty} R(\omega) = (-1)^m$$

since ρ_1 is the greatest zero of R. If all these results are substituted into (2.18) then (2.17) in the statement of the theorem follows immediately. $\triangle$

Examples for the application of this theorem can be found in *Theorem 2.28, 3.31* and *4.7*.

Theorem 2.16. Let the dimension n of the RFDE (1.6) be odd, that is $n = 2m+1$. If the corresponding characteristic function (2.1) has no zero on the imaginary axis, and (2.2) holds, then the number N of its zeros with positive real part is given by

$$N = m + \frac{1}{2} + (-1)^m \Big(\sum_{k=1}^{s-1} (-1)^k \operatorname{sgn} R(\sigma_k) + \frac{1}{2}(-1)^s \operatorname{sgn} R(0) \Big) \tag{2.19}$$

where $R(\omega) = \operatorname{Re} D(i\omega)$ and $\sigma_1 \geq \ldots \geq \sigma_s = 0$ are the non-negative real zeros of $S(\omega) = \operatorname{Im} D(i\omega)$.

Proof. *Lemma 2.12* and *Remark 2.13* imply that $s \geq 1$ and finite. Since the characteristic function has no pure imaginary zero, the functions R and S have no common real zero. As in the proof of *Theorem 2.15*, (2.18) gives N. Now, $n/2 = m + 1/2$ and $\Delta_{[0,\infty)}\varsigma$ will be determined by means of *Lemma 2.8*. In the interval $[0, \sigma_1]$, let the real zeros of R be denoted by $\rho_1 \geq \ldots \geq \rho_r > 0$, where $\sigma_1 > \rho_1$. Let l be the number of the zeros of S which are greater than ρ_r, that is $\sigma_l > \rho_r$ and $\sigma_{l+1} < \rho_r$. The application of *Lemma 2.8* results

$$\Delta_{[\rho_r,\infty)}\varsigma = \pi \operatorname{sgn} S(\sigma_1 + 0) \sum_{k=1}^{l} (-1)^k \operatorname{sgn} R(\sigma_k)$$

in exactly the same way as it was in the proof of *Theorem 2.15*. However, it needs special attention to calculate the change of ς in the interval $[0, \rho_r]$:

$$\Delta_{[0,\rho_r]}\varsigma = \frac{1}{2}\Delta_{[-\rho_r,\rho_r]}\varsigma$$

$$= \frac{1}{2}\pi \operatorname{sgn} S(\sigma_1 + 0)\Big(2 \sum_{k=l+1}^{s-1} (-1)^k \operatorname{sgn} R(\sigma_k) + (-1)^s \operatorname{sgn} R(\sigma_s)\Big).$$

Lemma 2.8 has been applied again when R is an even function, i.e. $R(\sigma_k) = R(-\sigma_k)$. Formula (2.16$b$) shows that

$$\operatorname{sgn} S(\sigma_1 + 0) = \operatorname{sgn} \lim_{\omega \to +\infty} S(\omega) = (-1)^m .$$

Since

$$\Delta_{[0,\infty)}\varsigma = \Delta_{[0,\rho_r]}\varsigma + \Delta_{[\rho_r,\infty)}\varsigma ,$$

the formula (2.19) in the statement of the theorem is true. $\triangle$

Corollary 2.17. Let the dimension n of the RFDE (1.6) be even, i.e. $n = 2m$, and let (2.2) hold. The characteristic function (2.1) is stable if and only if

$$S(\rho_k) \neq 0, \quad k = 1, \ldots, r \quad \text{and} \tag{2.20a}$$

$$\sum_{k=1}^{r} (-1)^k \operatorname{sgn} S(\rho_k) = (-1)^m m \tag{2.20b}$$

where $S(\omega) = \operatorname{Im} D(i\omega)$ and $\rho_1 \geq \ldots \geq \rho_r > 0$ are the real positive zeros of $R(\omega) = \operatorname{Re} D(i\omega)$.

Proof. *Corollary 2.5* and *Theorem 2.15* imply the statement. It is trivial that condition (2.5a) of *Corollary 2.5* is equivalent to (2.20a). Condition (2.5b) can be given by formula (2.17) in *Theorem 2.15* when the number of the zeros of the characteristic function (2.1) with positive real parts is zero, i.e. $N = 0$. $\triangle$

Corollary 2.18. Let the dimension n of the RFDE (1.6) be odd, i.e. $n = 2m+1$, and (2.2) hold. The characteristic function (2.1) is stable if and only if

$$R(\sigma_k) \neq 0, \quad k = 1, \ldots, s \quad \text{and} \tag{2.21a}$$

$$\sum_{k=1}^{s-1} (-1)^k \operatorname{sgn} R(\sigma_k) + \frac{1}{2}\big((-1)^s \operatorname{sgn} R(0) + (-1)^m\big) + (-1)^m m = 0 \tag{2.21b}$$

where $R(\omega) = \operatorname{Re} D(i\omega)$ and $\sigma_1 \geq \ldots \geq \sigma_s = 0$ are the non-negative real zeros of $S(\omega) = \operatorname{Im} D(i\omega)$.

Proof. This is a consequence of *Corollary 2.5* and formula (2.19) of *Theorem 2.16* when $N = 0$. $\triangle$

With the help of *Lemma 2.14*, formula (2.21*b*) can be simplified, since

$$D(0) = R(0) > 0$$

is a necessary condition of stability and in this case

$$\operatorname{sgn} R(0) = +1 .$$

Theorem 1.6 means that the stability of the characteristic function and the exponential stability of the trivial solution of the linear autonomous RFDE (1.6) are equivalent if condition (1.10) holds. However, this is identical to condition (2.2) for the characteristic function (2.1). If it holds, the necessary and sufficient condition of the stability of the characteristic function can be given by means of *Corollaries 2.17* and *2.18*. All these lead to the *main result* which is presented in the following theorem.

Theorem 2.19. Consider the n-dimensional linear autonomous RFDE

$$\dot{x}(t) = \int_{-\infty}^{0} [d\eta(\theta)] x(t+\theta)$$

and suppose that there exists a scalar $\nu > 0$ such that

$$\int_{-\infty}^{0} e^{-\nu\theta} |d\eta_{jk}(\theta)| < +\infty , \quad j, k = 1, \ldots, n .$$

The characteristic function assumes the form

$$D(\lambda) = \det\big(\lambda I - \int_{-\infty}^{0} e^{\lambda\theta} d\eta(\theta)\big) .$$

Let $\rho_1 \geq \ldots \geq \rho_r \geq 0$ and $\sigma_1 \geq \ldots \geq \sigma_s = 0$ denote the non-negative real zeros of R and S respectively, where

$$R(\omega) = \operatorname{Re} D(i\omega)\,, \quad S(\omega) = \operatorname{Im} D(i\omega)\,.$$

The trivial solution $x = 0$ of the RFDE is exponentially asymptotically stable if and only if

$$n = 2m\,,$$

$$S(\rho_k) \neq 0\,, \quad k = 1, \ldots, r\,, \tag{2.22a}$$

$$\sum_{k=1}^{r} (-1)^k \operatorname{sgn} S(\rho_k) = (-1)^m m\,; \tag{2.22b}$$

or

$$n = 2m + 1\,,$$

$$R(\sigma_k) \neq 0, \quad k = 1, \ldots, s-1\,, \tag{2.23a}$$

$$R(0) > 0\,, \tag{2.23b}$$

$$\sum_{k=1}^{s-1} (-1)^k \operatorname{sgn} R(\sigma_k) + \frac{1}{2}\big((-1)^s + (-1)^m\big) + (-1)^m m = 0 \tag{2.23c}$$

where m is integer.

Proof. The stability criterion is an immediate consequence of *Theorem 1.6, Lemma 2.14, Corollaries 2.17* and *2.18.* $\triangle$

For simple, representative applications of this theorem, see *Theorems 3.17, 3.24* or *3.28.*

With the help of the sign function, this criterion gives a complicated system of inequalities in a simple form. Similar mathematical formulation was used in the Yesipovich-Svirskii criterion (see formula (1.14)) but in a more complicated way, and for RDDEs only. The direct stability investigation in *Theorem 2.19* is a more general analytical criterion than the Pontyagin criterion since it works in the case of continuous delays or discrete delays of irrational ratios as well. It is valid for the same class of RFDEs as the geometrical criterion of Michailov, since both criteria are based on the application of the argument principle for the characteristic function directly. However, as the following section will show, this analytical criterion can be extended for a large class of NFDEs as well.

The examples in *Chapter 4* will demonstrate that the direct stability investigation is very convenient in the case of holonomic scleronomic dynamical systems of analytical mechanics, because the dimension of the corresponding RFDE is always even, and the number m in the stability condition (2.22b) is just the degree of freedom of the mechanical system.

2.2. Generalization to equations of neutral type

The direct stability investigation presented in *Theorem 2.19* is not applicable generally for the linear autonomous NFDE

$$\dot{x}(t) = \int_{-\infty}^{0} [d\eta_1(\theta)]\dot{x}(t+\theta) + \int_{-\infty}^{0} [d\eta_0(\theta)]x(t+\theta) .$$

It is not easy to give the "greatest" class of NFDEs where the stability criterion works without any change in its actual form. In order to avoid extra complications in notation, this class of NFDEs will be presented step-by-step.

First, let us consider the n-dimensional NFDE

$$\dot{x}(t) = \int_{-\infty}^{0} w(\theta)\dot{x}(t+\theta)d\theta + \int_{-\infty}^{0} [d\eta_0(\theta)]x(t+\theta) \tag{2.24}$$

and suppose that there exist scalars $\nu > 0$ and $K > 0$ such that

$$|w(\theta)| < Ke^{\nu\theta} , \quad \theta \in (-\infty, 0] \tag{2.25a}$$

and

$$\int_{-\infty}^{0} e^{-\nu\theta}|d\eta_{0,jk}(\theta)| < +\infty\,, \quad j,k=1,\dots,n\,. \tag{2.25b}$$

Theorem 2.20. Suppose that conditions (2.25*a*-*b*) hold. Then the characteristic function

$$D(\lambda) = \det\bigl(\lambda I - \lambda\int_{-\infty}^{0} w(\theta)e^{\lambda\theta}d\theta - \int_{-\infty}^{0} e^{\lambda\theta}d\eta_0(\theta)\bigr) \tag{2.26}$$

is stable if and only if condition (2.22) or (2.23) is satisfied in *Theorem 2.19.*

Proof. All the proofs of the lemmas and theorems in *Section 2.1* have to be repeated for (2.26). *Lemma 2.1* and *Theorem 2.4* hold trivially. *Lemmas 2.7* and *2.8* are independent from the actual form of the characteristic functions. However, *Lemma 2.9* is not trivial for (2.26). The variation of $\varsigma(\phi) = \arg D(He^{i\phi})$ has to be calculated as ϕ ranges from $-\pi/2$ to $+\pi/2$ and $H \to +\infty$. Since

$$D(\lambda) = \lambda^n \det\bigl(I - \int_{-\infty}^{0} w(\theta)e^{\lambda\theta}d\theta\bigr) + o(\lambda^n)\,,$$

$$R(\phi) = \operatorname{Re} D(He^{i\phi})\,; \quad S(\phi) = \operatorname{Im} D(He^{i\phi})$$

and

$$\lim_{H\to+\infty}\int_{-\infty}^{0} w_{jk}(\theta)e^{H\theta\cos\phi}\bigl(\cos(H\theta\sin\phi) + i\sin(H\theta\sin\phi)\bigr)d\theta = 0$$

for $\cos\phi \geq 0$ and $j,k = 1,\dots,n$, it can be shown that

$$\lim_{H\to+\infty}\Delta_{(g_1)}\varsigma = \lim_{H\to+\infty}\arctan\frac{S(\phi)}{R(\phi)}\bigg|_{\phi=-\pi/2}^{+\pi/2} = n\pi$$

as it was in *Lemma 2.9*. *Lemma 2.11* is valid for (2.26) without any change. In *Lemma 2.12*, the formulae (2.16) will be true again, since for $n = 2m$,

$$R(\omega) = \operatorname{Re} D(i\omega) = (-1)^m\omega^n \operatorname{Re}\det\Bigl(I - \int_{-\infty}^{0} w(\theta)\bigl(\cos(\omega\theta) + i\sin(\omega\theta)\bigr)d\theta\Bigr) + o(\omega^n)\,,$$

$$S(\omega) = \operatorname{Im} D(i\omega) = (-1)^m\omega^n \operatorname{Im}\det\Bigl(I - \int_{-\infty}^{0} w(\theta)\bigl(\cos(\omega\theta) + i\sin(\omega\theta)\bigr)d\theta\Bigr) + o(\omega^n)\,,$$

and the integrals tend to zero as $\omega \to +\infty$. Thus, R has real zeros of finite number if n is even, and in a similar way, S has real zeros of finite number if n is odd. *Lemma 2.14* remains true trivially. In the proof of *Theorem 2.15*, the only step which needs special attention is the calculation of

$$\lim_{\omega\to+\infty} \varsigma(\omega) = \lim_{\omega\to+\infty} \arctan \frac{S(\omega)}{R(\omega)} = 0,$$

but this follows from the above-mentioned forms of R and S. Thus, formulae (2.17) and (2.19) give the number of the zeros with positive real parts in the case of the characteristic function (2.26) as well. The stability conditions (2.22) and (2.23) in *Theorem 2.19* mean that this number has to be zero and there are no pure imaginary zeros of (2.26) either.
$\triangle$

Definition 2.21. Let C be an $n \times n$ constant matrix. The coefficient c_k, $(k = 1, \ldots, n)$ in the polynomial

$$\det(\lambda I + C) = \sum_{k=0}^{n} c_k \lambda^{n-k}$$

is called the kth scalar invariant of C. For example,

$$c_0 = 1, \quad c_1 = \operatorname{Tr} C, \quad \ldots, \quad c_n = \det C.$$

Lemma 2.22. If C is an $n \times n$ constant matrix and

$$\sum_{k=0}^{n} (-1)^k c_k \cos(k\phi) > 0, \quad \text{for all } \phi \in [0, 2\pi) \tag{2.27}$$

then the eigenvalues of C are situated in the open unit disc of the complex plane.

Proof. Let us consider the polynomial

$$P(\lambda) = \det(I - \lambda C) = \sum_{k=0}^{n} (-1)^k c_k \lambda^k$$

and the unit circle given by

$$\lambda(\phi) = e^{i\phi}, \quad \phi \in [0, 2\pi).$$

Thus,

$$R(\phi) = \operatorname{Re}\det(I - e^{i\phi}C) = \sum_{k=0}^{n}(-1)^k c_k \cos(k\phi),$$

$$S(\phi) = \operatorname{Im}\det(I - e^{i\phi}C) = \sum_{k=1}^{n}(-1)^k c_k \sin(k\phi),$$

$S(0) = S(2\pi) = 0$. The function R has no real zero because of (2.27). *Lemma 2.7* implies that the change of the argument of $P(e^{i\phi})$, as ϕ ranges from 0 to 2π, is zero, which means that there is no zero of P in the unit disc. If all the zeros of P are out of the unit disc, the zeros of the characteristic polynomial $\det(zI - C)$, $(z \in \mathbf{C})$ lie inside the unit circle. $\triangle$

Corollary 2.23. If C is an $n \times n$ constant matrix, condition (2.27) holds and τ is a positive scalar, then all the zeros λ_k, $k = (1, 2, \ldots)$ of the function d given by

$$d(\lambda) = \det(I - Ce^{-\lambda\tau})$$

have negative real parts.

Proof. $z_k = e^{\lambda_k \tau}$, $(k = 1, 2, \ldots)$ is an eigenvalue of C. *Lemma 2.22* implies that

$$|z_k| = |e^{\lambda_k \tau}| < 1$$

which means that $\operatorname{Re}\lambda_k < 0$, $(k = 1, 2, \ldots)$. $\triangle$

Lemma 2.24. If C is an $n \times n$ constant matrix and

$$\sum_{k=1}^{n} |c_k| < 1 \tag{2.28}$$

then the eigenvalues of C are situated in the open unit disc of the complex plane.

Proof. The inequality (2.28) implies that condition (2.27) is satisfied. The statement follows from *Lemma 2.22.* △

Let us consider the n-dimensional NFDE

$$\dot{x}(t) = C\dot{x}(t-\tau) + \int_{-\infty}^{0} [d\eta_0(\theta)]x(t+\theta) \tag{2.29}$$

where the discrete delay τ is positive and

$$\int_{-\infty}^{0} e^{-\nu\theta}|d\eta_{0,jk}(\theta)| < +\infty, \quad j,k = 1,\dots,n$$

for some scalar $\nu > 0$.

Theorem 2.25. The characteristic function

$$D(\lambda) = \det\big(\lambda I - \lambda e^{-\lambda\tau}C - \int_{-\infty}^{0} e^{\lambda\theta}d\eta_0(\theta)\big) \tag{2.30}$$

of the NFDE (2.29) is stable if

$$\sum_{k=1}^{n} |c_k| < 1$$

and condition (2.22) or (2.23) is satisfied in *Theorem 2.19.*

Proof. As in *Theorem 2.20,* only the critical steps of the proofs in *Section 2.1* will be checked.

In the proof of *Lemma 2.9,* $\lim_{H\to+\infty}\Delta_{(g_1)}\varsigma$ does not exist, where $\varsigma(\phi) = \arg D(He^{i\phi})$. This fact gives the limitation of Michailov's geometrical criterion for NFDEs. To solve this problem, a special sequence $\{(g^j)\}$, $(j = 1,2,\dots)$ of contours will be chosen:

(g_1^j) : $\lambda = H_j e^{i\phi}$, $H_j = \frac{2j\pi}{\tau}$, ϕ ranges from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$;

(g_2^j) : $\lambda = i\omega$, ω ranges from $\frac{2j\pi}{\tau}$ to 0;

(g_3^j) : $\lambda = i\omega$, ω ranges from 0 to $-\frac{2j\pi}{\tau}$, where $j = 1, 2, \ldots$ and

$$(g^j) = \bigcup_{k=1}^{3} (g_k^j)$$

as in *Definition 2.3.* Since

$$D(\lambda) = \lambda^n \sum_{k=0}^{n} (-1)^k c_k e^{-k\lambda\tau} + o(\lambda^n),$$

the change of its argument along the contour $\lim_{j\to+\infty}(g_1^j)$ is

$$\lim_{j\to+\infty} \Delta_{(g_1^j)} \arg D =$$

$$\lim_{j\to+\infty} \Delta_{(g_1^j)} \arg \lambda^n + \lim_{j\to+\infty} \Delta_{(g_1^j)} \arg \sum_{k=0}^{n} (-1)^k c_k e^{-k\lambda\tau}.$$

In this expression, the first term is just $n\pi$. If the second term is zero then *Lemma 2.9* is proved. This can be shown via the application of *Lemma 2.7,* since

$$\arctan \frac{\tilde{S}(\phi)}{\tilde{R}(\phi)}\bigg|_{\phi=-\pi/2}^{+\pi/2} = 0$$

for all $H_j = (2j\pi)/\tau$, $(j = 1, 2, \ldots)$ where

$$\tilde{R}(\phi) = \sum_{k=0}^{n} (-1)^k c_k e^{-k\tau H_j \cos\phi} \cos(k\tau H_j \sin\phi) \geq 1 - \sum_{k=1}^{n} |c_k| > 0,$$

i.e. $\tilde{R}$ has no real zero, and

$$\tilde{S}(\phi) = -\sum_{k=1}^{n} (-1)^k c_k e^{-k\tau H_j \cos\phi} \sin(k\tau H_j \sin\phi),$$

i.e.

$$\tilde{S}(-\frac{\pi}{2}) = \tilde{S}(+\frac{\pi}{2}) = 0.$$

Lemma 2.12 can also be proved with the help of condition (2.28). If $n = 2m$, for example, then

$$R(\omega) = \operatorname{Re} D(i\omega) = (-1)^m \omega^n \sum_{k=0}^{n} (-1)^k c_k \cos(k\omega\tau) + o(\omega^n),$$

$$S(\omega) = \operatorname{Im} D(i\omega) = (-1)^m \omega^n \sum_{k=1}^{n} (-1)^k c_k \sin(k\omega\tau) + o(\omega^n),$$

and $\lim_{\omega\to+\infty} R(\omega) = (-1)^m \infty$ follows from (2.28). Thus, the number of the real zeros of R is finite. This is true for S in a similar way if $n = 2m+1$.

The last difficult step is the calculation of $\lim_{\omega\to+\infty} \varsigma(\omega)$ in the proof of *Theorem 2.15*. This limit exists if ω is increased along $H_j = 2j\pi/\tau$ in accordance with the definition of (g_2^j):

$$\lim_{j\to+\infty} \varsigma(H_j) = \lim_{j\to+\infty} \arctan \frac{S(H_j)}{R(H_j)} = 0,$$

as it can be seen from the above-mentioned formulae of the functions R and S when $n = 2m$. In the case of *Theorem 2.16*, this can be repeated for the odd dimensions $n = 2m+1$. All these lead to the stability conditions (2.22) and (2.23) again. $\triangle$

Remark 2.26. *Theorem 1.9* and *Corollary 2.23* show that the characteristic function (2.30) does not have a sequence of zeros which tends to the imaginary axis if condition (2.28) holds. In this case, *Theorem 1.8* implies the exponential asymptotic stability of the trivial solution of the NFDE (2.29) when condition (2.22) or (2.23) is satisfied, since $\operatorname{Re}\lambda_k \le -\epsilon < 0$ is fulfilled for all the zeros λ_k of the characteristic function (2.30). $\triangle$

For representative applications of *Theorem 2.25*, see *Theorems 3.10* and *3.20*.

Let us consider the n-dimensional NFDE

$$\dot{x}(t) = \sum_{k=1}^{p} C_k \dot{x}(t-\tau_k) + \int_{-\infty}^{0} w(\theta)\dot{x}(t+\theta)d\theta + \int_{-\infty}^{0} [d\eta_0(\theta)]x(t+\theta), \qquad (2.31)$$

where the functions w and η_0 satisfy condition (2.25) and all the ratios of the discrete delays $\tau_k > 0$, $(k = 1, \dots, p)$ are rational numbers. C_k, $(k = 1, \dots, p)$ are constant $n \times n$ matrices.

Proposition 2.27. Consider the characteristic function

$$D(\lambda) = \det(\lambda I - \lambda \sum_{k=1}^{p} C_k e^{-\lambda \tau_k} - \lambda \int_{-\infty}^{0} w(\theta) e^{\lambda\theta} d\theta - \int_{-\infty}^{0} e^{\lambda\theta} d\eta_0(\theta)) \tag{2.32}$$

of the NFDE (2.31), where

$$\operatorname{Re}\det(I - \sum_{k=1}^{p} C_k e^{-i\omega\tau_k}) > 0 \tag{2.33}$$

for $\omega \in \mathbf{R}$. The characteristic function (2.32) is stable if condition (2.22) or (2.23) in *Theorem 2.19* is satisfied.

Proof. The proposition can be proved if the methods used in the proofs of *Theorems 2.20* and *2.25* are combined. Condition (2.28) (or (2.27)) can be substituted by the inequality (2.33). Since the discrete delays have rational ratios, it is possible to construct a sequence $\{(g^j)\}$ which can be used to calculate $\lim_{j\to+\infty} \Delta_{(g^j)} \arg D$. $\triangle$

It may cause difficulties to check assumption (2.33), but *Proposition 2.27* gives the most general case where the application of the direct stability investigation for NFDEs is still possible with conditions (2.22) and (2.23) in their original form.

The method can be extended even further when the stability of the function

$$d(\lambda) = \det(I - \sum_{k=1}^{p} C_k e^{-\lambda\tau_k})$$

is required instead of (2.33). As *Theorem 1.9* shows, this assumption ensures that the number of characteristic roots with non-negative real parts is finite. However, *Theorem 2.19* should be altered, since both the real and imaginary parts (R and S) of the characteristic function (2.32) at $\lambda = i\omega$ may simultaneously have real zeros of infinite number. This means that the numbers r or s of these zeros are not finite in the stability criterion. By means of *Theorem 1.9*, it is still possible to define finite number of relevant zeros of the functions R and S, but the method will in practice become too complicated and it loses its relatively simple form.

There is one, even theoretically, unsolved problem left. The direct stability investigation cannot be used for the general NFDE (1.8) when η_1 has discontinuities at τ_1 and τ_2, and τ_1/τ_2 is irrational. In this case, the contour $D(\lambda)|_{\lambda\in(g)}$ seems to behave "chaotically" as $H \to +\infty$ in the Bromwich contour (g), so we cannot calculate the limit of the variation of the argument of D along (g). However, we think that this is only a technical problem. Nevertheless, there are also open questions about the connection of the stability of the characteristic function and the asymptotic stability of the linear NFDE in these general cases.

2.3. Necessary conditions

By means of the theorems of the previous sections, some simple necessary conditions of stability can be constructed for characteristic functions.

Theorem 2.28. Consider the characteristic function

$$D(\lambda) = \det\left(\lambda I - \int_{-\infty}^{0} e^{\lambda\theta} d\eta(\theta)\right)$$

of the RFDE (1.6) where the function η satisfies assumption (2.2). The number of its zeros with positive real parts is odd if

$$\det\left(\lim_{\theta\to-\infty} \eta(\theta) - \eta(0)\right) < 0\,. \tag{2.34}$$

Proof. Condition (2.2) implies that $\lim_{\theta\to-\infty}\eta(\theta)$ exists and it is finite. Let the dimension n of the RFDE be even first, i.e. $n = 2m$. Condition (2.34) shows that

$$R(0) = \operatorname{Re} D(i0) = \det\left(0I - \int_{-\infty}^{0} e^{0\theta} d\eta(\theta)\right) < 0\,,$$

while

$$\lim_{\omega\to+\infty} R(\omega) = (-1)^m \infty$$

comes from the expression (2.16a) of R. Let us apply *Theorem 2.15*. If m is odd then the number r of the positive zeros of R is even, and if m is even then r is odd. This means that $m+r$ is always odd if (2.34) holds. As (2.17) shows in *Theorem 2.15*, the number of the zeros of the characteristic function in the positive half of the complex plane is given by

$$m + (-1)^m \sum_{k=1}^{r} (-1)^{k+1} \operatorname{sgn} S(\rho_k)\,.$$

This is always odd, since the number $(m+r)$ of its terms is odd and the terms are just $+1$ or -1. The proof of the theorem is similar when the dimension $n = 2m+1$. In this case, *Theorem 2.16* has to be used when $\operatorname{sgn} R(0) = -1$ in formula (2.19). △

Remark 2.29. When η in the characteristic function has a finite support, i.e. $\eta(\theta) =$ constant for $\theta \in (-\infty, -h]$, $h > 0$ then condition (2.34) has the simple form

$$\det(\eta(-h) - \eta(0)) < 0\,.$$

Corollary 2.30. Consider the RFDE

$$\dot{x}(t) = \int_{-\infty}^{0} [d\eta(\theta)] x(t+\theta) \tag{2.35}$$

where η satisfies the usual condition (2.2). Its trivial solution is not asymptotically stable if

$$\det\Big(\lim_{\theta \to -\infty} \eta(\theta) - \eta(0)\Big) \le 0\,. \tag{2.36}$$

Proof. The statement is a consequence of *Theorem 1.6* and *Lemma 2.14*. If the determinant in question is strictly negative, the characteristic function has zeros of odd number, i.e. it has at least 1 zero, in the right half of the complex plane, as this follows from *Theorem 2.28*. △

Remark 2.31. *Theorem 2.28* and *Corollary 2.30* can be proved in the same form for the characteristic function (2.32) of the NFDE (2.31). △

The following necessary conditions of stability are based on *Theorem 2.19.*

Theorem 2.32. Consider the n-dimensional RFDE (2.35) where η satisfies (2.2). The trivial solution of the RFDE is unstable if any of the following conditions is true:

$$s < m; \tag{2.37a}$$

$$n = 2m \quad \text{and} \quad r < m; \tag{2.37b}$$

$$n = 2m + 1 \quad \text{and} \quad r < m - 1. \tag{2.37c}$$

m is integer, r and s are the numbers of the non-negative real zeros of R and S respectively as in *Theorem 2.19.*

Proof. Let us consider, first, the case when the dimension of the RFDE is even, i.e. $n = 2m$. If $r < m$ then $(2.22b)$ in *Theorem 2.19* is not fulfilled since

$$\left|\sum_{k=1}^{r}(-1)^k \operatorname{sgn} S(\rho_k)\right| \leq r < m$$

on the left-hand side in $(2.22b)$, and the absolute value of the right-hand side is just m. As formula (2.17) in *Theorem 2.15* shows, the characteristic function has at least $m - r$ zeros with positive real parts, so the trivial solution is unstable and $(2.37b)$ is proved.

If $s < m$ then $(2.22b)$ is not fulfilled again since

$$\left|\sum_{k=1}^{r}(-1)^k \operatorname{sgn} S(\rho_k)\right|$$

$$= \left|\sum_{j=1}^{s} \sum_{\rho_k \in I_j} (-1)^k \operatorname{sgn} S(\rho_k)\right| \leq \sum_{j=1}^{s} 1 = s < m$$

where $I_j = [\sigma_j, \sigma_{j-1})$, $(j = 1, 2, \ldots, s)$, $\sigma_0 = +\infty$ and *Theorems 2.19* and *2.15* imply instability. The proof is similar for the case $n = 2m + 1$. If $s < m$ or $r < m - 1$ then $(2.23c)$ is not fulfilled in *Theorem 2.19.* $\triangle$

Theorem 2.32 is very useful in applications to prove instability since it is often very simple to show that the number of the zeros of either $R(\omega) = \operatorname{Re} D(i\omega)$ or $S(\omega) = \operatorname{Im} D(i\omega)$ is not enough to satisfy the conditions of stability.

Remark 2.33. *Theorem 2.32* is true for the NFDE (2.31) as well. $\triangle$

2.4. Construction of sufficient conditions

In this section, the sufficient conditions will be constructed by means of polynomials only, instead of the investigation of the functions R and S originating from the transcendental characteristic function D. In order to understand these sufficient criteria of stability well, it may be useful to recall how the direct stability investigation works in the case of characteristic polynomials of ordinary differential equations.

Example 2.34. Let us investigate the stability of the polynomial

$$\lambda^2 + a_1\lambda + a_2 .$$

In *Theorem 2.19*, we have $n = 2$, $m = 1$ and

$$R(\omega) = -\omega^2 + a_2 , \quad S(\omega) = a_1\omega .$$

The stability condition (2.22*b*) has the form

$$\sum_{k=1}^{r} (-1)^k \operatorname{sgn} S(\rho_k) = -1$$

which is satisfied if and only if R has a zero $\rho_1 = \sqrt{a_2}$, i.e. $a_2 > 0$ and $S(\rho_1) > 0$, i.e. $a_1 > 0$. $\triangle$

This elementary result is well known from the Routh-Hurwitz criterion.

Example 2.35. Let us consider the polynomial

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 .$$

When *Theorem 2.19* is used to investigate the stability of this polynomial then $n = 3$, $m = 1$ and

$$R(\omega) = -a_1\omega^2 + a_3 , \quad S(\omega) = -\omega^3 + a_2\omega .$$

The condition (2.23*b*) gives

$$R(0) = a_3 > 0 .$$

In order to satisfy condition (2.23*c*) as well, S has to have the zeros

$$\sigma_1 = \sqrt{a_2} \quad \text{and} \quad \sigma_2 = 0 ,$$

thus, $a_2 > 0$, $s = 2$ and (2.23*c*) has the form

$$-\operatorname{sgn} R(\sigma_1) + \frac{1}{2}(1-1) - 1 = 0 ,$$

that is

$$R(\sigma_1) = -a_1a_2 + a_3 < 0 .$$

△

The following lemma is also well known in the literature since the Routh-Hurwitz criterion implies it.

Lemma 2.36. The polynomial

$$P(\lambda) = \sum_{k=0}^{n} a_k\lambda^{n-k} , \quad (a_0 = 1)$$

is stable if and only if the polynomials $R(\omega) = \operatorname{Re} P(i\omega)$ and $S(\omega) = \operatorname{Im} P(i\omega)$ have real simple zeros of maximum number, which are alternating, $R(0) > 0$ and $S'(0) > 0$.

Proof. If $n = 2m$ then the conditions of the theorem result in the satisfaction of (2.22b) in *Theorem 2.19*, that is

$$\sum_{k=1}^{m} (-1)^k \operatorname{sgn} S(\rho_k) = (-1)^m m\,.$$

Conversely, if either $r < m$ (see *Theorem 2.32*) or $R(0) \leq 0$ (see *Theorem 2.28*) or $S'(0) \leq 0$ (i.e. $S(\rho_m) \leq 0$) or the zeros of R and S are not alternating, this equation is not satisfied. The proof is similar if $n = 2m + 1$. Δ

Now, let us examine the transcendental characteristic functions again.

Theorem 2.37. Consider the RFDE (2.35) where η satisfies (2.2). Suppose that there exist the polynomials R^+, R^-, S^+ and S^- such that

$$R^-(\omega) \leq R(\omega) = \operatorname{Re} D(i\omega) \leq R^+(\omega)\,, \tag{2.38a}$$

$$S^-(\omega) \leq S(\omega) = \operatorname{Im} D(i\omega) \leq S^+(\omega)\,, \tag{2.38b}$$

for $\omega \in [0, +\infty)$, where D is the corresponding characteristic function. Suppose that R^+ and R^- have the same number of real positive zeros. These zeros are denoted by $\rho_1^+ \geq \ldots \geq \rho_p^+ > 0$ and $\rho_1^- \geq \ldots \geq \rho_p^- > 0$ and they determine the intervals $I_{Rl} = [\min(\rho_l^-, \rho_l^+), \max(\rho_l^-, \rho_l^+)]$, $(l = 1, \ldots, p)$. In exactly the same way, the zeros of S^+ and S^- define the intervals I_{Sj}, $(j = 1, \ldots, q)$. Furthermore, let all these intervals be disjoint and let us choose the representative real numbers $\rho_l^0 \in I_{Rl}$, $(l = 1, \ldots, p)$ and $\sigma_j^0 \in I_{Sj}$, $(j = 1, \ldots, q-1)$. If either (2.22) or (2.23) in *Theorem 2.19* is satisfied by these numbers then the trivial solution of the RFDE is exponentially asymptotically stable.

Proof. For the non-negative zeros of R and S, it is true that

$$\rho_k \in \bigcup_{l=1}^{p} I_{Rl}\,, \quad k = 1, \ldots, r\,,$$

$$\sigma_k \in \bigcup_{j=1}^{q} I_{Sj}, \quad j = 1, \dots, q$$

as follows from (2.38). Moreover, the number of the zeros of R or S in any of the intervals I_{Rl} or I_{Sj} is odd. The intervals are disjoint which implies that

$$\sum_{\rho_k \in I_{Rl}} (-1)^k \operatorname{sgn} S(\rho_k) = (-1)^l \operatorname{sgn} S(\rho_l^0), \quad l = 1, \dots, p;$$

$$\sum_{\sigma_k \in I_{Sj}} (-1)^k \operatorname{sgn} R(\sigma_k) = (-1)^j \operatorname{sgn} R(\sigma_j^0), \quad j = 1, \dots, q.$$

This means that the conditions (2.22b) and (2.23c) can be rewritten as

$$\sum_{l=1}^{p} (-1)^l \operatorname{sgn} S(\rho_l^0) = (-1)^m m;$$

$$\sum_{j=1}^{q-1} (-1)^j \operatorname{sgn} R(\sigma_j^0) + \frac{1}{2}((-1)^q + (-1)^m) + (-1)^m m = 0,$$

and *Theorem 2.19* implies the stability. △

Corollary 2.38. Consider the n-dimensional RFDE (2.35) where η satisfies (2.2). Suppose that the degrees of the polynomials R^+, R^- and S^+, S^- in (2.38) are exactly n and $n-1$ respectively if n is even, and they are exactly $n-1$ and n respectively if n is odd. The trivial solution of the RFDE is exponentially asymptotically stable if R^+, R^- and S^+, S^- have real zeros of maximum number, the intervals, determined by them as in *Theorem 2.37*, are disjoint and alternating, $R^-(0) > 0$ and $S^-(\omega) > 0$ for $\omega \in I_{Rm}$ ($m = \operatorname{int}(n/2)$).

Proof. This sufficient condition of stability is a consequence of *Lemma 2.36* and *Theorem 2.37*. △

Remark 2.39. It is often easy to construct polynomials R^+, R^- and S^+, S^- in order to apply *Corollary 2.38*. As it has been shown in *Lemma 2.1*, the functions $a_k(\lambda)$, ($k = 1, \dots, n$) in the characteristic function

$$D(\lambda) = \sum_{k=0}^{n} (-1)^k a_k(\lambda) \lambda^{n-k}$$

are bounded if $\operatorname{Re}\lambda > -\nu$ ($\nu > 0$). Thus, there exist positive scalars A_k, ($k = 1, \ldots, n$) such that

$$|a_k(i\omega)| < A_k, \quad \omega \in [0, +\infty), \quad k = 1, \ldots, n.$$

In this way $\operatorname{Re} D(i\omega)$ and $\operatorname{Im} D(i\omega)$ can be estimated by polynomials based on the coefficients A_k. △

Remark 2.40. If the exact form of the above-mentioned $a_k(\lambda)$ is known, then much better estimations can be made. Usually, $a_k(i\omega)$ contains trigonometrical functions. If these are estimated by means of the inequalities

$$|\cos y| \leq 1, \quad |\sin y| \leq 1$$

then we may get sufficient conditions of the stability which are independent from the length h of the delay since h (or the discrete delays τ_k) appear only in the argument of the trigonometrical functions and they disappear from the formulae after the application of these estimations. These criteria are similar to those of Tzypkin [62]. If, for example, the estimation

$$|\sin y| \leq y, \quad y \in [0, +\infty)$$

is used then the length h of the delay does appear in the stability condition, which usually gives a critical maximum value of the length of the delay. △

For examples, see *Theorems 3.22, 3.23, 4.1* or *Corollaries 4.3, 4.5, 4.19.*

2.5. The basis of non-linear investigations

In the stability investigation of the trivial solutions of non-linear FDEs, the fundamental problem is to give the conditions when the asymptotic stability of a trivial solution follows from the asymptotic stability of the zero solution of the corresponding linear system of first approximation. This problem has been solved by the theory of stability by first approximation in a similar way as in the case of ordinary differential equations. We cite an early theorem of Halanay (see [23], Theorem 4.6) here for autonomous RFDEs with bounded delay, i.e. we consider finite h in the definition of space $\mathbf{B}$ in *Section 1.1.*

Theorem 2.41. Consider the RFDE

$$\dot{x}(t) = L(x_t) + f(x_t) \tag{2.39}$$

with bounded delay, where the functional $L : \mathbf{B} \to \mathbf{R}^n$ is linear and continuous, and the functional $f : \mathbf{B} \to \mathbf{R}^n$ is continuous with the property that $|f(x_t)| < \kappa ||x_t||$, κ being sufficiently small for $||x_t|| < K$. If the trivial solution of the first approximation linear system

$$\dot{y}(t) = L(y_t)$$

is asymptotically stable, then the trivial solution of the system (2.39) is likewise asymptotically stable.

Proof. See in [23]. $\triangle$

There are, of course, a lot of further, more general results in the literature about the connection of the non-linear and the corresponding first-approximation linear FDEs. We mention here the publication [46] of Ruiz-Claeyssen for finite delay equations, the first chapter of Cushing's book [11] for unbounded delays, and the references in Section 5.4 of the book [35] of Kolmanovskii and Nosov.

When a system is called oscillatory, it usually means that the corresponding characteristic function at one of its equilibria has a pair of relevant complex conjugate zeros close to the imaginary axis. The stability of the equilibrium is determined by these relevant zeros. If the parameters of the system are varied, these relevant characteristic roots may cross the imaginary axis, and the equilibrium will lose or get back its stability.

In the meantime, a periodic orbit can bifurcate from the equilibrium. This process is referred to as Hopf bifurcation of the non-linear system.

Formally, the Hopf Bifurcation Theorem for the infinite dimensional problem of RFDEs seems to be a straightforward generalization of its finite dimensional version for ordinary differential equations. But the proof of the theorem needs a special technique if the delay is not excluded from the bifurcation parameters [26]. As a matter of fact, it is usually the most interesting case when the length of the delay varies in the non-linear system since this gives the major difference from the finite dimensional systems.

The Hopf Bifurcation Theorem for RFDEs with delay of finite length is presented in [25] and, in a more general form, in [26]. Stech [51] gives a theorem for RFDEs with unbounded delays which involves a method to determine the senses of generic Hopf bifurcations as well as those of degenerate ones of co-dimension 1 (see also the classification of degenerate Hopf bifurcations in [21]). The book of Hassard et al. [27] also presents an efficient algorithm to organize the necessary calculations. This algorithm involves the ready-made formulae published in [1] for the simplest two-dimensional case.

The main point of the Hopf Bifurcation Theorem is to check that a pair of simple complex conjugate zeros of the characteristic function cross the imaginary axis with non-zero velocity as the bifurcation parameter ranges, while all the other zeros of the characteristic function remain in the left half of the complex plane. The D-subdivision method serves those critical parameters where the characteristic roots are pure imaginary numbers. But as it is noted in [26], one of the main difficulties always consists of analysing the characteristic equation, namely proving that all the characteristic roots except the critical ones have negative real parts. This is the only step of the long algorithm of the Hopf bifurcation calculation which needs special investigation in any example. *Theorems 2.15, 2.16* and *2.19* support this work, as it will be shown in the examples of *Chapter 4.*

3. Stability Charts

The so-called stability charts are constructed in the plane of two parameters of a dynamical system. A chart gives those regions in the parameter plane where the equilibrium of a system is asymptotically stable. These domains can be determined by means of *Theorem 2.19*. *Theorems 2.15* and *2.16* will help to determine the number of characteristic roots with positive real parts in those regions where the equilibrium is unstable. This makes it easier to investigate the types of the bifurcations of the equilibrium at the critical parameters of a nonlinear system. The stability chart provides a simple geometrical summarization of how the system parameters affect stability. It causes difficulties, of course, to get a similar clear geometrical picture when three parameters or more are to be analysed in a system, but even in these cases we often find a way to represent stability results geometrically. They are useful guides for engineers in design work. Moreover, the stability charts of linear FDEs have an important contribution in understanding the often peculiar physical behaviour of retarded dynamical systems.

This chapter will present some basic stability charts and stability results in the case of some mathematical examples related to simplified mechanical or biological models. However, real mechanical and biological examples will be discussed only in *Chapter 4*.

3.1. Undamped systems with discrete delays

If there is no viscous damping in a holonomic, scleronomic mechanical system of finite degree of freedom, the odd derivatives of the general coordinates usually do not appear in the equations of motion. For example, the linear, conservative oscillatory system of m degrees of freedom is described by

$$\ddot{q} + Kq = 0$$

where $q \in \mathbf{R}^m$ is the vector of general coordinates and K is the product of the inverse of the general mass matrix and the stiffness matrix. The corresponding characteristic polynomial does not contain any odd power of λ. In the case of anholonomic systems without viscous damping, it is possible that only odd derivatives of the general coordinates appear in the equations of motion, and the characteristic polynomial does not involve the even powers of λ. In this section, these undamped systems subjected to retarded forces are investigated. The characteristic functions analysed here do not contain every second power of λ in their polynomial parts.

Let us consider, first, the scalar RFDE of order $2m$ when there is one discrete delay in the system. It has the actual form

$$\sum_{j=0}^{m} a_{2j} \frac{d^{2j}x(t)}{dt^{2j}} = bx(t-1) \tag{3.1}$$

where $x \in \mathbf{R}$, $m > 0$ and $a_{2m} = 1$. The delay is 1, which does not mean the loss of generality; with the help of a linear transformation with respect to the time t, the single discrete delay can always be transformed into the unity. The following theorem gives the stability regions of (3.1) in the parameter space $(a_0, a_2, \ldots, a_{2m-2}, b)$.

Theorem 3.1. The trivial solution of the RDDE (3.1) is exponentially asymptotically stable if and only if

(i) m is even and there exist integers $0 \le k_1 < k_2 \le k_3 <\le \ldots \le k_{m-1} < k_m$ such that either

$$b > 0,$$

$$b < \sum_{j=0}^{m} (-1)^j a_{2j} (2k_{2l-1}\pi)^{2j}, \tag{3.2a}$$

$$b < -\sum_{j=0}^{m}(-1)^j a_{2j}((2k_{2l-1}+1)\pi)^{2j}, \tag{3.2b}$$

$$b < -\sum_{j=0}^{m}(-1)^j a_{2j}((2k_{2l}-1)\pi)^{2j}, \tag{3.2c}$$

$$b < \sum_{j=0}^{m}(-1)^j a_{2j}(2k_{2l}\pi)^{2j}, \tag{3.2d}$$

for all $l = 1, \ldots, m/2$ or

$$b < 0,$$

$$b > -\sum_{j=0}^{m}(-1)^j a_{2j}((2k_{2l-1}+1)\pi)^{2j},$$

$$b > \sum_{j=0}^{m}(-1)^j a_{2j}((2k_{2l-1}+2)\pi)^{2j},$$

$$b > \sum_{j=0}^{m}(-1)^j a_{2j}(2k_{2l}\pi)^{2j},$$

$$b > -\sum_{j=0}^{m}(-1)^j a_{2j}((2k_{2l}+1)\pi)^{2j},$$

for all $l = 1, \ldots, m/2$ or

(ii) m is odd and there exist integers $0 \le k_1 < k_2 \le k_3 <\le \ldots < k_{m-1} \le k_m$ such that either

$$b > 0,$$

$$b < \sum_{j=0}^{m}(-1)^j a_{2j}(2k_{2h-1}\pi)^{2j},$$

$$b < -\sum_{j=0}^{m}(-1)^j a_{2j}((2k_{2h-1}+1)\pi)^{2j},$$

$$b < -\sum_{j=0}^{m}(-1)^j a_{2j}((2k_{2l}-1)\pi)^{2j},$$

$$b < \sum_{j=0}^{m}(-1)^j a_{2j}(2k_{2l}\pi)^{2j},$$

for all $h = 1, \ldots, \frac{m+1}{2}$ and $l = 1, \ldots, \frac{m-1}{2}$ or

$$b < 0,$$

$$b > -\sum_{j=0}^{m}(-1)^j a_{2j}((2k_{2h-1}+1)\pi)^{2j},$$

$$b > \sum_{j=0}^{m}(-1)^j a_{2j}((2k_{2h-1}+2)\pi)^{2j},$$

$$b > \sum_{j=0}^{m}(-1)^j a_{2j}(2k_{2l}\pi)^{2j},$$

$$b > -\sum_{j=0}^{m}(-1)^j a_{2j}((2k_{2l}+1)\pi)^{2j},$$

for all $h = 1, \ldots, \frac{m+1}{2}$ and $l = 1, \ldots, \frac{m-1}{2}$.

Proof. Only the formulae (3.2*a*-*d*) will be proved here. The proofs of the other inequalities are similar. *Theorem 2.19* can be applied since the RDDE (3.1) satisfies condition (2.2) (see *Remark 2.2*). When m is even, that is the order of the scalar RDDE is $n = 2m = 4, 8, \ldots$, the actual form of the stability condition (2.22*b*) is as follows

$$\sum_{k=1}^{r}(-1)^k \operatorname{sgn} S(\rho_k) = m, \tag{3.3}$$

where the characteristic function of (3.1) is given by

$$D(\lambda) = \sum_{j=0}^{m} a_{2j}\lambda^{2j} - be^{-\lambda}$$

and

$$R(\omega) = \operatorname{Re} D(i\omega) = \sum_{j=0}^{m}(-1)^j a_{2j}\omega^{2j} - b\cos\omega,$$

$$S(\omega) = \operatorname{Im} D(i\omega) = b\sin\omega.$$

If $b > 0$ then S is positive in the intervals $(2p\pi, (2p+1)\pi)$ and it is negative in the intervals $((2p+1)\pi, (2p+2)\pi)$, $(p = 0, 1, \ldots)$. This means that S is positive in the intervals $I_l^+ = \left(2k_{2l-1}\pi, (2k_{2l-1}+1)\pi\right)$ and it is negative in $I_l^- = \left((2k_{2l}-1)\pi, 2k_{2l}\pi\right)$, $(l = 1, \ldots, m/2)$. The actual form of R shows that $(3.2a$-$d)$ are equivalent to

$$R(2k_{2l-1}\pi) > 0, \quad R\left((2k_{2l-1}+1)\pi\right) < 0,$$

$$R\left((2k_{2l}-1)\pi\right) < 0, \quad R(2k_{2l}\pi) > 0$$

for all $l = 1, \ldots, m/2$. This means that R has zeros of odd number in any of the intervals I_l^+ or I_l^-. In addition, R has more zeros of odd subscripts in I_l^- and it has more zeros of even subscripts in I_l^+. This fact, and

$$\lim_{\omega \to +\infty} R(\omega) = +\infty$$

imply that

$$\sum_{\rho_k \in I_l^+} (-1)^k \operatorname{sgn} S(\rho_k) = \sum_{\rho_k \in I_l^-} (-1)^k \operatorname{sgn} S(\rho_k) = 1\,.$$

Since the maximum value of l is $m/2$, the number of the intervals I_l^+ and I_l^- is just m, and

$$\sum_{l=1}^{m/2} \Big(\sum_{\rho_k \in I_l^+} (-1)^k \operatorname{sgn} S(\rho_k) + \sum_{\rho_k \in I_l^-} (-1)^k \operatorname{sgn} S(\rho_k) \Big) = m\,.$$

Thus, the inequalities $(3.2a$-$d)$ imply the satisfaction of the stability condition (3.3). However, $(3.2a$-$d)$ are necessary conditions as well. If there do not exist at least m intervals like I_l^+ and I_l^- where R changes its sign, then the sum on the left-hand side of (3.3) will be less than m. If there are m intervals I_l^+ and I_l^-, but they are not alternating as is described by the subscripts $k_1, \ldots, k_m$, then the sum $(-1)^k \operatorname{sgn} S(\rho_k)$ will give -1 in one of them, and (3.3) is not satisfied again. The maximum number of the extremums of the polynomial

$$\sum_{j=0}^{m} (-1)^j a_{2j} \omega^{2j}$$

is m in $[0, \infty)$. These extremums separate the intervals where the polynomial part of R is monotonous. The maximum number of these intervals is also m. This explains that R

cannot have zeros of odd number in any other interval except in the alternating I_l^+ and I_l^- $(l = 1, \ldots, m/2)$ if (3.3) is satisfied. Thus, (3.2a-d) are the necessary and sufficient conditions of exponential stability in the case of even m and $b > 0$. If $b < 0$ then S has opposite sign. This gives an explanation to the further inequalities in the statement (i). The proof is similar if m is odd. △

The stability conditions are linear inequalities for the parameters $a_0, a_2, \ldots, a_{2m-2}$ and b of the RDDE (3.1). They are linear because the function S is very simple, and its zeros ($p\pi$) can be calculated explicitly.

The following theorem investigates the scalar RDDE

$$\sum_{j=0}^{m} a_{2j+1} \frac{d^{2j+1}x(t)}{dt^{2j+1}} + bx(t-1) = 0 \tag{3.4}$$

where $m \geq 0$. It contains only the odd derivatives of the scalar x. The coefficient a_{2m+1} is assumed to be 1.

Theorem 3.2. The trivial solution of the RDDE (3.4) is exponentially asymptotically stable if and only if

$$b > 0\,, \tag{3.5a}$$

$$b < \sum_{j=0}^{m} (-1)^j a_{2j+1} \left(\frac{\pi}{2}\right)^{2j+1}, \tag{3.5b}$$

and either

(i) m is even and there exist integers $0 \leq k_1 < k_2 \leq k_3 <\leq \ldots \leq k_{m-1} < k_m$ such that

$$b < \sum_{j=0}^{m} (-1)^j a_{2j+1} \left(2k_{2l-1}\pi + \frac{\pi}{2}\right)^{2j+1}, \tag{3.6a}$$

$$b < -\sum_{j=0}^{m} (-1)^j a_{2j+1} \left((2k_{2l-1}+1)\pi + \frac{\pi}{2}\right)^{2j+1}, \tag{3.6b}$$

$$b < -\sum_{j=0}^{m} (-1)^j a_{2j+1} \left((2k_{2l}-1)\pi + \frac{\pi}{2}\right)^{2j+1}, \tag{3.6c}$$

$$b < \sum_{j=0}^{m} (-1)^j a_{2j+1} \big(2k_{2l}\pi + \frac{\pi}{2}\big)^{2j+1}, \tag{3.6d}$$

for all $l = 1, \dots, m/2$; or

(ii) m is odd and there exist integers $0 \le k_1 < k_2 \le k_3 <\le \dots < k_{m-1} \le k_m$ such that

$$b < \sum_{j=0}^{m} (-1)^j a_{2j+1} \big(2k_{2h-1}\pi + \frac{\pi}{2}\big)^{2j+1},$$

$$b < -\sum_{j=0}^{m} (-1)^j a_{2j+1} \big((2k_{2h-1}+1)\pi + \frac{\pi}{2}\big)^{2j+1},$$

$$b < -\sum_{j=0}^{m} (-1)^j a_{2j+1} \big((2k_{2l}-1)\pi + \frac{\pi}{2}\big)^{2j+1},$$

$$b < \sum_{j=0}^{m} (-1)^j a_{2j+1} \big(2k_{2l}\tau + \frac{\pi}{2}\big)^{2j+1},$$

for all $h = 1, \dots, \frac{m+1}{2}$ and $l = 1, \dots, \frac{m-1}{2}$.

Proof. The characteristic function of (3.4) has the form

$$D(\lambda) = \sum_{j=0}^{m} a_{2j+1}\lambda^{2j+1} + be^{-\lambda}$$

and the functions R and S are as follows

$$R(\omega) = \operatorname{Re} D(i\omega) = b\cos\omega\,,$$

$$S(\omega) = \operatorname{Im} D(i\omega) = \sum_{j=0}^{m} (-1)^j a_{2j+1}\omega^{2j+1} - b\sin\omega\,.$$

Condition (2.23b) of *Theorem 2.19* means that

$$R(0) = b > 0$$

which is equivalent to (3.5a). If m is even, that is the order of the RDDE (3.4) is $n = 2m+1 = 1, 5, 9, \dots$, then the stability condition (2.23c) in *Theorem 2.19* has the actual form

$$\sum_{k=1}^{s-1} (-1)^k \operatorname{sgn} R(\sigma_k) + \frac{1}{2}\big((-1)^s + 1\big) + m = 0\,. \tag{3.7}$$

The inequalities (3.6*a*-*d*) are equivalent to

$$S\left(2k_{2l-1}\pi + \frac{\pi}{2}\right) > 0\,, \quad S\left((2k_{2l-1}+1)\pi + \frac{\pi}{2}\right) < 0\,,$$

$$S\left((2k_{2l}-1)\pi + \frac{\pi}{2}\right) < 0\,, \quad S\left(2k_{2l}\pi + \frac{\pi}{2}\right) > 0\,;$$

$(l = 1, \ldots, m/2)$. Since R is positive in the intervals $I_l^+ = \big((2k_{2l}-1)\pi+\pi/2\,,\, 2k_{2l}\pi+\pi/2\big)$ and it is negative in $I_l^- = \big(2k_{2l-1}\pi+\pi/2\,,\,(2k_{2l-1}+1)\pi+\pi/2\big)$, these inequalities result

$$\sum_{\sigma_k \in I_l^+} (-1)^k \operatorname{sgn} R(\sigma_k) = \sum_{\sigma_k \in I_l^-} (-1)^k \operatorname{sgn} R(\sigma_k) = -1\,.$$

Hence,

$$\sum_{l=1}^{m/2}\Big(\sum_{\sigma_k \in I_l^+} (-1)^k \operatorname{sgn} R(\sigma_k) + \sum_{\sigma_k \in I_l^-} (-1)^k \operatorname{sgn} R(\sigma_k)\Big) = -m\,.$$

However, formulae (3.6*a*-*d*) do not exclude the existence of further zeros σ_k of S outside the intervals I_l^+ and I_l^-, $(l = 1, \ldots, m/2)$. If inequality (3.5*b*) holds then $S(\pi/2) > 0$. This means that S may have one more zero at the most, and it is in $(0, \pi/2)$. Thus,

$$\sum_{k=1}^{s-1} (-1)^k \operatorname{sgn} R(\sigma_k)$$

$$= \sum_{\sigma_k \in (0,\pi/2)} (-1)^k \operatorname{sgn} R(\sigma_k) + \sum_{l=1}^{m/2}\Big(\sum_{\sigma_k \in I_l^+} (-1)^k \operatorname{sgn} R(\sigma_k) + \sum_{\sigma_k \in I_l^-} (-1)^k \operatorname{sgn} R(\sigma_k)\Big)\,.$$

If there is a zero $\sigma_{s-1} \in (0, \pi/2)$ then s is even and

$$\sum_{\sigma_k \in (0,\pi/2)} (-1)^k \operatorname{sgn} R(\sigma_k) = (-1)^{s-1} 1 = -1\,.$$

The stability condition (3.7) is satisfied:

$$-1 - m + \frac{1}{2}((-1)^s + 1) + m = 0\,.$$

If there is no zero in $(0, \pi/2)$ then s is odd and

$$\sum_{\sigma_k \in (0,\pi/2)} (-1)^k \operatorname{sgn} R(\sigma_k) = 0\,.$$

The stability condition (3.7) is satisfied again:

$$0 - m + \frac{1}{2}((-1)^s + 1) + m = 0\,.$$

Thus, *Theorem 2.19* proves that the trivial solution is exponentially stable if (3.5*a*), (3.5*b*) and (3.6*a*-*d*) hold. There are no more regions of stability in the parameter space. This can be proved analysing the polynomial part of S as it was done in the proof of *Theorem 3.1*. The proof is similar when m is odd. △

In *Chapter 4*, *Theorems 3.1* and *3.2* will often be used to solve practical problems. Moreover, these stability results serve as a good basis for the investigation of systems with slight damping, i.e. when the terms of every second power of λ in the characteristic function have small coefficients relative to the others. In the case of first and second order scalar RDDEs, *Theorems 3.1* and *3.2* give some well-known results of the literature. Let us see these special cases, and also some further stability charts for third and fourth order equations.

Corollary 3.3. The trivial solution of the scalar RDDE

$$\dot{x}(t) + bx(t-1) = 0 \tag{3.8}$$

is exponentially asymptotically stable if and only if

$$0 < b < \frac{\pi}{2}\,.$$

Proof. This is a consequence of (3.5*a*) and (3.5*b*) in *Theorem 3.2* when $m = 0$ and $a_1 = 1$. The conditions (3.6*a*-*d*) vanish. △

This result has been known for a long time in the literature [26].

Corollary 3.4. The trivial solution of the scalar RDDE

$$\ddot{x}(t) + a_0 x(t) = bx(t-1) \tag{3.9}$$

is exponentially asymptotically stable if and only if there exists an integer $k_1 \geq 0$ such that either

$$b > 0, \quad b < a_0 - 4k_1^2\pi^2 \quad \text{and} \quad b < -a_0 + (2k_1+1)^2\pi^2$$

or

$$b < 0, \quad b > -a_0 + (2k_1+1)^2\pi^2 \quad \text{and} \quad b > a_0 - (2k_1+2)^2\pi^2 .$$

Proof. This is a consequence of the conditions in paragraph (ii) of *Theorem 3.1* when $m = 1$ and $a_2 = 1$. △

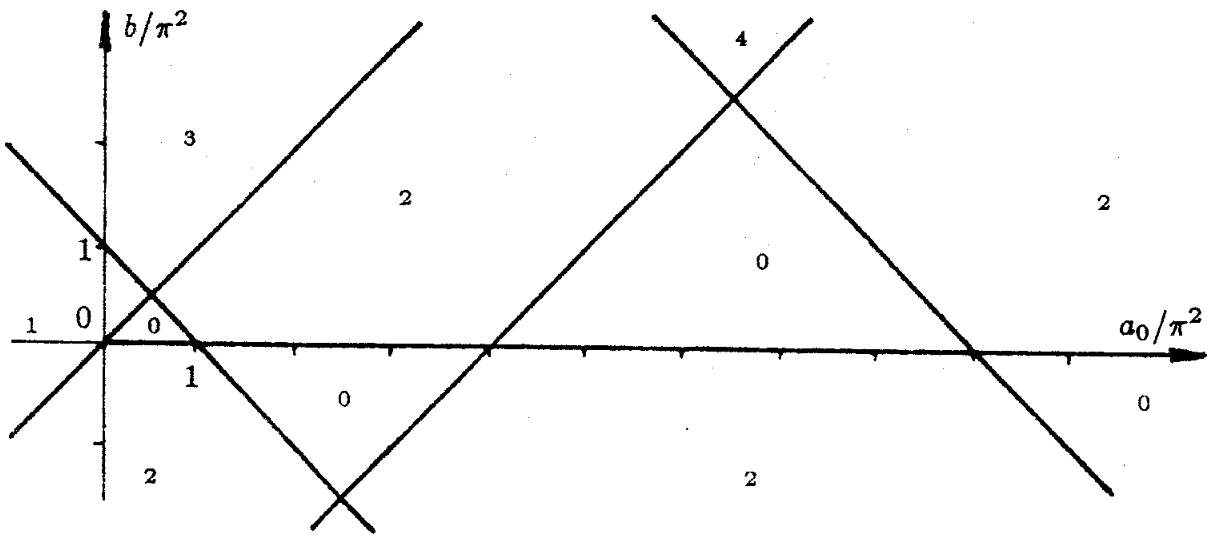

Fig. 3.1. Stability chart of RDDE (3.9)

The stability chart on the plane of the parameters a_0 and b is presented in *Fig. 3.1.* By means of *Theorem 2.15*, we have also determined the number of the characteristic roots with positive real parts in the actual regions of the plane (a_0, b). Thus, the regions of stability are denoted by 0. They can also be found, for example, in [3,33,53]. The number of these disjoint stability domains is infinite. The number of the characteristic roots with positive real parts is odd if $b > a_0$. In this case, there is at least one positive real characteristic root, as follows from *Theorem 2.28* and *Corollary 2.30.*

Corollary 3.5. The trivial solution of the scalar RDDE

$$\frac{d^3}{dt^3}x(t) + a_1\frac{d}{dt}x(t) + bx(t-1) = 0 \tag{3.10}$$

is exponentially asymptotically stable if and only if there exists an integer $k_1 \geq 0$ such that

$$b > 0, \quad b < a_1\frac{\pi}{2} - (\frac{\pi}{2})^3, \quad b < a_1(2k_1\pi + \frac{\pi}{2}) - (2k_1\pi + \frac{\pi}{2})^3$$
$$\text{and} \quad b < -a_1(2k_1\pi + \frac{3\pi}{2}) + (2k_1\pi + \frac{3\pi}{2})^3 .$$

Proof. This is a consequence of *Theorem 3.2* when $m = 1$ and $a_3 = 1$. $\triangle$

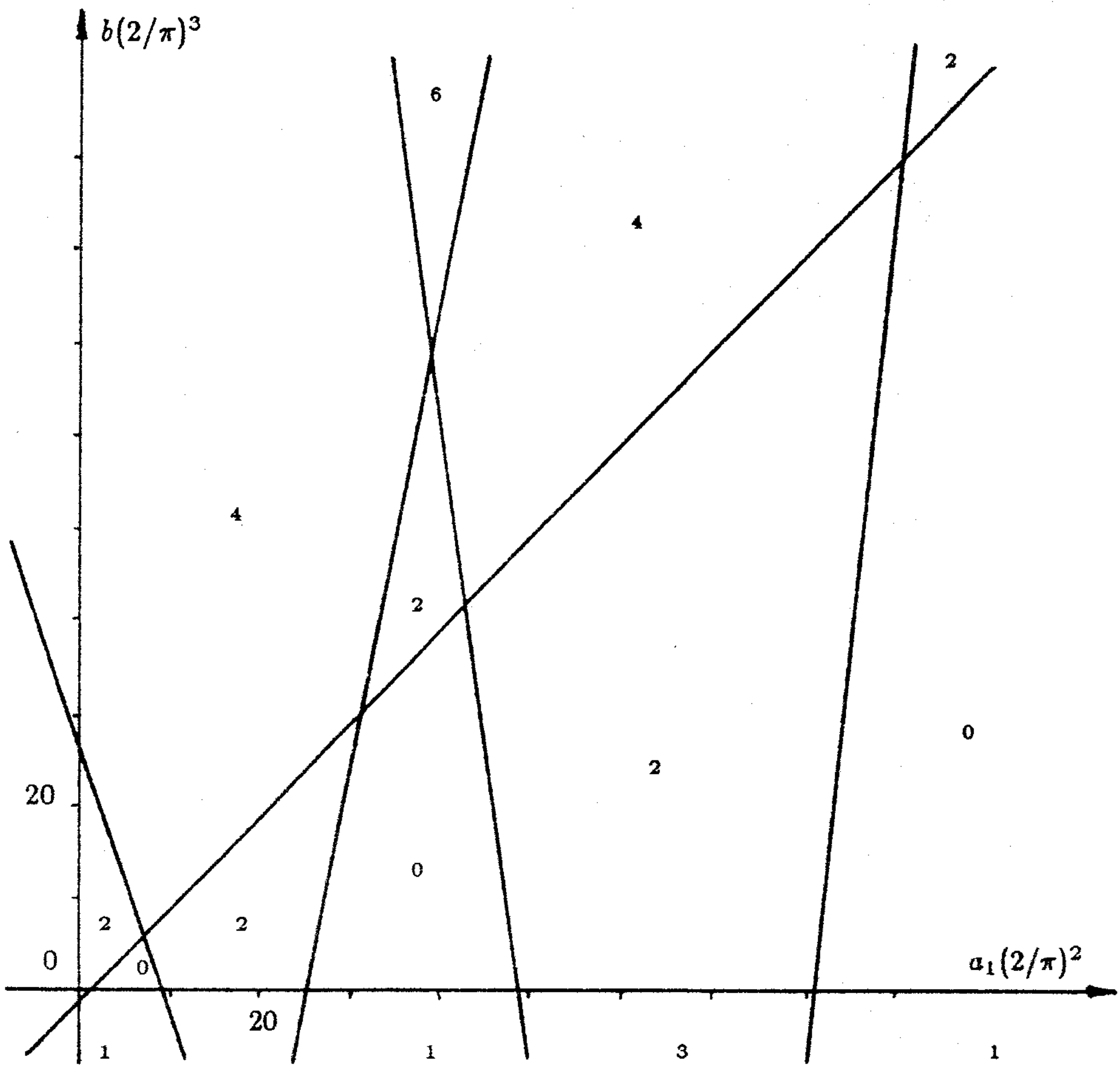

Fig. 3.2. Stability chart of RDDE (3.10)

The stability chart of (3.10) is shown in *Fig. 3.2.* The regions of stability are denoted by o, the other numbers present the number of characteristic roots with positive real parts. The number of the disjoint stability regions is infinite, but they exist only in the positive quadrant of the parameter plane (a_1, b).

Corollary 3.6. The trivial solution of the scalar RDDE

$$\frac{d^4}{dt^4}x(t) + a_2\frac{d^2}{dt^2}x(t) + a_0x(t) = bx(t-1) \tag{3.11}$$

is exponentially asymptotically stable if

$$0 < b < a_0, \quad b < -a_0 + \pi^2 a_2 - \pi^4 \quad \text{and} \quad b < a_0 - 4\pi^2 a_2 + 16\pi^4 .$$

Proof. This sufficient stability condition is a result of the inequalities (3.2*a*-*d*) in *Theorem 3.1* when $m = 2$, $a_4 = 1$ and the integers have the smallest possible value, that is $k_1 = 0$ and $k_2 = 1$. In this case, the inequalities (3.2*b*) and (3.2*c*) are identical. △

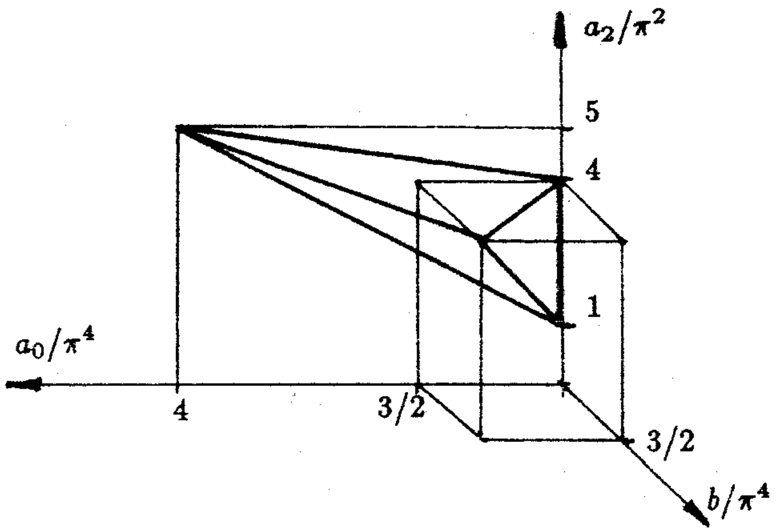

Fig. 3.3. A stability region of RDDE (3.11)

All the stability regions of (3.11) can, of course, be given by means of *Theorem 3.1*, although they are described by a rather complicated structure of inequalities. In the parameter space (a_0, a_2, b), *Fig. 3.3* shows the stability domain determined in *Corollary 3.6*. *Fig. 3.4* shows some further regions of stability denoted by + and − in the plane (a_0, a_2) when b has a small positive or negative value, respectively.

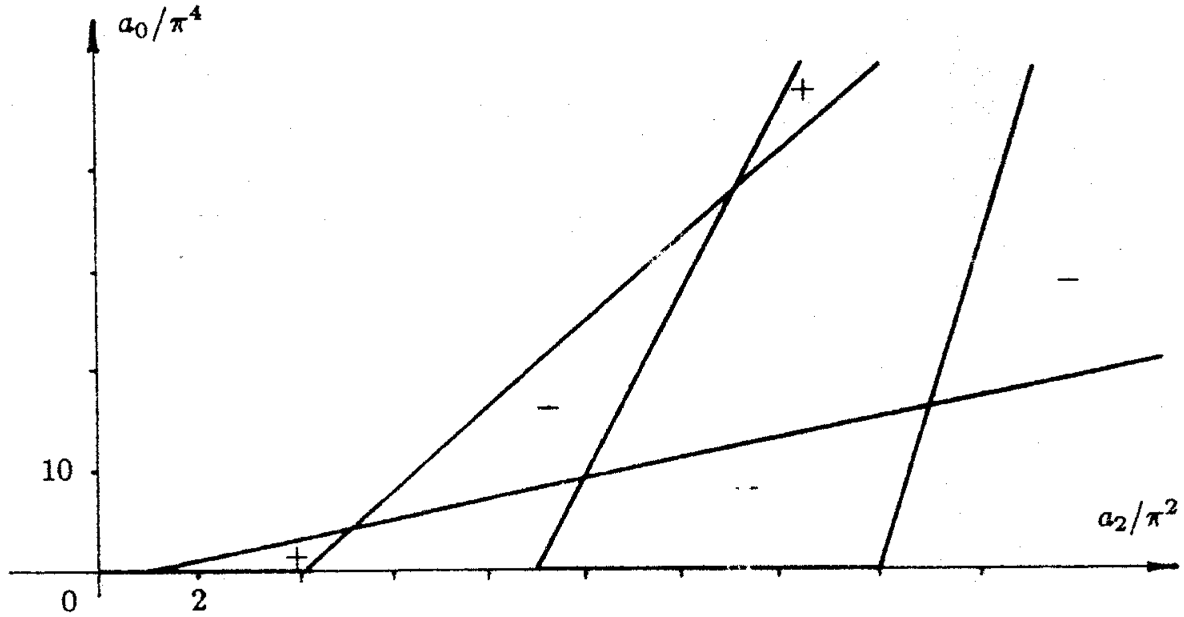

Fig. 3.4. Stability regions of RDDE (3.11) for small $|b|$

Let us examine some simplified mechanical problems where these results can be applied.

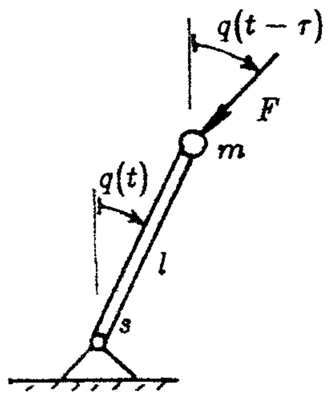

Fig. 3.5. A mechanical system subjected to retarded follower force

Example 3.7. The linearized equation of motion of a single degree of freedom mechanical system subjected to a retarded follower force (see *Fig. 3.5*) is as follows:

$$ml^2\ddot{q}(t) + (s - Fl)q(t) = -Flq(t - \tau)\,. \tag{3.12}$$

The scalar q is the general coordinate, s is the torsional stiffness at the pin, m is the mass at the end of the light beam, l denotes the length of the beam, and F stands for the constant magnitude of the applied retarded force. The constant delay at the angle of the force is τ. After a linear time transformation with respect to τ, the corresponding characteristic function has the form

$$D(\lambda) = \lambda^2 + (1 - f)T^2 + fT^2 e^{-\lambda}\,,$$

where the following notation has been used:

$$f = \frac{Fl}{s}\,, \quad \alpha = \sqrt{\frac{s}{ml^2}}\,, \quad T = \alpha\tau\,. \tag{3.13}$$

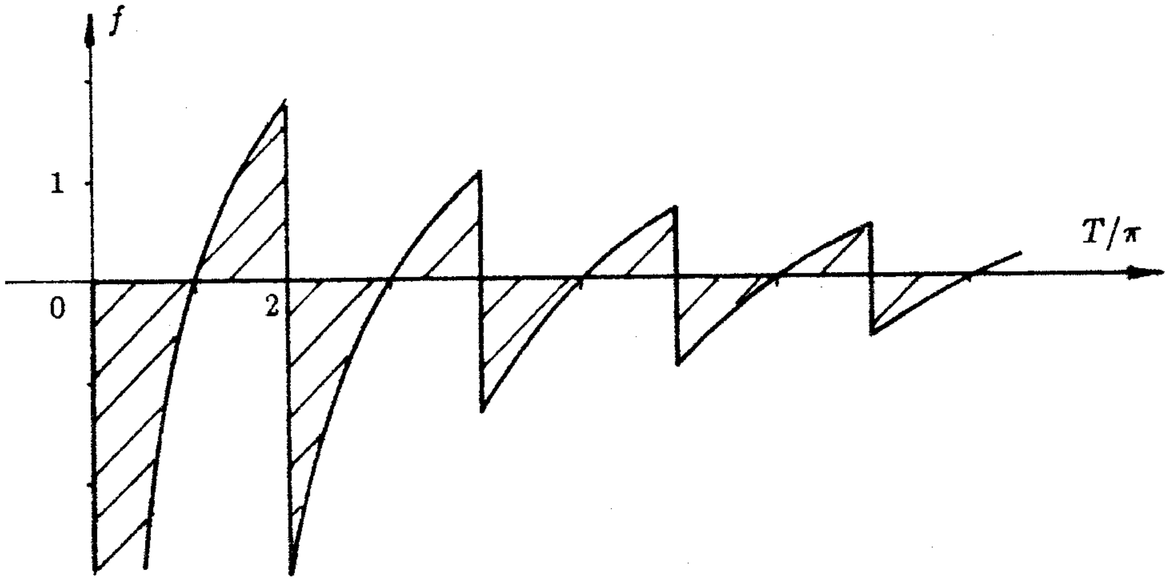

Fig. 3.6. Stability chart of RDDE (3.12)

If

$$a_0 = (1 - f)T^2 \quad \text{and} \quad b = -fT^2$$

are substituted into the inequalities of *Corollary 3.4*, we get the stability regions in the plane of the parameters T and f, where T is the dimensionless delay, and f is the dimensionless load. The $q = 0$ position of the system in *Fig. 3.5* is exponentially asymptotically stable if and only if there exists a scalar $k_1 \geq 0$ such that either

$$2k_1\pi < T < (2k_1+1)\pi \quad \text{and} \quad 0 > f > \frac{T^2 - (2k_1+1)^2\pi^2}{2T^2}$$

or

$$(2k_1+1)\pi < T < (2k_1+2)\pi \quad \text{and} \quad 0 < f < \frac{T^2 - (2k_1+1)^2\pi^2}{2T^2}\,.$$

These domains are shaded in *Fig. 3.6*. This example presents a simple case when the equilibrium is not asymptotically stable if there is no delay ($\tau = 0$), but it can be asymptotically stable if the delay appears in the system.

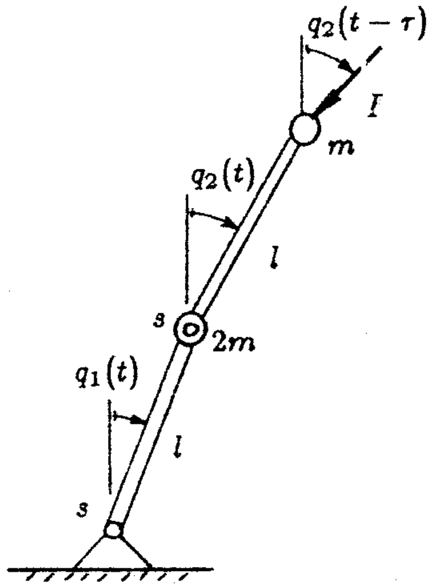

Fig. 3.7. Mechanical model with two degrees of freedom

The linearized equations of motion of the mechanical system shown in *Fig. 3.7* has the form

$$\begin{pmatrix} 3ml^2 & ml^2 \\ ml^2 & ml^2 \end{pmatrix} \begin{pmatrix} \ddot{q}_1(t) \\ \ddot{q}_2(t) \end{pmatrix} + \begin{pmatrix} 2s - Fl & -s \\ -s & s - Fl \end{pmatrix} \begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix} + \begin{pmatrix} 0 & Fl \\ 0 & Fl \end{pmatrix} \begin{pmatrix} q_1(t-\tau) \\ q_2(t-\tau) \end{pmatrix} = 0\,, \tag{3.14}$$

and the corresponding characteristic function is given by

$$D(\lambda) = 2\lambda^4 + (7-4f)T^2\lambda^2 + (1-3f+f^2)T^4 + (2\lambda^2 + (3-f)T^2)fT^2e^{-\lambda}\,,$$

where the notation (3.13) has been used again. Unfortunately, *Theorem 3.1* or *Corollary 3.6* cannot be applied here directly, in spite of the fact that λ^3 and λ^1 do not appear in the characteristic function. However, the same method can be used as in the proof of *Theorem 3.1*, since the real, non-negative zeros of

$$S(\omega) = \operatorname{Im} D(i\omega) = -fT^2(-2\omega^2 + (3-f)T^2)\sin\omega$$

are known explicitly: they are $p\pi$, $(p = 0, 1, \ldots)$ and $T\sqrt{(3-f)/2}$. A complicated system of inequalities, like that of (3.2) in *Theorem 3.1*, serves the stability chart in *Fig. 3.8.* This stability chart has also been determined by means of the Pontryagin method in [34], and by the τ-decomposition method in [32]. $\triangle$

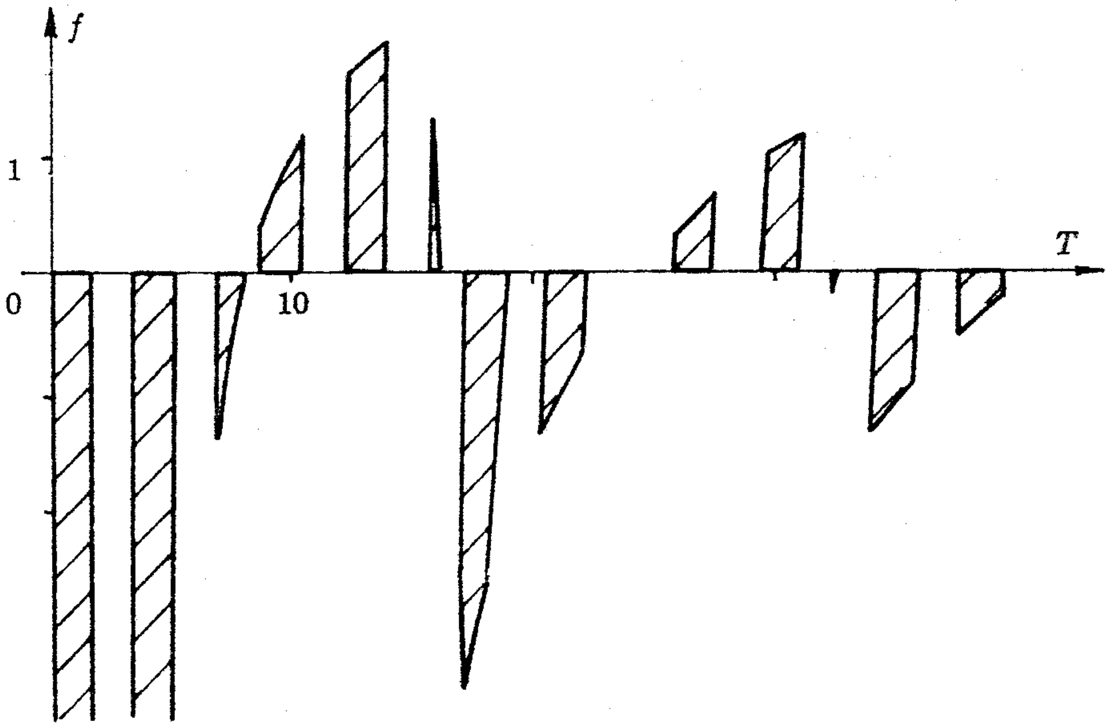

Fig. 3.8. Stability chart of RDDE (3.14)

In mechanical examples, the feed-back force is often proportional to the delayed values of the general velocities or accelerations. It is possible to construct general theorems, like *Theorems 3.1* and *3.2*, when the delayed value of a derivative of x appears in (3.1) or (3.4), but we mention only two simple cases here.

Theorem 3.8. The trivial solution of the scalar RDDE

$$\ddot{x}(t) + a_0 x(t) = b\dot{x}(t-1) \tag{3.15}$$

is exponentially asymptotically stable if and only if

$$b < 0, \quad a_0 > 0 \quad \text{and} \quad b > \frac{2}{\pi}a_0 - \frac{\pi}{2}$$

or there exists an integer $k \geq 0$ such that either

$$b > 0, \quad b < \frac{2}{(4k+1)\pi}a_0 - (4k+1)\frac{\pi}{2} \quad \text{and} \quad b < -\frac{2}{(4k+3)\pi}a_0 + (4k+3)\frac{\pi}{2}$$

or

$$b < 0, \quad b > -\frac{2}{(4k+3)\pi}a_0 + (4k+3)\frac{\pi}{2} \quad \text{and} \quad b > \frac{2}{(4k+5)\pi}a_0 - (4k+5)\frac{\pi}{2}.$$

Proof. The characteristic function has the form

$$D(\lambda) = \lambda^2 + a_0 - b\lambda e^{-\lambda},$$

thus,

$$R(\omega) = \operatorname{Re} D(i\omega) = -\omega^2 + a_0 - b\omega \sin\omega,$$

$$S(\omega) = -b\omega\cos\omega.$$

The zeros of S can be determined explicitly. In this way, it is easy to check that the stability condition (2.22b) in *Theorem 2.19* is satisfied if either

$$b < 0, \quad R(0) > 0 \quad \text{and} \quad R(\frac{\pi}{2}) < 0$$

or there is an integer $k \geq 0$ such that

$$b > 0, \quad R((4k+1)\frac{\pi}{2}) > 0 \quad \text{and} \quad R((4k+3)\frac{\pi}{2}) < 0$$

or

$$b < 0, \quad R\big((4k+3)\frac{\pi}{2}\big) > 0 \quad \text{and} \quad R\big((4k+5)\frac{\pi}{2}\big) < 0\,.$$

These inequalities are equivalent to those of the statement of the theorem. The special characteristic of the polynomial $-\omega^2 + a_0$ in R explains that there are no more stability regions. $\triangle$

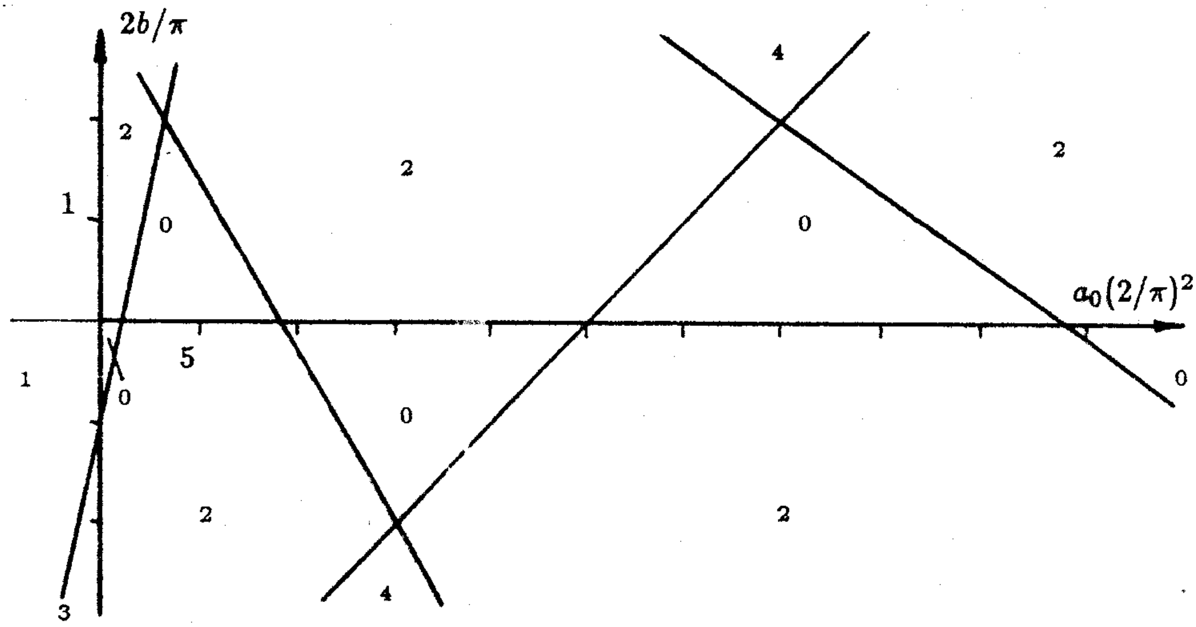

Fig. 3.9. Stability chart of RDDE (3.15)

The number of the characteristic roots with positive real parts is shown in the stability chart of (3.15) in *Fig. 3.9.* Before we give the stability regions of the NDDE

$$\ddot{x}(t) + a_0 x(t) = b\ddot{x}(t-1) \tag{3.16}$$

in the parameter plane (a_0, b), we prove a lemma which is very useful in the case of the scalar NFDE

$$\sum_{j=0}^{n} a_j \frac{d^j}{dt^j} x(t) = b\frac{d^n}{dt^n} x(t-\tau) + \sum_{l=0}^{n-1} \int_{-\infty}^{0} \frac{d^l}{dt^l} x(\theta) d\eta_l(\theta) \tag{3.17}$$

where $x \in \mathbf{R}$, $\tau > 0$, $n > 0$, $a_n = 1$, the scalar η_l, $(l = 0, \ldots, n-1)$ are functions of bounded variation and they satisfy the usual condition (2.2).

Lemma 3.9. The characteristic function of the scalar NFDE (3.17) has zeros of infinite number in the positive half of the complex plane if

$$|b| > 1\,. \tag{3.18}$$

Proof. If (3.17) is transformed into an n-dimensional system of first order NFDEs like (2.29) then the matrix C in (2.29) has the form

$$C = \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \dots & b \end{pmatrix}$$

which means that the eigenvalues of C are 0 with multiplicity $(n-1)$, and b. If $|b| > 1$ then C has an eigenvalue outside the unit disc, and

$$d(\lambda) = \det(I - Ce^{-\lambda\tau})$$

has zeros of infinite number in the right half of the complex plane. *Theorem 1.9* implies that the characteristic function of (3.17) also has zeros of infinite number there. $\triangle$

Because of this lemma, and since exponential stability is implied by (2.22) or (2.23) and

$$\sum_{k=1}^{n} |c_k| = |\mathrm{Tr}C| = |b| < 1\,,$$

these conditions of *Theorem 2.25* are not only sufficient but also necessary for the exponential asymptotical stability of the zero solution of the scalar NFDE (3.17) having one discrete delay in its neutral term.

Theorem 3.10. The trivial solution of the scalar NDDE (3.16) is exponentially asymptotically stable if and only if

$$b < 0\,, \quad a_0 > 0 \quad \text{and} \quad b > \frac{a_0}{\pi^2} - 1$$

or there exists an integer $k \geq 1$ such that either

$$b > 0\,, \quad b < \frac{a_0}{(2k-1)^2\pi^2} - 1 \quad \text{and} \quad b < -\frac{a_0}{(2k\pi)^2} + 1$$

or

$$b < 0\,, \quad b > -\frac{a_0}{(2k\pi)^2} + 1 \quad \text{and} \quad b > \frac{a_0}{(2k+1)^2\pi^2} - 1\,.$$

Proof. It is easy to see that these inequalities give stability regions where $|b| < 1$. Since the inequalities are equivalent to conditions (2.22), the statement of the theorem follows from *Lemma 3.9* and *Theorem 2.25.* △

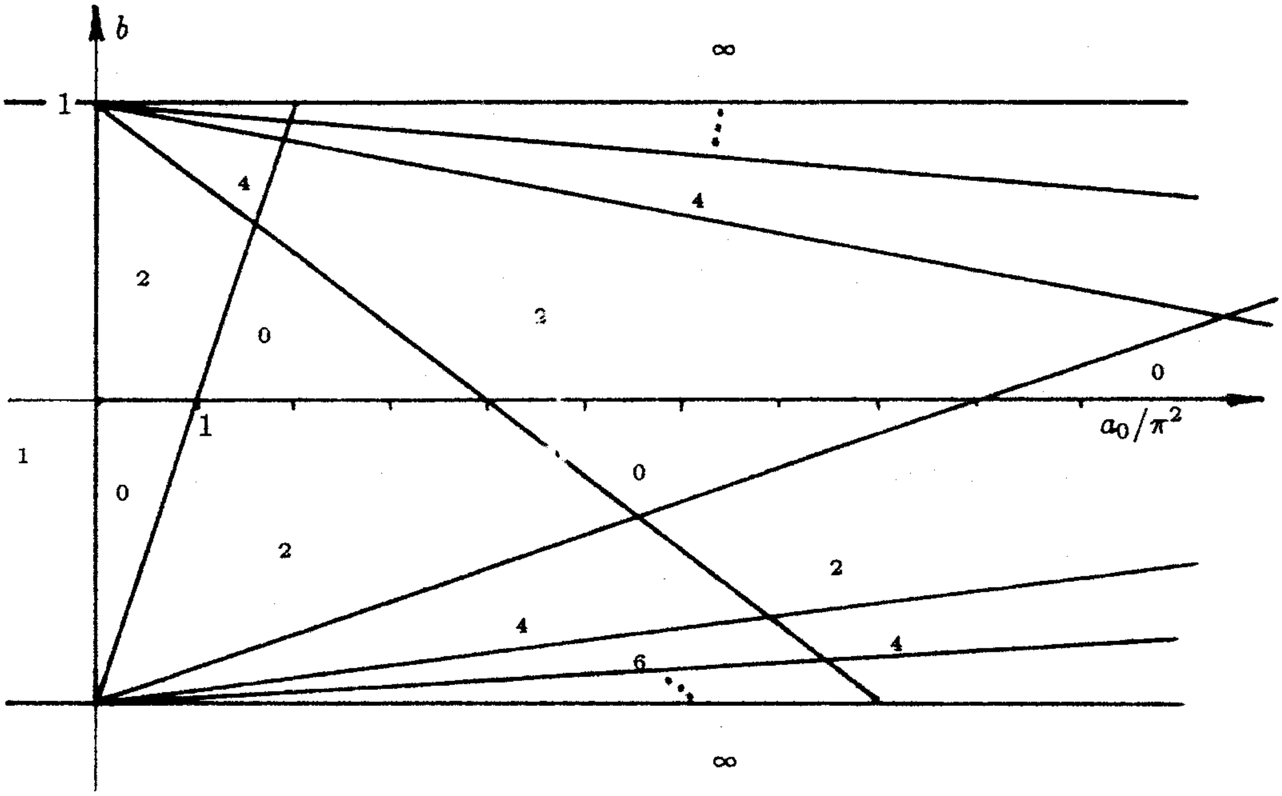

Fig. 3.10. Stability chart of NDDE (3.16)

The numbers of the characteristic roots with positive real parts are shown in the stability chart of (3.16) in *Fig. 3.10.*

The stability charts of the second order DDEs (3.9), (3.15) and (3.16) were correctly constructed in the papers [5,33] by means of the Pontryagin method. Similar, simple charts are often cited as Vyshnegradskii diagrams [35].

As the references show, the low (first and second) order DDEs with a single discrete delay have been analysed in the literature by means of several methods mentioned in *Section 1.3*, since all these methods can be applied in the case of one discrete delay. It is more complicated to investigate DDEs with two discrete delays or more. The special literature, of course, presents a lot of results in this line as well, but mainly for the first order scalar RDDE only [26,27,46], and the results are usually gained by a special analysis of the actual characteristic functions. Let us examine such problems now, using the direct stability investigation of the characteristic function presented in *Theorem 2.19.*

Theorem 3.11. The trivial solution of the scalar RDDE

$$\dot{x}(t) + bx(t-\tau_1) + bx(t-\tau_2) = 0 \tag{3.19}$$

is exponentially asymptotically stable if and only if

$$0 < b < \frac{\pi}{2(\tau_1+\tau_2)\cos\left(\frac{\tau_1-\tau_2}{\tau_1+\tau_2}\frac{\pi}{2}\right)}\,.$$

Proof. In order to apply *Theorem 2.1*, we need the characteristic function

$$D(\lambda) = \lambda + be^{-\tau_1\lambda} + be^{-\tau_2\lambda}$$

and the functions

$$\begin{aligned} R(\omega) &= \operatorname{Re} D(i\omega) = b\cos(\tau_1\omega) + b\cos(\tau_2\omega) \\ &= 2b\cos\left(\frac{\tau_1+\tau_2}{2}\omega\right)\cos\left(\frac{\tau_1-\tau_2}{2}\omega\right), \\ S(\omega) &= \operatorname{Im} D(i\omega) = \omega - b\sin(\tau_1\omega) - b\sin(\tau_2\omega) \\ &= \omega - 2b\sin\left(\frac{\tau_1+\tau_2}{2}\omega\right)\cos\left(\frac{\tau_1-\tau_2}{2}\omega\right). \end{aligned}$$

In *Theorem 2.19*, condition (2.23*b*) gives

$$R(0) = 2b > 0\,. \tag{3.20}$$

If the number s of the non-negative zeros of S is even then

$$S'(0) < 0 \quad \Rightarrow \quad b > \frac{1}{\tau_1+\tau_2}\,, \tag{3.21}$$

and the stability condition (2.23c) with $m = 0$ has the actual form

$$\sum_{k=1}^{s-1} (-1)^k \operatorname{sgn} R(\sigma_k) = -1 .$$

This is equivalent to

$$S\Big(\frac{\pi}{\tau_1+\tau_2}\Big) > 0 \quad \Leftrightarrow \quad \frac{\pi}{\tau_1+\tau_2} - 2b\cos\Big(\frac{\tau_1-\tau_2}{\tau_1+\tau_2}\frac{\pi}{2}\Big) > 0 , \tag{3.22}$$

where $\pi/(\tau_1+\tau_2)$ is the smallest positive zero of R. If the number s of the zeros of S is odd, then

$$S'(0) \le 0 \quad \Rightarrow \quad b \le \frac{1}{\tau_1+\tau_2} \tag{3.23}$$

which means that S has one zero only. Hence, $s = 1$. The stability condition (2.23c) is satisfied trivially:

$$\sum_{k=1}^{s-1} (-1)^k \operatorname{sgn} R(\sigma_k) = 0 .$$

The inequalities (3.20) and (3.22) are just equivalent to the condition in the statement of the theorem, while (3.21) and (3.23) show that this result is independent from the actual number s of the zeros S. Δ

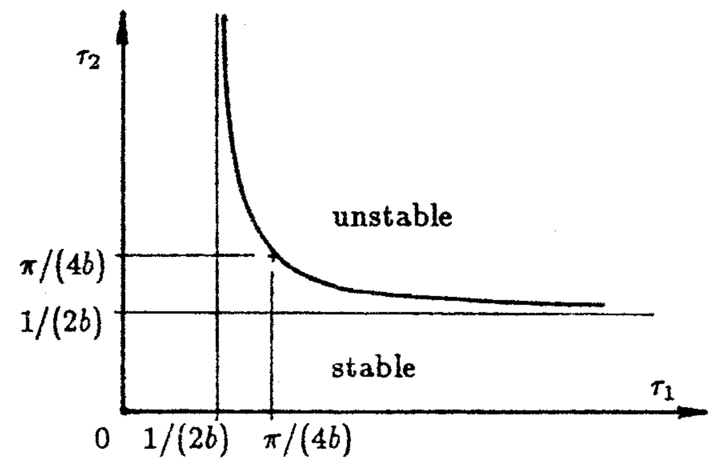

Fig. 3.11. Stability chart of RDDE (3.19)

Fig. 3.11 shows the corresponding stability chart of the RDDE (3.19) in the plane of the delay-parameters (τ_1, τ_2). With the help of *Theorem 2.16*, it is easy to prove that two complex conjugate characteristic roots cross the imaginary axis from the left to the right, as τ_1 and/or τ_2 are increased through the limit of stability. This has been proved in [46] as well. If $b < 0$ then there is at least one positive real characteristic root of (3.19).

As a matter of fact, this stability chart of (3.19) can easily be constructed by means of a previous result in *Corollary 3.3* and the D-subdivision method. In this case, those parameters have to be determined in the parameter space (τ_1, τ_2, b) where the characteristic equation of (3.19) has roots with zero real parts. These are given by

$$R(\omega) = 0, \quad S(\omega) = 0; \quad \omega \in [0, \infty) .$$

For example, when b is assumed to be a constant, they give a sequence of the so-called D-curves in the parameter plane (τ_1, τ_2). The limit of stability is given by some of them, and they can be chosen by means of *Corollary 3.3* applied to the case $\tau_1 = \tau_2$, i.e. to the special form

$$\dot{x}(t) + 2bx(t - \tau_1) = 0$$

of the RDDE (3.19). Its zero solution is exponentially asymptotically stable if

$$2b\tau_1 < \frac{\pi}{2} \quad \Rightarrow \quad \tau_1 = \tau_2 < \frac{\pi}{4b} .$$

This point can also be found in *Fig. 3.11*. It determines the critical curve, the limit of stability, since the characteristic roots are continuous functions of the parameters in the characteristic function. However, it always needs additional work if we want to give a correct mathematical proof, that is to prove that there are no more regions of stability, to investigate the structure of the D-curves and the continuous dependence of the characteristic roots on system parameters.

The stability charts of second and higher order systems may become extremely complicated in the presence of more discrete delays than one. The investigation of the

RDDE

$$\ddot{x}(t) + a_0 x(t) = bx(t-\tau_1) + bx(t-\tau_2) \tag{3.24}$$

will represent this. Instead of the reconstruction of the complicated system of inequalities from the stability condition (2.22*b*) in *Theorem 2.19*, some simple necessary or sufficient conditions will be given first, and an interesting stability chart will be presented for special parameters a_0 and b in the parameter plane of the delays τ_1 and τ_2.

Corollary 3.12. The trivial solution of (3.24) is not asymptotically stable if

$$a_0 \le 2b\,.$$

Proof. *Lemma 2.14* and *Corollary 2.30* imply the statement. △

Theorem 3.13. The zero solution of (3.24) is exponentially asymptotically stable if

$$b > 0\,, \quad a_0 > 2b\,, \quad |\tau_1 - \tau_2| < \frac{\pi}{\sqrt{a_0}}$$

and

$$\tau_1 > 3\tau_2 \quad \text{or} \quad \tau_2 > 3\tau_1\,.$$

Proof. The method followed here is similar to the one in the proof of *Theorem 2.37*. We need the functions

$$D(\lambda) = \lambda^2 + a_0 - be^{-\lambda\tau_1} - be^{-\lambda\tau_2}\,,$$

$$R(\omega) = -\omega^2 + a_0 - 2b\cos\left(\frac{\tau_1+\tau_2}{2}\omega\right)\cos\left(\frac{\tau_1-\tau_2}{2}\omega\right),$$

$$S(\omega) = 2b\sin\left(\frac{\tau_1+\tau_2}{2}\omega\right)\cos\left(\frac{\tau_1-\tau_2}{2}\omega\right).$$

If $\tau_1 > 3\tau_2$ or $\tau_2 > 3\tau_1$ then

$$\frac{\pi}{|\tau_1 - \tau_2|} < \frac{2\pi}{\tau_1 + \tau_2}\,.$$

Thus, the smallest positive zero of S is $\pi/|\tau_1 - \tau_2|$. Since $b > 0$, the function S is positive if

$$\omega \in \left(0, \frac{\pi}{|\tau_1 - \tau_2|}\right).$$

The inequalities $a_0 > 2b$ and $|\tau_1 - \tau_2| < \pi/\sqrt{a_0}$ are equivalent to

$$R(0) > 0 \quad \text{and} \quad R\left(\frac{\pi}{|\tau_1 - \tau_2|}\right) < 0$$

which means that the number r of the positive zeros ρ_k of R is odd, and these zeros are all in the interval where S is positive. Thus,

$$\sum_{k=1}^{r} (-1)^k \operatorname{sgn} S(\rho_k) = -1,$$

and the stability conditions (2.22) in *Theorem 2.19* are satisfied. $\triangle$

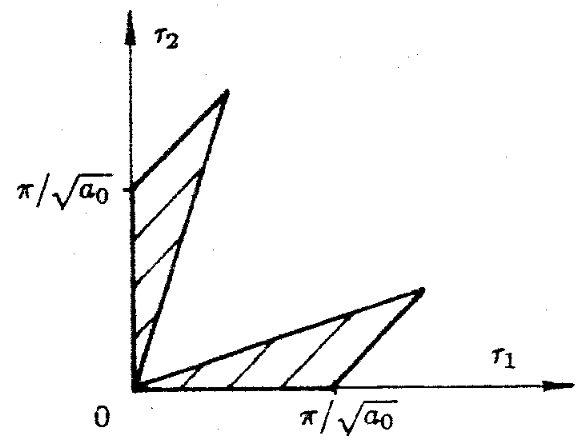

Fig. 3.12. Some stability regions of RDDE (3.24) for $0 < b < a_0/2$

The stability domains given in the statement of *Theorem 3.13* are shaded in *Fig. 3.12*.

In *Fig. 3.13*, all the domains of stability are shaded in the actual quadrant of the plane (τ_1, τ_2) when $a_0 = 6$ and $b = 1$. This stability chart has been constructed by means of the D-curves

$$R(\omega) = 0, \quad S(\omega) = 0; \quad \omega \in [0, \infty)$$

of the D-subdivision method and with the help of the following

Corollary 3.14. The zero solution of (3.24) is exponentially asymptotically stable if $a_0 = 6$, $b = 1$ and

$$\tau_2 = 0\,; \quad 0 < \tau_1 < \frac{\pi}{\sqrt{6}} \quad \text{or} \quad \pi < \tau_1 < \frac{3\pi}{\sqrt{6}} \quad \text{or} \quad 2\pi < \tau_1 < \frac{5\pi}{\sqrt{6}}$$

or

$$\tau_2 = \tau_1\,; \quad 0 < \tau_1 < \frac{\pi}{\sqrt{8}} \quad \text{or} \quad \pi < \tau_1 < \frac{3\pi}{\sqrt{8}}\,.$$

Proof. If either $\tau_2 = 0$ or $\tau_2 = \tau_1$, the RDDE (3.24) contains one delay only, and it can be transformed into the form of (3.9). If the actual parameters are substituted into the inequalities of *Corollary 3.4*, the statement is implied directly. $\triangle$

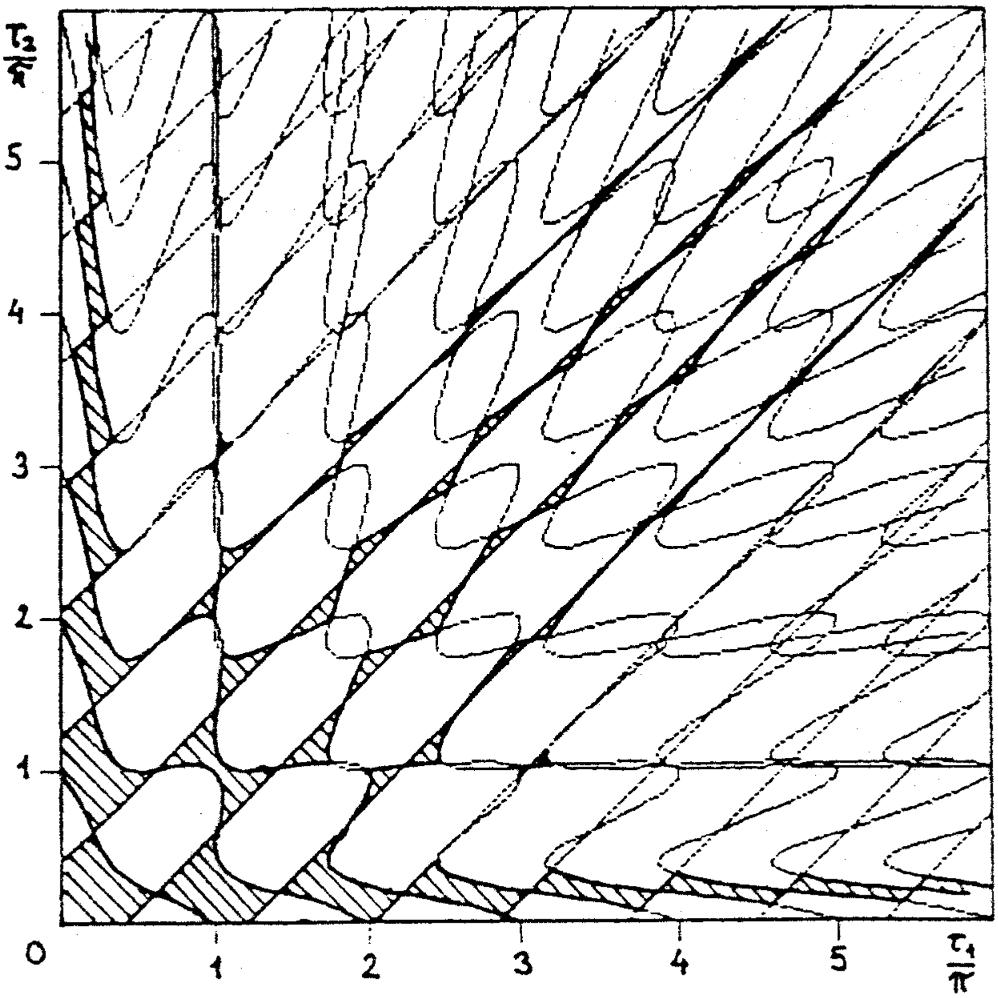

Fig. 3.13. Stability chart of RDDE (3.24) for $a_0 = 6$, $b = 1$

3.2. Damping in low-order systems

If there is viscous damping in a holonomic or anholonomic mechanical system of finite degrees of freedom then all the derivatives of the general coordinates may appear in the equations of motion. However, this viscous damping is often slight. This makes it possible to construct stability charts based on the stability results obtained for undamped systems. For example, the stability charts of the mechanical systems presented in the *Example 3.6* can easily be determined if there are terms of $\dot{q}(t)$ or $\dot{q}_1(t)$, $\dot{q}_2(t)$ with small positive coefficients of damping in the equations (3.12) or (3.14) respectively. The D-curves given by

$$R(\omega) = \operatorname{Re} D(i\omega) = 0\,, \quad S(\omega) = \operatorname{Im} D(i\omega) = 0\,; \quad \omega \in [0, \infty)$$

determine disjunct domains in the parameter plane. Since the characteristic roots are continuous functions of the damping parameters, the domains of stability can be chosen with the help of the stability charts of *Figs. 3.6* or *3.8*, where these parameters were zero. Such stability charts are presented in [32] for the system in *Fig. 3.7* with viscous damping. Hsu also calls attention for a parameter domain where a stability region appears at higher values of viscous damping, independently from the basic structure of the stability regions determined in the absence of damping. This example in [32] shows an important limitation of the D-subdivision method.

Nevertheless, the following stability charts are not always accompanied with theorems giving the exact necessary and sufficient conditions of stability. Its reason is mainly the complexity of the conditions. When all the powers of λ are present in the characteristic functions, neither the zeros of the function R nor those of S can be calculated explicitly, as was possible in the case of undamped systems. In these cases, we shall often refer to the D-subdivision method and the corresponding theorems of *Section 3.1* only.

Example 3.15. The familiar stability chart of the scalar RDDE

$$\dot{x}(t) + a_0 x(t) + b x(t-1) = 0 \tag{3.25}$$

in *Fig. 3.14* was determined first in [28]. It contains the result of *Corollary 3.3* when $a_0 = 0$. $\triangle$

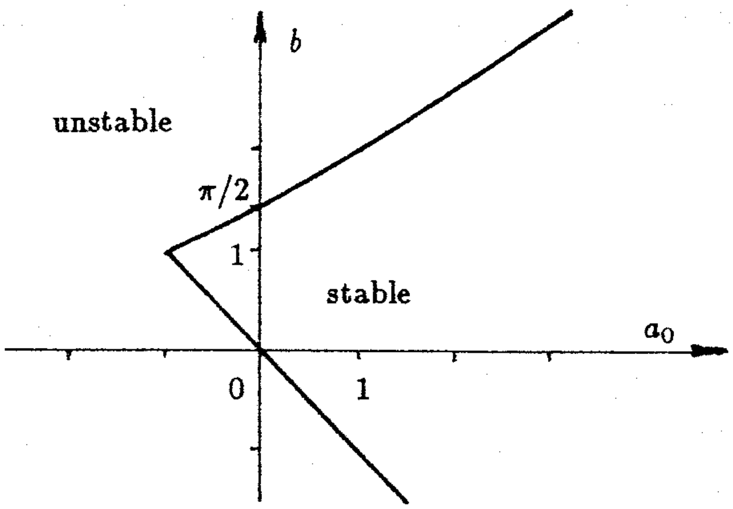

Fig. 3.14. Stability chart of RDDE (3.25)

Example 3.16. *Figures 3.15* and *3.16* show the stability regions of the scalar RDDE

$$\ddot{x}(t) + a_1 \dot{x}(t) + a_0 x(t) = b x(t-1) \tag{3.26}$$

when $a_1 = 1$ and $a_1 = -1$ respectively. These charts are also known from [33]. The stability regions become greater than those of *Fig. 3.1* as the "damping" a_1 is increased from zero. However, there are still small domains of stability when $a_1 < 0$, that is when the system is unstable in the absence of the delay (see the shaded areas in *Fig. 3.16*). $\triangle$

It is a well-known special case of (3.26) when $a_0 = 0$. The following statement is proved by means of the Pontryagin method in [25]. *Theorem 2.19* gives its proof in a much simpler way.

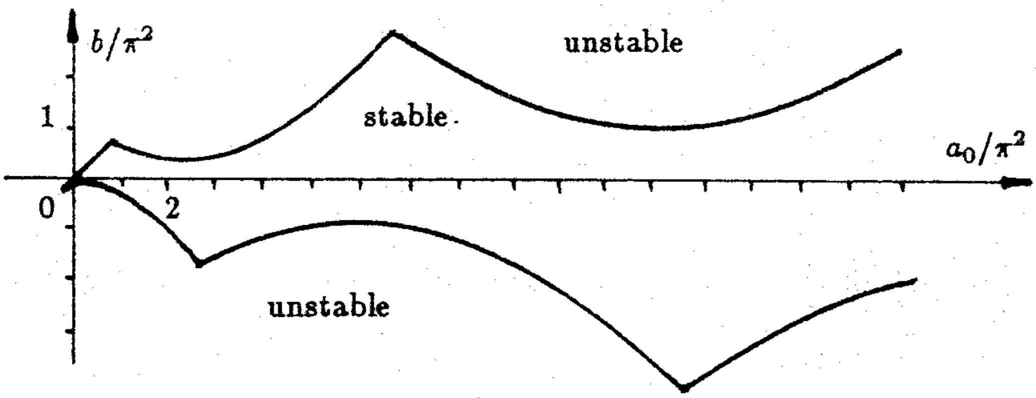

Fig. 3.15. Stability chart of RDDE (3.26) when $a_1 = 1$

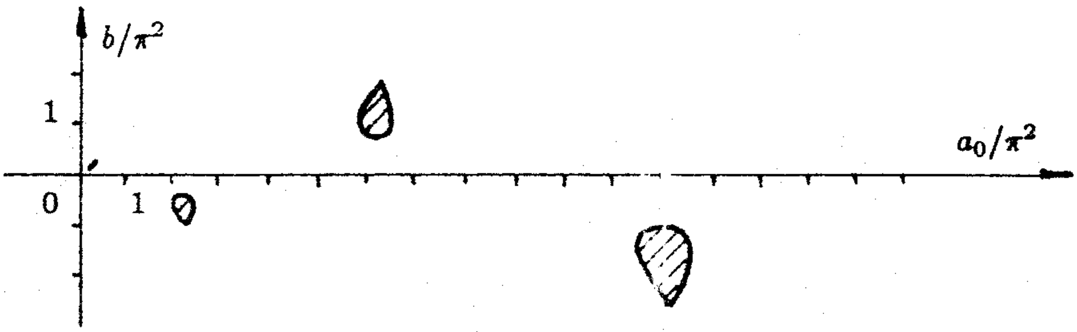

Fig. 3.16. Stability chart of RDDE (3.26) when $a_1 = -1$

Theorem 3.17. The zero solution of the scalar RDDE

$$\ddot{x}(t) + a_1\dot{x}(t) = -x(t-1)$$

is exponentially asymptotically stable if and only if

$$a_1 > \frac{\sin \rho_1}{\rho_1}$$

where ρ_1 is the only real positive root of the equation

$$\omega^2 = \cos\omega\,.$$

Proof. The characteristic function has the form

$$D(\lambda) = \lambda^2 + a_1\lambda + e^{-\lambda}\,.$$

Thus,

$$R(\omega) = -\omega^2 + \cos\omega\,,$$

$$S(\omega) = a_1\omega - \sin\omega\,.$$

Since $m = 1$, and R has only one positive zero ρ_1, the stability condition (2.22*b*) in *Theorem 2.19* gives

$$-\operatorname{sgn} S(\rho_1) = -1 \quad \Leftrightarrow \quad a_1\rho_1 - \sin\rho_1 > 0\,.$$

This is equivalent to the statement. △

The structures of the stability charts (see e.g. [33]) of the scalar DDEs

$$\ddot{x}(t) + a_1\dot{x}(t) + a_0x(t) = bx^{(p)}(t-1)\,, \quad p = 0, 1, 2 \tag{3.27}$$

are similar to each other. They can be determined for $p = 1$ and 2 as in *Example 3.16.* There always exist stability domains for $a_1 < 0$, where the order p of the derivative of the delayed term may be 0, 1 or 2 as well. However, these stability regions disappear below a critical negative value of a_1.

Theorem 3.18. The zero solution of the DDE (3.27) is unstable if

$$a_1 < -2\,,$$

where $p = 0, 1$ or 2.

Proof. The proof is shown, first, for the case $p = 0$, when

$$D(\lambda) = \lambda^2 + a_1\lambda + a_0 - be^{-\lambda}\,,$$

$$R(\omega) = -\omega^2 + a_0 - b\cos\omega\,,$$

$$S(\omega) = a_1\omega + b\sin\omega\,,$$

and the stability condition (2.22b) in *Theorem 2.19* has the actual form

$$\sum_{k=1}^{r}(-1)^k \operatorname{sgn} S(\rho_k) = -1 .$$

Since

$$R'(\omega) = -2\omega + b\sin\omega ,$$

and because $a_1 < -2$, the function S is always negative in those intervals where R is decreasing, that is between a maximum and a subsequent minimum of R determined by $R'(\omega) = 0$. R may have zeros of odd subscript in these intervals only. Thus,

$$(-1)^{2j+1} \operatorname{sgn} S(\rho_{2j+1}) = 1, \quad \text{for all} \quad j = 0, 1, \ldots, \operatorname{int}\left(\frac{r}{2}\right),$$

and condition (2.22b) is not satisfied. *Theorem 2.19* implies that the trivial solution is not asymptotically stable. *Theorem 2.15* shows that 'here is at least one characteristic root with positive real parts in this case, so the trivia' solution is unstable.

The statement can also be proved if we try to find the critical values of a_1 when the D-curves

$$R(\omega) = 0, \quad S(\omega) = 0; \quad \omega \in [0, \infty)$$

have "peaks". This calculation will be shown in the case of $p = 1$ in (3.27). Now,

$$R(\omega) = -\omega^2 + a_0 - b\omega \sin\omega ,$$

$$S(\omega) = a_1\omega - b\omega\cos\omega$$

and the D-curves in the plane of the parameters a_0 and b are given by

$$a_0 = \omega^2 + a_1\omega\tan\omega ,$$

$$b = \frac{a_1}{\cos\omega}; \quad \omega \in [0, \infty) .$$

The stability regions disappear when

$$\frac{da_0}{d\omega} = 2\omega + a_1\tan\omega + a_1\frac{\omega}{\cos^2\omega} = 0$$

and

$$\frac{db}{d\omega} = a_1 \frac{\sin\omega}{\cos^2\omega} = 0 .$$

These equations can easily be solved for ω and a_1. They give the critical value $a_1 = -2$. The zero solution is stable (but not asymptotically stable) if

$$a_1 = -2 , \quad a_0 = j^2\pi^2 , \quad b = (-1)^{j+1}2 ; \quad j = 1, 2, \ldots .$$

The result is similar when $p = 2$, i.e. when (3.27) is a NDDE. However, the trivial solution is unstable in this case for all the parameters a_0 and b if $a_1 = -2$. $\triangle$

The analysis of the D-curves in the plane (a_0, b) shows that the greater the damping is, the greater the stability regions are when (3.27) is a RDDE, i.e. when $p = 0$ or 1. This leads to the following

Proposition 3.19 If the zero solution of the RDDE (3.27) ($p = 0, 1$) is exponentially asymptotically stable for the parameters (a_0, b, a_1^*) then it is exponentially asymptotically stable for (a_0, b, a_1) as well, where $a_1 \geq a_1^*$. $\triangle$

This statement is not valid for the NDDE (3.27) ($p = 2$) as is shown below.

Theorem 3.20. The trivial solution of the NDDE

$$\ddot{x}(t) + a_1\dot{x}(t) + \frac{\pi^2}{4}x(t) = -\frac{3}{4}\ddot{x}(t-1)$$

is exponentially asymptotically stable if and only if

$$-\frac{3\pi}{8} < a_1 < a_1^* \quad \text{or} \quad 0 < a_1$$

where

$$a_1^* = -\frac{3}{4}\rho_2 \sin\rho_2$$

and ρ_2 is the only solution of the equation

$$\omega^2\left(1 + \frac{3}{4}\cos\omega\right) = \frac{\pi^2}{4}$$

in the interval $(\pi/2, \pi)$.

Proof. As was shown in *Lemma 3.9*, *Theorem 2.25* can be applied as a necessary and sufficient stability criterion since the absolute value of the coefficient of the neutral term is less than 1. The positive zeros of

$$R(\omega) = -\omega^2 + \frac{\pi^2}{4} - \frac{3}{4}\omega^2 \cos\omega$$

are

$$\rho_1 = \pi, \quad \rho_2 \in \left(\frac{\pi}{2}, \pi\right) \quad \text{and} \quad \rho_3 = \frac{\pi}{2}.$$

Since

$$S(\omega) = a_1\omega + \frac{3}{4}\omega^2 \sin\omega,$$

the stability condition (2.22*b*)

$$\sum_{k=1}^{r} (-1)^k \operatorname{sgn} S(\rho_k) = -1$$

yields

$$-\operatorname{sgn} a_1 + \operatorname{sgn}\left(a_1 + \frac{3}{4}\rho_2 \sin\rho_2\right) - \operatorname{sgn}\left(a_1 + \frac{3\pi}{8}\right) = -1.$$

This is just equivalent to the inequalities in the statement of the theorem. △

Let us examine, now, the modification of the stability chart in *Fig. 3.2* when the "damping" appears in the third order equation (3.10).

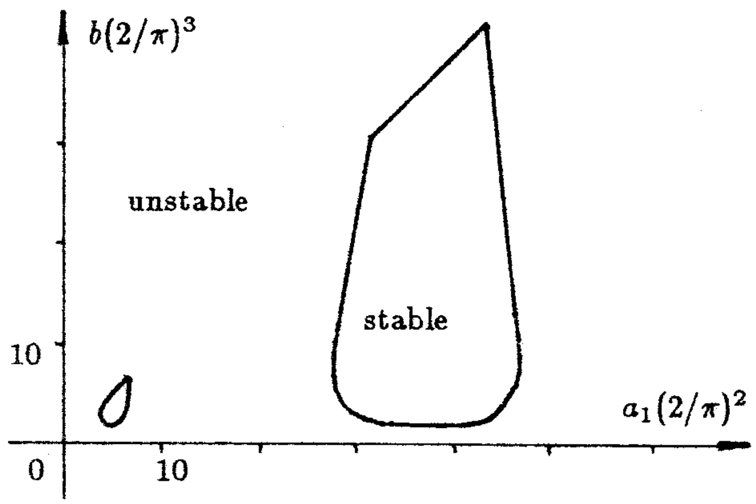

Fig. 3.17. Stability chart of RDDE (3.28) when $a_0 = 5$

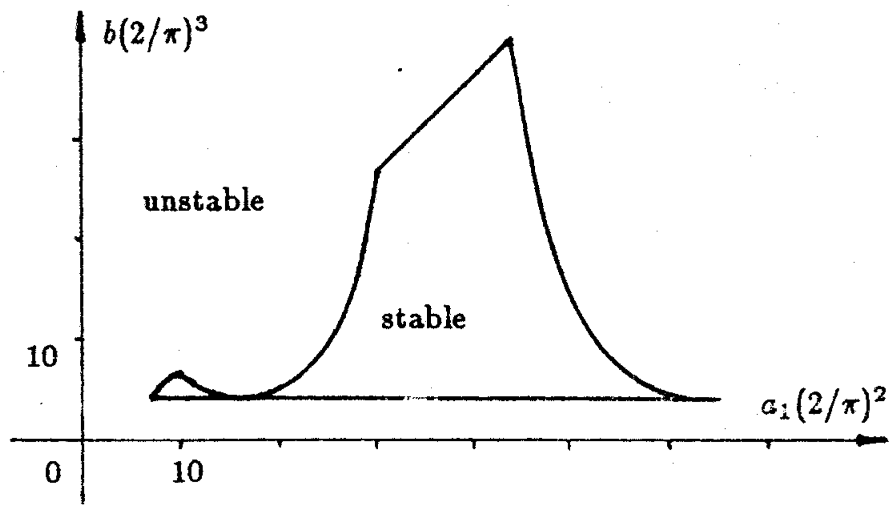

Fig. 3.18. Stability chart of RDDE (3.28) when $a_0 = -17$

Example 3.21. The stability charts of the scalar RDDE

$$\frac{d^3}{dt^3}x(t) + a_1\frac{d}{dt}x(t) + a_0 x(t) + bx(t-1) = 0 \tag{3.28}$$

are shown in *Figs. 3.17* and *3.18* when $a_0 = 5$ and $a_0 = -17$, respectively. They have been based on *Corollary 3.5* and on the application of the D-subdivision method. △

Simple sufficient stability conditions can be given for the scalar second order RDDE

$$\ddot{x}(t) + a_1\dot{x}(t) + a_0 x(t) = \sum_{j=1}^{p} b_j x(t-\tau_j)\,. \tag{3.29}$$

As the following theorems show, it is easy to estimate a minimum value of the "damping" a_1 or maximum values of the delays τ_j, $(j = 1,\dots,p)$ where the zero solution is still stable.

Theorem 3.22. The trivial solution of (3.29) is exponentially asymptotically stable for all the values of the delays, i.e. for $\tau_j \in [0,\infty)$, $(j = 1,\dots,p)$ if

$$a_0 > \sum_{j=1}^{p} |b_j| \quad \text{and} \quad a_1 > \frac{\sum_{j=1}^{p} |b_j|}{\sqrt{a_0 - \sum_{j=1}^{p} |b_j|}}\,.$$

Proof. In order to apply *Theorem 2.37* (or its special form given in *Corollary 2.38*), we shall give some estimations of the functions R and S calculated from the characteristic function of (3.29):

$$D(\lambda) = \lambda^2 + a_1\lambda + a_0 - \sum_{j=1}^{p} b_j e^{-\tau_j\lambda},$$

$$R(\omega) = -\omega^2 + a_0 - \sum_{j=1}^{p} b_j \cos(\tau_j\omega),$$

$$S(\omega) = a_1\omega + \sum_{j=1}^{p} b_j \sin(\tau_j\omega).$$

It is easy to see that

$$R^-(\omega) \le R(\omega) \le R^+(\omega), \quad \omega \in [0, +\infty),$$

$$R^-(\omega) = -\omega^2 + a_0 - \sum_{j=1}^{p} |b_j|,$$

$$R^+(\omega) = -\omega^2 + a_0 + \sum_{j=1}^{p} |b_j|,$$

and

$$S^-(\omega) \le S(\omega) \le S^+(\omega), \quad \omega \in [0, +\infty),$$

$$S^-(\omega) = a_1\omega - \sum_{j=1}^{p} |b_j|,$$

$$S^+(\omega) = a_1\omega + \sum_{j=1}^{p} |b_j|.$$

Since $a_0 > \sum_{j=1}^{p} |b_j|$, all the positive zeros of R are in the interval

$$I_{R1} = [\sqrt{a_0 - \sum_{j=1}^{p} |b_j|}, \sqrt{a_0 + \sum_{j=1}^{p} |b_j|}\ ],$$

and the non-negative zeros of S are in the interval

$$I_{S1} = [0, \frac{1}{a_1} \sum_{j=1}^{p} |b_j|].$$

These intervals are disjoint since

$$a_1 > \frac{\sum_{j=1}^{p} |b_j|}{\sqrt{a_0 - \sum_{j=1}^{p} |b_j|}} \quad \Rightarrow \quad \frac{1}{a_1}\sum_{j=1}^{p} |b_j| < \sqrt{a_0 - \sum_{j=1}^{p} |b_j|} \,.$$

If a representative number ρ_1^0 is chosen from I_{R1}, where S is always positive, the condition (2.22b) is satisfied:

$$\sum_{k=1}^{1} (-1)^k \operatorname{sgn} S(\rho_k^0) = -1 \,,$$

and *Theorem 2.37* implies the statement of the theorem. △

As a matter of fact, the estimations R^+ and S^+ have had no role in the proof, and *Theorem 2.19* can also be applied directly.

Theorem 3.23. The trivial solution of (3.29) is exponentially asymptotically stable if

$$a_0 > \sum_{j=1}^{p} b_j \quad \text{and} \quad a_1 > \sum_{j=1}^{p} |b_j| \tau_j \,.$$

Proof. Let us estimate the function S in the following way:

$$S(\omega) = \operatorname{Im} D(i\omega) > S^-(\omega) = (a_1 - \sum_{j=1}^{p} |b_j| \tau_j)\omega \,, \quad \omega \in (0, \infty) \,.$$

Since $a_1 > \sum_{j=1}^{p} |b_j| \tau_j$, the function S is positive in $(0, \infty)$. Thus, the stability condition (2.22b) is satisfied independently from the polynomial estimation of R (which can be done as in the proof of *Theorem 3.22* anyway). *Theorem 2.37* implies the statement. △

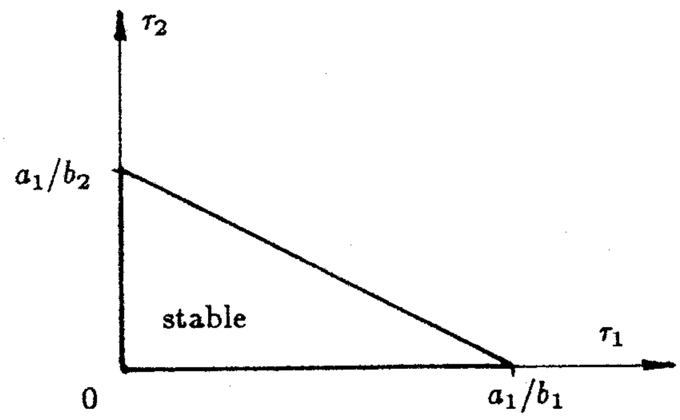

Fig. 3.19. A stability region of RDDE (3.29), $p = 2$, $a_1, b_1, b_2 > 0$, $a_0 > b_1 + b_2$

3.3. Systems with finite continuous delay

In this section, the second order scalar RFDE

$$\ddot{x}(t) + a_0 x(t) = b \int_{-1}^{0} w(\theta) x(t+\theta) d\theta \tag{3.30}$$

is investigated. The delay is continuous, its length h is finite, and it can be assumed to be 1 without the loss of generality. We suppose that the weight function w satisfies

$$\left| \int_{-1}^{0} w(\theta) d\theta \right| = 1 . \tag{3.31}$$

The parameter

$$\epsilon = \int_{-1}^{0} \theta w(\theta) d\theta \tag{3.32}$$

will also be used to characterize the function w. Let us see, first, a simple introductory result.

Theorem 3.24. Consider the weight function

$$w(\theta) \equiv 1 , \quad \theta \in [-1, 0] \tag{3.33}$$

in (3.30). The $x = 0$ solution of the RFDE (3.30) is exponentially asymptotically stable if and only if

$$0 < b < a_0 \quad \text{and} \quad a_0 \neq 4p^2\pi^2$$

for all the integers $p = 1, 2, \ldots$.

Proof. According to *Remark 2.2*, the RFDE with delay of finite length satisfies condition (2.2), and *Theorem 2.19* can be applied again. The characteristic function has the form

$$D(\lambda) = \begin{cases} \lambda^2 + a_0 - b\frac{1-e^{-\lambda}}{\lambda} , & \lambda \neq 0 \\ a_0 - b , & \lambda = 0 \end{cases} .$$

Thus, the functions R and S are as follows:

$$R(\omega) = \begin{cases} -\omega^2 + a_0 - b\frac{\sin\omega}{\omega}, & \omega \neq 0 \\ a_0 - b., & \omega = 0 \end{cases},$$

$$S(\omega) = \begin{cases} b\frac{1-\cos\omega}{\omega}, & \omega \neq 0 \\ 0, & \omega = 0 \end{cases}.$$

The condition $a_0 > b$ implies $R(0) > 0$. This means that R has positive zeros of odd number. The function S is positive in $(0, \infty)$ if $b > 0$ and $\omega \neq 2p\pi$ for all $p = 1, 2, \ldots$. In this way,

$$\sum_{k=1}^{r}(-1)^k \operatorname{sgn} S(\rho_k) = -1$$

and the stability condition (2.22b) is satisfied when $m = 1$. The condition $a_0 \neq 4p^2\pi^2$ is equivalent to (2.22a) in *Theorem 2.19*. If either $R(0) < 0$ or $b < 0$, it is easy to see that (2.22b) is not satisfied. △

Remark 3.25. In the RFDE (3.30) with (3.33), let us increase the parameter a_0 through $4\pi^2$ while b is fixed somewhere in $(0, a_0)$. When a_0 is just $4\pi^2$, there are two pure imaginary characteristic roots and all the others have negative real parts. However, the relevant pair of roots do not cross the imaginary axis as a_0 is increased further. They turn back to the left, after touching the imaginary axis. The conditions of the Hopf Bifurcation Theorem are not satisfied in this case. △

The following theorems will give the stability charts of the RFDE (3.30) with the weight function

$$w(\theta) = \frac{\pi}{2}\sin(\pi\theta) + \pi(1 - 2\epsilon)\sin(2\pi\theta), \quad \theta \in [-1, 0] \tag{3.34}$$

which satisfies (3.31) and (3.32) as well. To prepare the proofs of the theorems below, and to avoid long and tedious calculations, the following expressions present the formulae of the characteristic function and its real and imaginary part when $\lambda = i\omega$:

$$D(\lambda) = \begin{cases} \lambda^2 + a_0 + b\pi^2\left(\frac{1+e^{-\lambda}}{2(\lambda^2+\pi^2)} + 2(1-2\epsilon)\frac{1-e^{-\lambda}}{\lambda^2+4\pi^2}\right), & \lambda \neq \pm i\pi, \pm i2\pi \\ -\pi^2 + a_0 + \frac{4}{3}(1-2\epsilon)b \mp i\frac{\pi}{4}b, & \lambda = \pm i\pi \\ -4\pi^2 + a_0 - \frac{1}{3}b \mp i\frac{\pi}{2}(1-2\epsilon)b, & \lambda = \pm i2\pi \end{cases},$$

$$R(\omega) = \begin{cases} -\omega^2 + a_0 + b\pi^2\left(\frac{1+\cos\omega}{2(\pi^2-\omega^2)} + 2(1-2\epsilon)\frac{1-\cos\omega}{4\pi^2-\omega^2}\right), & \omega \neq \pi, 2\pi \\ -\pi^2 + a_0 + \frac{4}{3}(1-2\epsilon)b, & \omega = \pi \\ -4\pi^2 + a_0 - \frac{1}{3}b, & \omega = 2\pi \end{cases},$$

$$S(\omega) = \begin{cases} b\pi^2 \sin\omega\left(-\frac{1}{2(\pi^2-\omega^2)} + 2\frac{1-2\epsilon}{4\pi^2-\omega^2}\right), & \omega \neq \pi, 2\pi \\ -\frac{\pi}{4}b, & \omega = \pi \\ -\frac{\pi}{2}(1-2\epsilon)b, & \omega = 2\pi \end{cases}. \tag{3.35}$$

Theorem 3.26. The $x = 0$ solution of the RFDE (3.30) with the weight function

$$w(\theta) = \frac{\pi}{2}\sin(\pi\theta)$$

(i.e. when $\epsilon = 1/2$ in (3.34)) is exponentially asymptotically stable if and only if

$$b < 0, \quad b > -a_0 \quad \text{and} \quad b > 3(a_0 - 4\pi^2)$$

or there exists an integer $j \geq 1$ such that either

$$b > 0, \quad b < 3(a_0 - 4\pi^2), \quad b < (4j^2 - 1)(a_0 - 4j^2\pi^2) \quad \text{and} \quad a_0 < (2j+1)^2\pi^2$$

or

$$b < 0, \quad b > -a_0, \quad b > ((2j+2)^2 - 1)(a_0 - (2j+2)^2\pi^2) \quad \text{and} \quad a_0 > (2j+1)^2\pi^2.$$

Proof. Let us consider, first, the case of $b > 0$. When $\epsilon = 1/2$, the formula (3.35) of S shows that S is positive in the intervals $(2j\pi, (2j+1)\pi)$, $(j = 1, 2, \ldots)$. For $b > 0$, the inequalities in the theorem are equivalent to

$$R(2\pi) > 0, \quad R(2j\pi) > 0 \quad \text{and} \quad R((2j+1)\pi) < 0$$

for some $j = 1, 2, \ldots$, as is shown by the actual form of R at $\epsilon = 1/2$. On the basis of all these, R has zeros of odd number only in an interval where S is positive, that is

$$\sum_{k=1}^{r} (-1)^k \operatorname{sgn} S(\rho_k) = -1,$$

and *Theorem 2.19* implies the stability of the zero solution. The polynomial part of R is a simple polynomial of second degree and the transcendental part is bounded in $[0, \infty)$. Thus, there are no other regions of stability if $b > 0$. If $b < 0$, then R has to have zeros of odd number either in $(0, 2\pi)$ or in one of the intervals $((2j+1)\pi, (2j+2)\pi)$, $(j = 1, 2, \ldots)$, where S is positive. This is equivalent to the further inequalities in the theorem. △

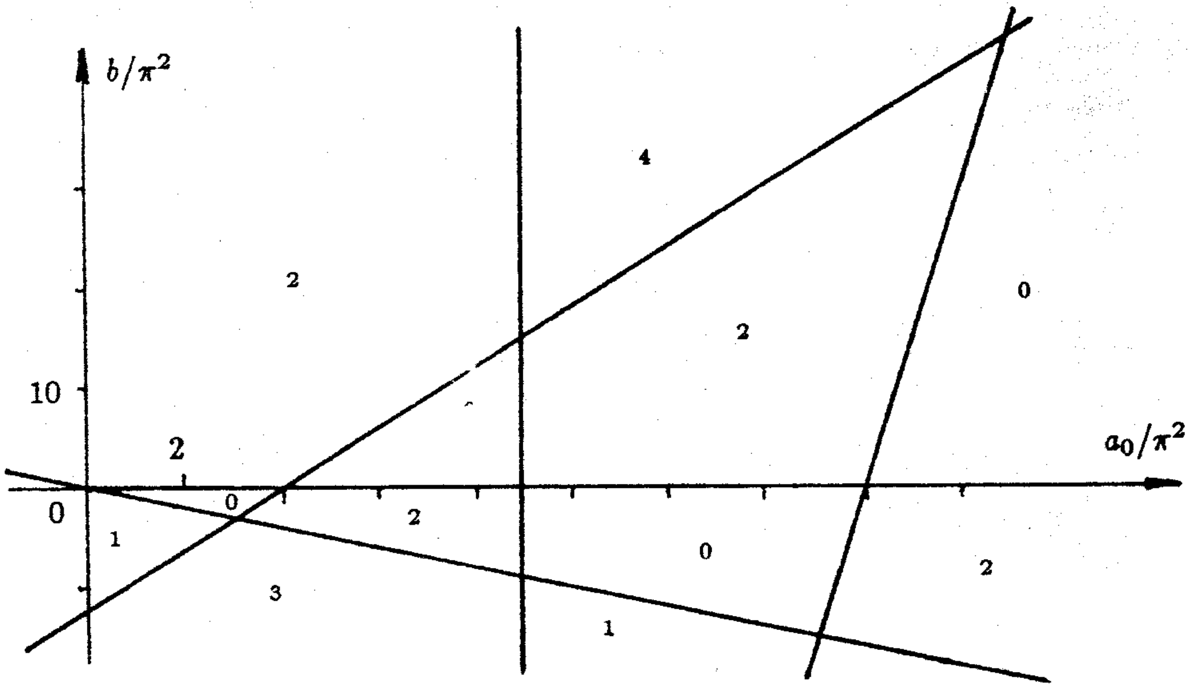

Fig. 3.20. Stability chart of RFDE (3.30) with weight function (3.34), $\epsilon = \frac{1}{2}$

The stability chart of the RFDE (3.30) presents the number of characteristic roots having positive real parts in *Fig. 3.20.* These numbers have been determined by means of *Theorem 2.15.* The structure of the stability regions (denoted by o) is similar to that of the RDDE (3.9) in *Fig. 3.1.*

Theorem 3.27. The zero solution of the RFDE (3.30) with the weight function (3.34), where $\epsilon = 1/4$, is exponentially asymptotically stable if and only if

$$b < 0\,, \quad b > -a_0 \quad \text{and} \quad b > 5(a_0 - 9\pi^2)$$

or there exists an integer $j \geq 2$ such that either

$$b > 0\,, \quad b < \frac{5}{2}(a_0 - 9\pi^2)\,, \quad b < \frac{1}{2}((2j-1)^2 - 4)(a_0 - (2j-1)^2\pi^2) \quad \text{and}$$

$$b > (4j^2 - 1)(a_0 - 4j^2\pi^2)$$

or

$$b < 0\,, \quad b > -a_0\,, \quad b < (4j^2 - 1)(a_0 - 4j^2\pi^2) \quad \text{and}$$

$$b > \frac{1}{2}((2j+1)^2 - 4)(a_0 - (2j+1)^2\pi^2)\,.$$

Proof. The proof of *Theorem 3.26* serves as a guide-line to prove this theorem. The only important difference is that S is positive in $((2j-1)\pi, 2j\pi)$, and it is negative in $(0, 3\pi)$ and $(2j\pi, (2j+1)\pi)$, $(j = 2, 3, \ldots)$ when $b > 0$ and $\epsilon = 1/4$ in (3.35). $\triangle$

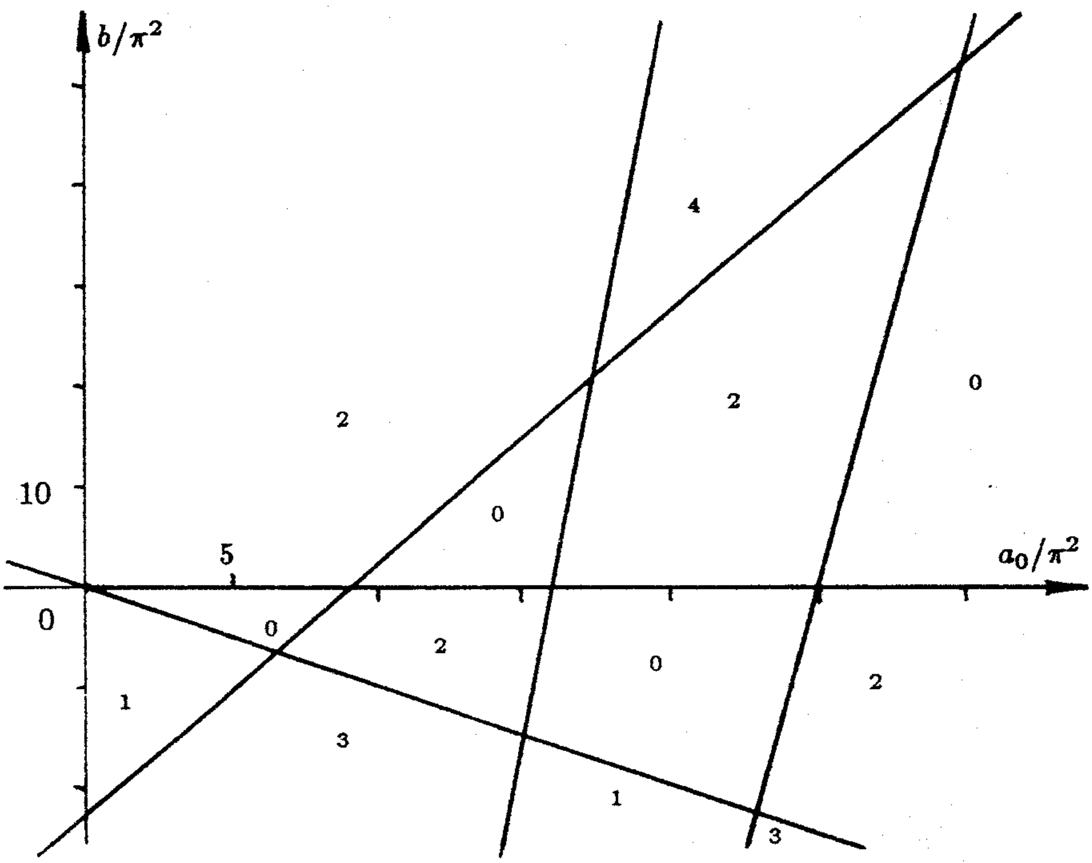

Fig. 3.21. Stability chart of RFDE (3.30) with weight function (3.34), $\epsilon = \frac{1}{4}$

The stability chart in *Fig. 3.21* presents the number of characteristic roots having positive real parts when $\epsilon = 1/4$ in (3.34).

The comparison of the stability charts in *Figs. 3.20* and *3.21* emphasizes that the regions of stability moved globally to the opposite sides of the axis a_0 in the parameter plane (a_0, b) as the parameter ϵ changed from 1/2 to 1/4 in the weight function (3.34), although the main characteristic of the weight function did not change substantially.

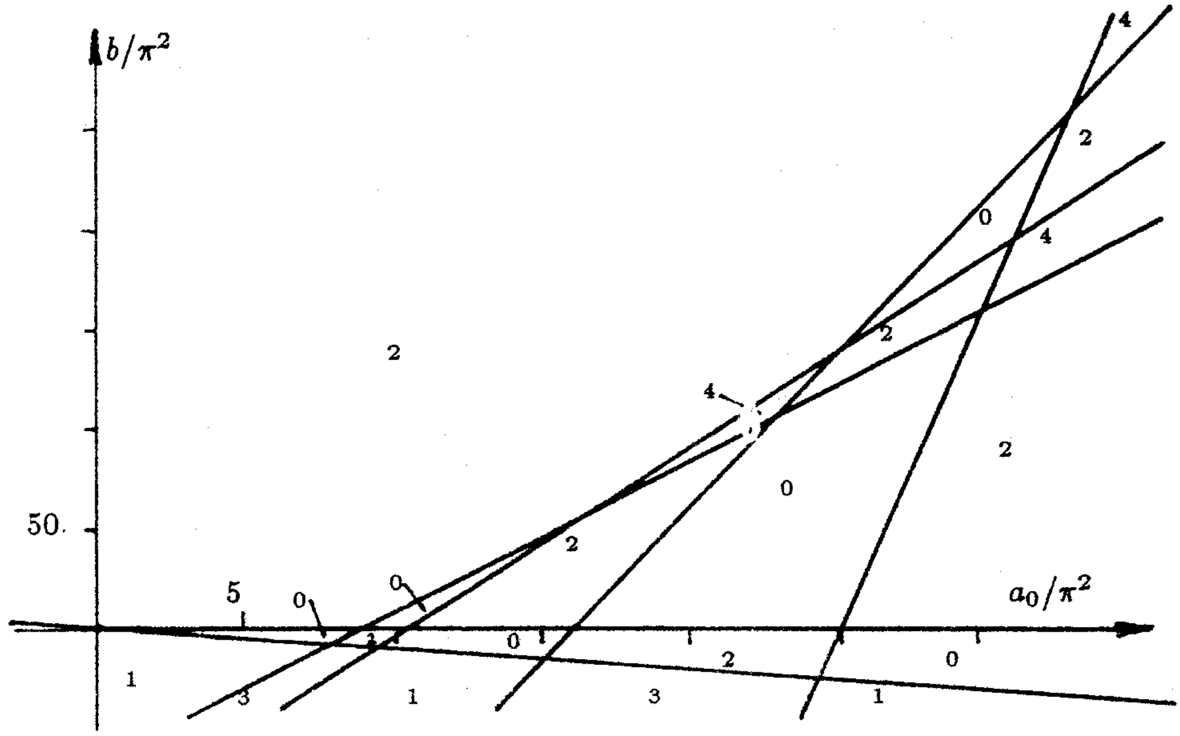

Fig. 3.22. Stability chart of RFDE (3.30) with weight function (3.34), $\epsilon = \frac{32}{77}$

If $\epsilon \in (\frac{3}{8}, \frac{1}{2})$ in (3.34), then the function S has positive zeros not only at $p\pi$, $(p = 3, 4, \ldots)$, but also at

$$\sigma^* = 2\pi\sqrt{\frac{2\epsilon}{8\epsilon - 3}}\,.$$

This can easily be calculated from (3.35). Thus, the stability conditions of (3.30) are the combinations of the inequalities of the *Theorems 3.26* and *3.27* together with an additional one related to σ^*. *Fig. 3.22* shows a stability chart of (3.30) when $\epsilon = 32/77$ in (3.34). In this case, the special zero of S is $\sigma^* = 16\pi/5$. The figure gives the numbers of characteristic roots having positive real parts. There is a domain of stability which is substantially separated from the others.

3.4. Unbounded delay

Let us consider the first order scalar equation

$$\dot{x}(t) = \int_{-\infty}^{0} x(t+\theta)d\eta(\theta)\,, \tag{3.36}$$

where $x \in \mathbf{R}$, η is a scalar function of bounded variation, and it satisfies the usual condition (2.2). This equation has been studied in the literature when η has a special characteristic, for example, when η is monotonous. Such equations often occur in biological applications [11]. The first theorem is about a sufficient condition of stability when η is non-increasing in (3.36).

Theorem 3.28. Suppose that η is a non-constant and non-increasing function in the scalar RFDE (3.36). The trivial solution of (3.36) is exponentially asymptotically stable if

$$\int_{-\infty}^{0} \theta d\eta(\theta) < 1\,.$$

Proof. The characteristic function of (3.36) is as follows

$$D(\lambda) = \lambda - \int_{-\infty}^{0} e^{\lambda\theta} d\eta(\theta)\,.$$

Thus,

$$R(\omega) = \operatorname{Re} D(i\omega) = -\int_{-\infty}^{0} \cos(\omega\theta)d\eta(\theta)\,,$$

$$S(\omega) = \operatorname{Im} D(i\omega) = \omega - \int_{-\infty}^{0} \sin(\omega\theta)d\eta(\theta)\,.$$

Since η is non-constant and non-increasing function, we have

$$R(0) = -\int_{-\infty}^{0} d\eta = \lim_{\theta \to -\infty} \eta(\theta) - \eta(0) > 0 .$$

This means that condition (2.23*b*) is satisfied in *Theorem 2.19*. Since η is non-increasing and $\int_{-\infty}^{0} \theta d\eta(\theta) < 1$, the following estimation of S is true:

$$S(\omega) \geq \omega - \int_{-\infty}^{0} \omega\theta d\eta(\theta) = \omega\big(1 - \int_{-\infty}^{0} \theta d\eta(\theta)\big) > 0 ,$$

for all $\omega \in (0, \infty)$. Thus, S has only $s = 1$ non-negative zero, and this is $\sigma_1 = 0$. This yields the trivial satisfaction of condition (2.23*c*) with $m = 0$:

$$\sum_{k=1}^{0} (-1)^k \operatorname{sgn} R(\sigma_k) + \frac{1}{2}((-1)^1 + (-1)^0) + (-1)^0 0 = 0 ,$$

and *Theorem 2.19* implies the statement of the theorem. △

This theorem immediately presents a result of [50] in connection with a problem in bioecology:

Corollary 3.29. Consider the scalar RFDE

$$\dot{x}(t) = -\alpha \int_{-\infty}^{-h} w(\theta + h) x(t + \theta) d\theta \tag{3.37}$$

where $h \geq 0$, $\alpha > 0$ and the weight function w is non-negative, continuous on $(-\infty, 0]$,

$$\int_{-\infty}^{0} w(\theta) d\theta = 1 ,$$

and there exist the scalars $\nu > 0$ and $K > 0$ such that

$$|w(\theta)| \leq K e^{\nu\theta} , \quad \text{for all} \quad \theta \in (-\infty, 0] .$$

The zero solution of (3.37) is exponentially asymptotically stable if

$$\int_{-\infty}^{0} \theta w(\theta) d\theta > -\frac{1}{\alpha} + h .$$

Proof. The RFDE (3.37) is, of course, a special case of (3.36) where

$$\eta(\theta) = \begin{cases} -\alpha \int_{-\infty}^{\theta} w(\chi + h) d\chi\,, & \theta \in (-\infty, -h] \\ -\alpha \int_{-\infty}^{0} w(\chi) d\chi\,, & \theta \in (-h, 0] \end{cases} .$$

According to *Theorem 3.28,*

$$\int_{-\infty}^{0} \theta d\eta(\theta) < 1$$

implies the stability of the zero solution. The calculation of this integral gives

$$-\alpha \int_{-\infty}^{-h} \theta w(\theta + h) d\theta = -\alpha \Big(\int_{-\infty}^{0} \chi w(\chi) d\chi - h \Big) < 1\,,$$

which proves the statement. △

It is obvious that $\int_{-\infty}^{0} \theta w(\theta) d\theta$ is negative, and it does not satisfy the condition of *Corollary 3.29* if $\alpha h > 1$, i.e. when the past-effect is great enough

The case of non-increasing η has been considered in *Theorem 3.28.* The following theorem investigates the RFDE (3.36) with "almost" non-decreasing function η. More precisely, (3.36) is investigated with

$$\eta(\theta) = \begin{cases} \mu(\theta)\,, & \theta \in (-\infty, 0) \\ -\mu_0 + \lim_{\theta \to -0} \mu(\theta)\,, & \theta = 0 \end{cases} ,$$

where μ_0 is a real number, $\mu : (-\infty, 0) \mapsto \mathbf{R}$ is of bounded variation, satisfies the usual condition (2.2), and μ is a non-decreasing function. Such RFDEs may describe, for example, compartmental systems in biology [22], or single species models in population dynamics.

Theorem 3.30. Consider the scalar RFDE

$$\dot{x}(t) = -\mu_0 x(t) + \int_{-\infty}^{0} x(t + \theta) d\mu(\theta) \tag{3.38}$$

where μ is a non-decreasing function and sastisfies (2.2). Its trivial solution is exponentially asymptotically stable if and only if

$$\mu_0 > \int_{-\infty}^{0} d\mu\,.$$

Proof. If the characteristic function

$$D(\lambda) = \lambda + \mu_0 - \int_{-\infty}^{0} e^{\lambda\theta} d\mu(\theta)$$

is not positive at zero, that is if

$$D(0) = \mu_0 - \int_{-\infty}^{0} d\mu \leq 0,$$

then the zero solution is not asymptotically stable as follows from *Corollary 2.30.* It will be shown that $D(0) > 0$ implies the exponential asymptotical stability. Since μ is a non-decreasing function in

$$R(\omega) = \operatorname{Re} D(i\omega) = \mu_0 - \int_{-\infty}^{0} \cos(\omega\theta) d\mu(\theta),$$

R can be estimated as follows:

$$R(\omega) \geq \mu_0 - \int_{-\infty}^{0} d\mu > 0.$$

The stability condition (2.23*b*) is satisfied then, and (2.23*c*) is also fulfilled since

$$\sum_{k=1}^{s-1} (-1)^k \operatorname{sgn} R(\sigma_k) + \frac{1}{2}((-1)^s + (-1)^m) + (-1)^m$$

$$= \sum_{k=1}^{s-1} (-1)^k + \frac{1}{2}((-1)^s + (-1)^0) + (-1)^0 0 = 0$$

is true independently from the number s of the non-negative zeros of S. Thus, the statement is proved by *Theorem 2.19.* $\triangle$

The RFDE (3.38) has also been investigated by Plant. In [42], he proved that the zero solution is asymptotically stable when $\mu(\theta) = \arctan\theta$ and $\mu_0 > \int_{-\infty}^{0} d\mu = \frac{\pi}{2}$. This function μ does not satisfy condition (2.2) since $\mu'(\theta) = (1 + \theta^2)^{-1}$ cannot be bounded by any exponential function of the form $Ke^{\nu\theta}$, $\nu > 0$. However, the asymptotic stability of the zero solution is not exponential in this case, as it was also pointed out in [42].

Krisztin [36] has investigated the critical case $\mu_0 = \int_{-\infty}^{0} d\mu$, giving the necessary and sufficient condition of the stability of the solutions. These results of [36] or [42] cannot be achieved by the direct stability investigation of the characteristic functions only, because our method is applicable to prove exponential stability if it exists.

In this section, we shall speak, finally, about the scalar RFDE (3.36) when η is an exponential function. In applications when there is no exact information about the "weights" of the past states of a dynamical system, it is very convenient to choose an exponential weight function. In this way, the RFDE with unbounded delay can be transformed into a system of ordinary differential equations. Since the characteristic function is polynomial, all the difficulties are avoided. These mathematical models can be handled easily, but the qualitative results obtained from them may be incorrect for the real system having an infinite spectrum, since its weight function is different from the approximative exponential one.

This problem has generally been investigated in [17]. The following theorem deals with a special equation only, but it gives explicit estimations for the location of the characteristic roots of a linear RFDE (3.36) where η is close enough to an exponential function. The results are partly related to those of Driver [12].

Theorem 3.31. Consider the scalar RFDE

$$\dot{x}(t) = \int_{-\infty}^{0} x(t+\theta)d\eta(\theta)$$

where

$$\eta(\theta) = \begin{cases} -e^{\theta/h}, & \theta \in (-\infty, -\tau] \\ -\epsilon - e^{\theta/h}, & \theta \in (-\tau, 0] \end{cases},$$

$\tau > 0$, $0 < h < 1/4$ and $\epsilon \in \mathbf{R}$. When $\epsilon = 0$, there are two characteristic roots only, these are denoted by $\tilde{\lambda}_{1,2}$. For any small $\rho > 0$ and large $M > 0$, there exists an $\epsilon^*(\rho, M)$ such that for all $|\epsilon| < \epsilon^*$ there are exactly two characteristic roots $\lambda_{1,2}$ satisfying

$$|\lambda_j - \tilde{\lambda}_j| < \rho, \quad j = 1, 2$$

and all the further characteristic roots have real parts less than $-M$, that is

$$\mathrm{Re}\,\lambda_j < -M\,, \quad j = 3, 4, \ldots\,.$$

ϵ^* can be given as

$$\epsilon^* = \min\Big(\rho\frac{\tau}{h}\sqrt{\frac{1}{4} - h}\; e^{-\tau/h}, \frac{1}{\tau}e^{-\tau M}\Big)\,. \tag{3.39}$$

Proof. Substituting η into the RFDE, we get

$$\dot{x}(t) = -\int_{-\infty}^{0} \frac{1}{h}e^{\theta/h}x(t+\theta)d\theta - \epsilon x(t-\tau)\,.$$

Let us introduce the new variable $y \in \mathbf{R}$ by

$$y(t) = \int_{-\infty}^{0} \frac{1}{h}e^{\theta/h}x(t+\theta)d\theta\,.$$

Since the initial functions can be chosen from the space $\mathbf{B}$ only (see *Definition 1.2*), the derivative of y can be calculated by partial integration. This yields

$$\dot{y}(t) = \frac{1}{h}x(t) - \frac{1}{h}y(t)\,,$$

and the RFDE with unbounded delay can be transformed into a two-dimensional system of RDDEs:

$$\dot{x}(t) = -y(t) - \epsilon x(t-\tau)\,,$$

$$\dot{y}(t) = \frac{1}{h}x(t) - \frac{1}{h}y(t)\,.$$

Its characteristic function

$$D(\lambda) = \lambda^2 + \frac{1}{h}\lambda + \frac{1}{h} + \epsilon(\lambda + \frac{1}{h})e^{-\tau\lambda}$$

has only two zeros if $\epsilon = 0$. They are

$$\tilde{\lambda}_{1,2} = \frac{1}{2h}(-1 \pm \sqrt{1-4h})$$

which are real and negative since $0 < h < 1/4$. Without the loss of generality, we can suppose that

$$\rho < \min\left(\frac{1}{2h}\sqrt{1-4h}, |\tilde{\lambda}_1|\right).$$

In this case, the analysis of the transcendental part of D shows that

$$|\epsilon(\lambda + \frac{1}{h})e^{-\tau\lambda}| \leq |\epsilon|\frac{1}{\tau}e^{\tau/h}, \quad \lambda \in [-\frac{1}{h}, +\infty).$$

Thus, (3.39) gives

$$|\lambda^2 + \frac{1}{h}\lambda + \frac{1}{h}| \leq \frac{\rho}{h}\sqrt{\frac{1}{4} - h}$$

for the characteristic roots lying in $[-1/h, +\infty)$. The elementary examination of this inequality proves that there are only two real zeros $\lambda_{1,2}$ of D in $[-1/h, +\infty)$, and they are in the ρ-neighbourhoods of $\tilde{\lambda}_{1,2}$.

We shall prove, now, that all the other characteristic roots are on the left-hand side of the line $\lambda = -M$ in the complex plane. Without the loss of generality, we can suppose that

$$M > \frac{1}{h} + \frac{1}{\tau}.$$

The function

$$D_M(z) = (z-M)^2 + \frac{1}{h}(z-M) + \frac{1}{h} + \epsilon((z-M) + \frac{1}{h})e^{-\tau(z-M)}$$

has the zeros

$$z_j = \lambda_j + M, \quad j = 1, 2, \ldots,$$

where λ_j are the zeros of D. Since $\lambda_{1,2} > -1/\tau$, D_M has at least 2 positive real zeros $z_{1,2} = \lambda_{1,2} + M$. By means of *Theorem 2.15*, it will be shown that D_M has exactly 2 zeros in the right half plane if $|\epsilon| < \epsilon^*$. When $m = 1$, the number N of the characteristic roots with positive real parts are given by (2.17):

$$N = 1 + \sum_{k=1}^{r}(-1)^k \operatorname{sgn} S(\rho_k), \tag{3.40}$$

where

$$S(\omega) = \operatorname{Im} D_M(i\omega) = -\omega(2M - \frac{1}{h}) + \epsilon e^{\tau M}\omega\cos(\tau\omega) - \epsilon(\frac{1}{h} - M)e^{\tau M}\sin(\tau\omega)$$

and ρ_k $(k = 1, \dots, r)$ are the positive real zeros of

$$R(\omega) = \operatorname{Re} D_M(i\omega) = -\omega^2 + (M^2 - \frac{M}{h} + \frac{1}{h}) + \epsilon e^{\tau M}\omega\sin(\tau\omega) + \epsilon(\frac{1}{h} - M)e^{\tau M}\cos(\tau\omega)\,.$$

It is obvious that

$$\lim_{\omega\to+\infty} R(\omega) = -\infty\,.$$

With the help of (3.39), it can also be seen that

$$R(0) = M^2 - \frac{M}{h} + \frac{1}{h} - \epsilon(M - \frac{1}{h})e^{\tau M} > M^2 - \frac{M}{h} + \frac{1}{h} - \frac{M}{\tau} + \frac{1}{\tau h} > 0\,.$$

Thus, the number r of the positive zeros of R is odd. In the same time, (3.39) yields the estimation

$$S(\omega) \le -\omega\big((2M - \frac{1}{h}) - |\epsilon|e^{\tau M} - |\epsilon|(M - \frac{1}{h})\tau e^{\tau M}\big)$$
$$< -\omega\big((2M - \frac{1}{h}) - \frac{1}{\tau} - (M - \frac{1}{h})\big) = -\omega(M - \frac{1}{\tau}) < 0$$

for $\omega \in (0, \infty)$. This means that $S(\rho_k) < 0$ $(k = 1, \dots, r)$ and (3.40) gives that D_M has

$$N = 1 - \sum_{k=1}^{r}(-1)^k = 2$$

zeros $z_{1,2}$ with positive real parts, and

$$\operatorname{Re} z_j < 0 \quad \Rightarrow \quad \operatorname{Re}\lambda_j < -M\,, \quad j = 3, 4, \dots$$

for the further zeros λ_j of D. This completes the proof of the theorem. $\triangle$

As formula (3.39) shows, the greater the discrete delay τ is, the smaller ϵ^* is. If we want to push the non-relevant characteristic roots λ_j $(j = 3, 4, \dots)$ far to the left, the "perturbation" ϵ of the pure exponential function η has to be decreased exponentially as a function of M. *Theorem 3.31* was also an example of how to use *Theorems 2.15* and *2.16* to separate the relevant and non-relevant part of the spectrum in the case of RFDEs.

4. Applications

In the previous section, *Example 3.7* has presented some stability problems of retarded mechanical systems. These simplified mechanical models were used in the sixties to investigate the stability of space vehicles [34]. A lot of interesting stability problems caused by delays can be mentioned from the different fields of control theory. However, the greatest difficulties in analytical investigations occur when an oscillatory system is subjected to a retarded feed-back force, especially when the delay is continuous, or there are more delays than one. There are such examples in mechanical engineering, in machine dynamics. The analytical stability investigation of some models of man-machine systems, robotics and machine tool vibrations will be presented in *Sections 4.2, 4.3* and *4.4*, respectively. But first, a bioecological problem will be examined. This is a good introduction to the above-mentioned field of engineering applications, since from a mathematical viewpoint, the oscillations in a predator-prey system of population dynamics are very similar to the vibrations of machine tools.

4.1. Bioecology

The time history of the sizes of two populations living in a well-separated environment is often described (cf. [8,11,37]) by the two-dimensional RFDE

$$\dot{u}(t) = f(u_t, v_t)\,, \tag{4.1a}$$

$$\dot{v}(t) = g(u_t, v_t)\,. \tag{4.1b}$$

The scalar u and v stand for the population densities of the two species, so they must be non-negative. The functions u_t , $v_t \in \mathbf{B}$ are defined by (1.3) in *Definition 1.1.* We assume that (4.1) has a trivial solution (U, V) in the positive quadrant of the plane (u, v), and $f, g : \mathbf{B} \times \mathbf{B} \to \mathbf{R}$ are continuous and the variational system with respect to (U, V) exists.

In this section, the variational systems of two models of Freedman will be examined, as well as the non-linear Lotka-Volterra-MacDonald model of a predator-prey system. The models proposed by Freedman contain discrete delays, while the Lotka-Volterra model modified by McDonald introduces continuous delay into the equations.

Using the Nyquist criterion, Freedman and Rao [19] have achieved some general stability and instability results with respect to (U, V) in (4.1). These results were applied then to a model of two competing populations. The linearisation of the equations at the equilibrium (U, V) yields

$$\dot{x}_1(t) = -a_{11}x_1(t) - a_{12}x_2(t) + b_{11}x_1(t - \tau), \qquad (4.2a)$$

$$\dot{x}_2(t) = -a_{21}x_1(t) - a_{22}x_2(t) + b_{22}x_2(t - \xi), \qquad (4.2b)$$

where $x_1 = u - U$, $x_2 = v - V$ and all the constant parameters are positive except the intrinsic delays τ and ξ which are non-negative. The parameters a_{12} and a_{21} can be assumed to be the competition strengths, a_{11} and a_{22} are the intrinsic death rates, while b_{11} and b_{22} are the intrinsic birth rates [19].

The following stability result gives an estimation of the delays τ and ξ for which stability will persist.

Theorem 4.1. The trivial solution of the RDDE (4.2) is exponentially asymptotically stable if

$$(b_{11} - a_{11})(b_{22} - a_{22}) > a_{12}a_{21} \qquad (4.3a)$$

and

$$b_{11}(b_{22} + 0.22a_{22})\tau + b_{22}(b_{11} + 0.22a_{11})\xi < a_{11} + a_{22} - b_{11} - b_{22}\,. \qquad (4.3b)$$

Proof. The same method will be used as in the proof of *Theorem 3.23.* The characteristic function is given by

$$D(\lambda) = \lambda^2 + (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21})$$

$$-b_{11}\lambda e^{-\tau\lambda} - b_{22}\lambda e^{-\xi\lambda} - a_{22}b_{11}e^{-\tau\lambda} - a_{11}b_{22}e^{-\xi\lambda} + b_{11}b_{22}e^{-(\tau+\xi)\lambda},$$

thus,

$$R(\omega) = -\omega^2 + (a_{11}a_{22} - a_{12}a_{21}) - b_{11}\omega\sin(\tau\omega) - b_{22}\omega\sin(\xi\omega)$$

$$-a_{22}b_{11}\cos(\tau\omega) - a_{11}b_{22}\cos(\xi\omega) + b_{11}b_{22}\cos\big((\tau+\xi)\omega\big),$$

$$S(\omega) = (a_{11} + a_{22})\omega - b_{11}\omega\cos(\tau\omega) - b_{22}\omega\cos(\xi\omega)$$

$$+a_{22}b_{11}\sin(\tau\omega) + a_{11}b_{22}\sin(\xi\omega) - b_{11}b_{22}\sin\big((\tau+\xi)\omega\big).$$

For $\omega \in (0,\infty)$, the following inequalities hold:

$$|\cos(\tau\omega)| \leq 1, \quad -\sin\big((\tau+\xi)\omega\big) > -(\tau+\xi)\omega$$

and

$$\sin(\tau\omega) > -0.22\tau\omega.$$

With the help of them, S can be estimated in the following way:

$$S(\omega) \geq S^-(\omega)$$

$$= \Big(a_{11} + a_{22} - b_{11} - b_{22} - 0.22a_{22}b_{11}\tau - 0.22a_{11}b_{22}\xi - b_{11}b_{22}(\tau+\xi)\Big)\omega.$$

Thus, $(4.3b)$ yields

$$S(\omega) \geq S^-(\omega) > 0, \quad \text{for} \quad \omega \in (0,\infty).$$

On the other hand,

$$R(0) = a_{11}a_{22} - a_{12}a_{21} - a_{22}b_{11} - a_{11}b_{22} + b_{11}b_{22} > 0$$

follows from (4.3a). Since

$$\lim_{\omega\to\infty} R(\omega) = -\infty\,,$$

the number r of the positive zeros ρ_k of R is odd, and

$$\sum_{k=1}^{r}(-1)^k \operatorname{sgn} S(\rho_k) = -1\,.$$

This means that the stability condition (2.22b) in *Theorem 2.19* is satisfied, and the proof is completed. △

Remark 4.2. The RDDE (4.2) can be transformed into the second order RDDE

$$\ddot{x}_1(t) + (a_{11} + a_{22})\dot{x}_1(t) + (a_{11}a_{22} - a_{12}a_{21})x_1(t)$$

$$= b_{11}\dot{x}_1(t-\tau) + b_{22}\dot{x}_1(t-\xi) + a_{22}b_{11}x_1(t-\tau) + a_{11}b_{22}x_1(t-\xi) - b_{11}b_{22}x_1(t-(\tau+\xi))\,.$$

This equation contains 3 discrete delays: τ, ξ and $\tau+\xi$. *Theorem 3.23* cannot be applied here directly because of the presence of the delayed values of the first derivative of x_1.

We also note that (4.3b) is not satisfied if

$$a_{11} + a_{22} - b_{11} - b_{22} < 0\,,$$

since the left-hand side of (4.3b) is non-negative. This means that the condition (4.3b) is not satisfied if the trivial solution is unstable when $\tau = \xi = 0$ ((4.2) is an ordinary differential equation). △

The condition of asymptotic stability given by Freedman and Rao in [19] is more restrictive with respect to the delays than (4.3b) is in *Theorem 4.1*.

Another special variational system of (4.1) has the form

$$\dot{x}_1(t) = -a_{11}x_1(t) - b_{12}x_2(t-\xi)\,, \tag{4.4a}$$

$$\dot{x}_2(t) = -a_{22}x_2(t) + b_{21}x_1(t-\tau)\,, \tag{4.4b}$$

which may describe the dynamics of a predator-prey system around the equilibrium (U, V). If u and v are the population densities of the prey and the predator, respectively, then $x_1 = u - U$, $x_2 = v - V$. There are two discrete delays in this model. The derivation of the parameters is presented in [19]. The coefficients a_{22}, b_{12} and b_{21} are positive.

Corollary 4.3. The trivial solution of (4.4) is exponentially asymptotically stable for any time lags $\tau \geq 0$ and $\xi \geq 0$ if

$$a_{11}a_{22} > b_{12}b_{21} \quad \text{and} \quad a_{11} + a_{22} > \frac{b_{12}b_{21}}{\sqrt{a_{11}a_{22} - b_{12}b_{21}}} .$$

Proof. The elimination of x_2 in the RDDE (4.4) leads to the equation

$$\ddot{x}_1(t) + (a_{11} + a_{22})\dot{x}_1(t) + a_{11}a_{22}x_1(t) = -b_{12}b_{21}x_1(t - (\tau + \xi))$$

which is identical to the RDDE (3.29) if

$$a_1 = a_{11} + a_{22} , \quad a_0 = a_{11}a_{22} , \quad b_1 = -b_{12}b_{21} ,$$

$$b_j = 0, \ (j = 2, 3, \ldots, p), \quad \text{and} \quad \tau_1 = \tau + \xi \tag{4.5}$$

is substituted into (3.29). With these parameters, *Theorem 3.22* implies the statement. △

Remark 4.4. With the help of a different technique, Freedman and Rao have obtained a better result in [18]. It can be proved that $a_{11}a_{22} > b_{12}b_{21}$ implies asymptotic stability in (4.4), even if the second inequality in *Corollary 4.3* does not hold. △

Corollary 4.5. The trivial solution of the RDDE (4.4) is exponentially asymptotically stable if

$$a_{11}a_{22} > -b_{12}b_{21} \quad \text{and} \quad \tau + \xi < \frac{a_{11} + a_{22}}{b_{12}b_{21}} .$$

Proof. The statement follows from *Theorem 3.23* after the substitution of the parameters (4.5) into (3.29). △

As a matter of fact, it is not difficult to determine the exact stability regions in the parameter space of the RDDE (4.4). In *Example 3.16*, similar stability charts were presented. If

$$a_1 = (a_{11} + a_{22})(\tau + \xi), \quad a_0 = a_{11}a_{22}(\tau + \xi)^2$$

$$\text{and} \quad b = -b_{12}b_{21}(\tau + \xi)^2$$

is substituted into the RDDE (3.26), then *Fig. 3.15* gives a stability chart for the parameters a_0, a_1 and b, which can be transformed into the space of some chosen parameters of (4.4).

In the cases of the models of the competing populations and the predator-prey system, we have investigated neither the conditions of the existence of (U, V), nor the nonlinear vibrations in the RFDE (4.1). Results in this line can be found in [18]. However, we shall give a deeper analysis of the Lotka-Volterra model of predator-prey systems, which has been modified by MacDonald [37]. The mathematical model (4.1) gets the following actual form:

$$\dot{u}(t) = u(t)\big(\epsilon - \frac{\epsilon}{K}u(t) - \alpha v(t)\big), \tag{4.6a}$$

$$\dot{v}(t) = v(t)\big(-\gamma + \kappa \int_{-\infty}^{0} u(t+\theta)d\eta(\theta)\big). \tag{4.6b}$$

In these equations, u and v are the population densities of the prey and the predator, respectively. All the parameters are positive. K is the carrying capacity of the prey, α is the rate of predation per predator, κ is the rate of conversion of prey into predator, γ is the specific mortality of predator in absence of prey and ϵ is the specific growth rate of prey at zero density in absence of predators. η is a scalar function of bounded variation. It is non-decreasing on $(-\infty, 0]$, and

$$\int_{-\infty}^{0} d\eta = 1.$$

In this way, the model takes into account the density of the prey in past with a positive weight. We also suppose that the parameters satisfy the following inequality:

$$K\kappa > \gamma. \tag{4.7}$$

This implies that the RFDE (4.6) has a trivial solution in the positive quadrant of the plane (u, v), i.e. there exists an equilibrium of the predator-prey system where the species may exist together. This equilibrium is given by

$$(U, V) = \left(\frac{\gamma}{\kappa}, \frac{\epsilon}{\alpha}\left(1 - \frac{\gamma}{K\kappa}\right)\right)$$

as it can easily be calculated from the right-hand side of (4.6). In order to investigate the stability of this equilibrium and to analyse the non-linear vibrations around it, the new variables $x_1 = u - U$ and $x_2 = v - V$ are introduced, and the RFDE (4.6) is transformed into the new form of

$$\dot{x}_1(t) = -\frac{\epsilon\gamma}{K\kappa}x_1(t) - \frac{\alpha\gamma}{\kappa}x_2(t) - \frac{\epsilon}{K}x_1^2(t) - \alpha x_1(t)x_2(t), \qquad (4.8a)$$

$$\dot{x}_2(t) = \frac{\kappa\epsilon}{\alpha}\left(1 - \frac{\gamma}{K\kappa}\right)\int_{-\infty}^{0} x_1(t+\theta)d\eta(\theta) + \kappa x_2(t)\int_{-\infty}^{0} x_1(t+\theta)d\eta(\theta). \qquad (4.8b)$$

Although the linear part of (4.8) involves the delayed values of x_1, the stability investigation of the trivial solution is simple if the kernel is exponential, more exactly when

$$\eta(\theta) = e^{\theta/\tau}, \quad \theta \in (-\infty, 0]$$

where τ can be considered as the measure of the influence of the past. The following theorem has been proved first in [15] by Farkas.

Theorem 4.6. The trivial solution of the RFDE (4.8) with $\eta(\theta) = e^{\theta/\tau}$ is exponentially asymptotically stable for any $\tau \geq 0$ if

$$(\gamma <)K\kappa \leq \gamma\left(\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{\epsilon}{\gamma}}\right).$$

The trivial solution is also exponentially asymptotically stable if

$$K\kappa > \gamma\left(\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{\epsilon}{\gamma}}\right) \quad \text{and} \quad \tau < \tau_{cr}$$

where

$$\tau_{cr} = \frac{K\kappa}{(K\kappa)^2 - \gamma K\kappa - \gamma\epsilon}.$$

The trivial solution is unstable if $\tau > \tau_{cr}$.

Proof. Let us introduce the new variable

$$x_3(t) = \int_{-\infty}^{0} \frac{1}{\tau} e^{\theta/\tau} x_1(t+\theta) d\theta.$$

Since the initial functions are chosen from the space **B** in accordance with *Definition 1.2*, it can be shown that

$$\dot{x}_3 = \frac{1}{\tau}x_1 - \frac{1}{\tau}x_3.$$

Thus, the variational system can be transformed into the system of ordinary differential equations

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} -\frac{\epsilon\gamma}{K\kappa} & -\frac{\alpha\gamma}{\kappa} & 0 \\ 0 & 0 & \frac{\kappa\epsilon}{\alpha}(1-\frac{\gamma}{K\kappa}) \\ \frac{1}{\tau} & 0 & -\frac{1}{\tau} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

and the characteristic function becomes polynomial:

$$D(\lambda) = \lambda^3 + (\frac{1}{\tau} + \frac{\epsilon\gamma}{K\kappa})\lambda^2 + \frac{1}{\tau}\frac{\epsilon\gamma}{K\kappa}\lambda + \frac{1}{\tau}\gamma\epsilon(1 - \frac{\gamma}{K\kappa}).$$

The application of the Routh-Hurwitz criterion results in the critical value τ_{cr} of the measure of the delay if $(K\kappa)^2 - \gamma K\kappa - \gamma\epsilon > 0$. As $K\kappa$ goes to $\gamma(1/2 + \sqrt{1/4 + \epsilon/\gamma}) + 0$, τ_{cr} tends to ∞. $\triangle$

If a better approximation of the past-effect is necessary, functions like

$$\eta(\theta) = \left(1 - \frac{\theta}{\tau}\right)e^{\theta/\tau}$$

can be used in (4.8). Then the same method should be used as in *Theorem 4.6*, since the characteristic function D remains a polynomial of fourth degree. For more details, see references [4,13]. However, if η is different from these exponential functions, then the characteristic function of (4.8) is transcendental:

$$D(\lambda) = \det \begin{pmatrix} \lambda + \frac{\epsilon\gamma}{K\kappa} & \frac{\alpha\gamma}{\kappa} \\ -\frac{\kappa\epsilon}{\alpha}(1 - \frac{\gamma}{K\kappa}) \int_{-\infty}^{0} e^{\lambda\theta} d\eta(\theta) & \lambda \end{pmatrix},$$

and the stability investigation requires the application of *Theorem 2.19*. The following theorems give similar results to those of *Theorem 4.6* when the length of the delay is finite. Since the delay is the bifurcation parameter in the investigation of the non-linear system, the stability conditions are usually expressed with the help of the critical maximum value of the delay.

Theorem 4.7. Consider the function η given by

$$\eta(\theta) = \begin{cases} 0, & \theta \in (-\infty, -h] \\ \frac{1}{2}(1+\cos(\frac{\pi}{h}\theta)), & \theta \in (-h, 0] \end{cases}$$

in the RFDE (4.8), where $h > 0$. The trivial solution of (4.8) is asymptotically stable if

$$h < h_{cr}$$

where

$$h_{cr} = \frac{2}{\pi^2(K\kappa - \gamma)} \frac{\rho_1(\pi^2 - \rho_1^2)}{\sin \rho_1},$$

and ρ_1 is the only positive root of the equation

$$\omega^2 = h^2 \epsilon\gamma(1 - \frac{\gamma}{K\kappa})\frac{\pi^2}{2}\frac{1+\cos\omega}{\pi^2 - \omega^2}$$

in the interval $(0, \pi)$. The trivial solution is unstable if $h > h_{cr}$.

Proof. If the actual function η is substituted into (4.8), the calculation of

$$\int_{-\infty}^{0} e^{\lambda\theta} d\eta(\theta) = -\int_{-h}^{0} \frac{\pi}{2h} \sin(\frac{\pi}{h}\theta) e^{\lambda\theta} d\theta$$

leads to the characteristic function

$$D(\lambda) = \begin{cases} \lambda^2 + \frac{\epsilon\gamma}{K\kappa}h\lambda + \epsilon\gamma(1 - \frac{\gamma}{K\kappa})h^2\frac{\pi^2}{2}\frac{1+e^{-\lambda}}{\lambda^2+\pi^2}, & \lambda \neq \pm i\pi \\ -\pi^2 + i\pi(\pm\frac{\epsilon\gamma}{K\kappa}h \mp \frac{1}{4}\epsilon\gamma(1 - \frac{\gamma}{K\kappa})h^2), & \lambda = \pm i\pi \end{cases}.$$

Thus, its real and imaginary parts at $\lambda = i\omega$ give

$$R(\omega) = \begin{cases} -\omega^2 + \epsilon\gamma(1 - \frac{\gamma}{K\kappa})h^2\frac{\pi^2}{2}\frac{1+\cos\omega}{\pi^2-\omega^2}, & \omega \neq \pi \\ -\pi^2, & \omega = \pi \end{cases},$$

$$S(\omega) = \begin{cases} \frac{\epsilon\gamma}{K\kappa} h\left(\omega - h(K\kappa - \gamma)\frac{\pi^2}{2}\frac{\sin\omega}{\pi^2-\omega^2}\right), & \omega \neq \pi \\ \frac{\epsilon\gamma}{K\kappa} h\pi\left(1 - \frac{1}{4}h(K\kappa - \gamma)\right), & \omega = \pi \end{cases} .$$

The analysis of $(1+\cos\omega)/(\pi^2-\omega^2)$ shows that it is positive and decreasing for $\omega \in [0,\pi)$, and it is non-positive for $\omega \in [\pi,\infty)$. This means that R has $r = 1$ zero only, and this zero is $\rho_1 \in (0,\pi)$ defined in the statement of the theorem. The stability condition (2.22*b*) of *Theorem 2.19* results in

$$\sum_{k=1}^{r}(-1)^k \operatorname{sgn} S(\rho_k) = -\operatorname{sgn} S(\rho_1) = -1 .$$

Since $K\kappa > \gamma$ according to (4.7), $S(\rho_1) > 0$ is exactly equivalent to $h < h_{cr}$, and *Theorem 2.41* implies the asymptotic stability of the zero solution of the non-linear RFDE (4.8). If $h > h_{cr}$, i.e. $S(\rho_1) < 0$, then formula (2.17) in *Theorem 2.15* shows that there are

$$N = 1 - \operatorname{sgn} S(\rho_1) = 2$$

characteristic roots with positive real parts, and the trivial solution is unstable. △

For the time being, we do not state anything about the critical case $h = h_{cr}$, when the stability of the trivial solution of (4.8) is determined by the non-linear terms. It will be examined later.

The following theorem deals with discrete delays in (4.8).

Theorem 4.8. Consider the function η given by

$$\eta(\theta) = \begin{cases} 0, & \theta \in (-\infty, -\tau) \\ \frac{1}{2}, & \theta \in [-\tau, 0) \\ 1, & \theta = 0 \end{cases}$$

in the RFDE (4.8), where the delay $\tau \geq 0$. The trivial solution of (4.8) is asymptotically stable for any $\tau \in [0,\infty)$ if

$$(\gamma <) K\kappa \leq \gamma\left(\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{\epsilon}{\gamma}}\right). \tag{4.9}$$

The trivial solution is also asymptotically stable if

$$K\kappa > \gamma\Big(\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{\epsilon}{\gamma}}\Big) \quad \text{and} \quad \tau < \tau_{cr} \tag{4.10}$$

where

$$\tau_{cr} = \frac{K\kappa}{\sqrt{\epsilon\gamma((K\kappa)^2 - \gamma K\kappa - \epsilon\gamma)}} \arccos\Big(1 - 2\frac{\epsilon\gamma}{(K\kappa)^2 - \gamma K\kappa}\Big).$$

The trivial solution is unstable if $\tau > \tau_{cr}$.

Proof. After substituting η into the linear part of (4.8), we get the following RDDE with one discrete delay:

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} -\frac{\epsilon\gamma}{K\kappa} & -\frac{\alpha\gamma}{\kappa} \\ \frac{1}{2}\frac{\kappa\epsilon}{\alpha}(1 - \frac{\gamma}{K\kappa}) & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \frac{1}{2}\frac{\kappa\epsilon}{\alpha}(1 - \frac{\gamma}{K\kappa}) & 0 \end{pmatrix} \begin{pmatrix} x_1(t-\tau) \\ x_2(t-\tau) \end{pmatrix}.$$

The corresponding characteristic function has the form

$$D(\lambda) = \lambda^2 + \frac{\epsilon\gamma}{K\kappa}\tau\lambda + \frac{1}{2}\epsilon\gamma(1 - \frac{\gamma}{K\kappa})\tau^2(1 + e^{-\lambda}),$$

and the functions R and S are given by

$$R(\omega) = -\omega^2 + \frac{1}{2}\epsilon\gamma(1 - \frac{\gamma}{K\kappa})\tau^2(1 + \cos\omega),$$

$$S(\omega) = \frac{\epsilon\gamma}{K\kappa}\tau(\omega - \frac{1}{2}(K\kappa - \gamma)\tau\sin\omega).$$

There is at least one positive zero of R and the number r of its zeros is uneven. The zeros ρ_k having even subscript k are situated in some of the intervals $((2j-1)\pi, 2j\pi)$, $(j = 1, 2, \ldots)$. In these intervals, S is positive, that is

$$(-1)^k \operatorname{sgn} S(\rho_k) = 1, \quad \text{for all} \quad k = 2, 4, \ldots, r-1.$$

The stability condition (2.22b) in *Theorem 2.19* gives

$$\sum_{k=1}^{r} (-1)^k \operatorname{sgn} S(\rho_k) = -1$$

which is satisfied if and only if

$$\operatorname{sgn} S(\rho_k) = +1\,, \quad \text{for all} \quad k = 1, 3, \ldots, r\,.$$

Using the actual form of the functions S and R, this condition means that

$$\rho_k - \frac{1}{2}(K\kappa - \gamma)\tau \sin\rho_k > 0\,, \tag{4.11a}$$

$$-\rho_k^2 + \frac{1}{2}\epsilon\gamma(1 - \frac{\gamma}{K\kappa})\tau^2(1 + \cos\rho_k) = 0\,, \tag{4.11b}$$

for all $k = 1, 3, \ldots, r$. The elimination of τ from (4.11) results

$$\sqrt{\frac{2\epsilon\gamma}{K\kappa(K\kappa - \gamma)}}\sqrt{1 + \cos\rho_k} > \sin\rho_k\,, \quad \text{for all} \quad k = 1, 3, \ldots, r\,.$$

A detailed analysis of these inequalities shows that they are equivalent to the following conditions:

$$\frac{\epsilon\gamma}{K\kappa(K\kappa - \gamma)} \geq 1 \quad \text{and} \quad \rho_k \neq \pi, 3\pi, 5\pi, \ldots\,, \tag{4.12}$$

or

$$\rho_k < (r - k)\pi + \arccos(1 - \frac{2\epsilon\gamma}{K\kappa(K\kappa - \gamma)}) \tag{4.13}$$

for all $k = 1, 3, \ldots, r$. (4.12) is equivalent to (4.9) in the statement of the theorem. If the inequalities (4.13) are substituted into (4.11*b*), we obtain conditions with respect to the delay τ. The strictest condition arises when $k = r$ in (4.13). Since $\rho_r \in (0, \pi)$, this gives

$$\tau = \sqrt{\frac{2K\kappa}{\epsilon\gamma(K\kappa - \gamma)}}\frac{\rho_r}{\sqrt{1 + \cos\rho_r}} < \sqrt{\frac{2K\kappa}{\epsilon\gamma(K\kappa - \gamma)}}\frac{\arccos(1 - \frac{2\epsilon\gamma}{K\kappa(K\kappa - \gamma)})}{\sqrt{2 - \frac{2\epsilon\gamma}{K\kappa(K\kappa - \gamma)}})}\,,$$

which is equivalent to $\tau < \tau_{cr}$ in (4.10). The asymptotic stability of the zero solution of the non-linear RFDE (4.8) is implied by *Theorem 2.41*. If $\tau > \tau_{cr}$, then $S(\rho_r) < 0$. In accordance with formula (2.17) in *Theorem 2.15*, the number of the characteristic roots with positive real parts is determined by

$$N = 1 + \sum_{k=1}^{r}(-1)^k \operatorname{sgn} S(\rho_k)\,.$$

Since $S(\rho_k) > 0$, $(k = 2, 4, \ldots, r-1)$ and $S(\rho_r) < 0$, we get

$$N = 2 + \frac{r-1}{2} + \sum_{j=1}^{(r-1)/2} (-1)^{2j-1} \operatorname{sgn} S(\rho_{2j-1}) \geq 2$$

if $\tau > \tau_{cr}$, and the trivial solution is unstable. As τ is increased further, more and more complex conjugate characteristic roots cross the imaginary axis of the complex plane. $\triangle$

Although the stability conditions have more or less the same structures in *Theorems 4.6, 4.7* and *4.8*, it is not easy to compare the critical lengths of the delays in the cases of the three different weight functions in the RFDE (4.8). The following numerical results may help in this matter. If the parameters are fixed at

$$\alpha = \gamma = \kappa = \epsilon = 1 \quad \text{and} \quad K = 2 \tag{4.14}$$

then *Theorem 4.6* gives $\tau_{cr} = 2$ when η is exponential, *Theorem 4.7* gives $h_{cr} = 2.27\ldots$ when η has finite support and it is trigonometrical, and *Theorem 4.8* gives $\tau_{cr} = \pi$ when there is one discrete delay (η is a step-function) in (4.8). Some numerical results with respect to further kernels are given in [55].

All the above-mentioned results seem to prove the rule of thumb, that increasing delay destabilizes any dynamical system. However, the following example calls attention to rather peculiar phenomena. Let us consider the predator-prey model (4.8) with the fixed parameters (4.14) and with the function η given by

$$\eta(\theta) = \begin{cases} 0, & \theta \in (-\infty, -\tau) \\ 1-p, & \theta \in [-\tau, 0) \\ 1, & \theta = 0 \end{cases} , \quad 0 \leq p \leq 1 . \tag{4.15}$$

With the help of the parameter p, the ratio of the weights of the present and past can be expressed. There is a single discrete delay τ in the system. Since all the other parameters are fixed, we can determine the stability chart on the plane of the two parameters p and τ. *Fig. 4.1* shows this chart in a narrow but interesting interval of p. It has been constructed by means of the D-subdivision method, *Theorem 4.8*, and the following

Theorem 4.9. Consider the RFDE (4.8) with (4.14) and (4.15):

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -1 \\ \frac{p}{2} & 0 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \frac{1-p}{2} & 0 \end{pmatrix} \begin{pmatrix} x_1(t-\tau) \\ x_2(t-\tau) \end{pmatrix} . \qquad (4.16)$$

Its trivial solution is exponentially asymptotically stable for any $\tau \in [0, \infty)$ if

$$16p^2 - 40p + 17 < 0 .$$

Proof. The zeros of the characteristic function

$$D(\lambda) = \lambda^2 + \frac{\tau}{2}\lambda + p\frac{\tau^2}{2} + (1-p)\frac{\tau^2}{2}e^{-\lambda}$$

depend continuously on the parameters. There are pure imaginary characteristic roots $\pm i\omega$ if

$$R(\omega) = -\omega^2 + p\frac{\tau^2}{2} + (1-p)\frac{\tau^2}{2}\cos\omega = 0 ,$$

$$S(\omega) = \frac{\tau}{2}(\omega - \tau(1-p)\sin\omega) = 0 .$$

If τ is eliminated from these equations, we obtain the following equation of second degree for $\cos\omega$:

$$2(1-p)^2\cos^2\omega + (1-p)\cos\omega + p + 2(1-p)^2 = 0 .$$

It has no real solution for $\cos\omega$ if $16p^2 - 40p + 17 < 0$. Thus, the assumption of the theorem implies that there are no pure imaginary characteristic roots. It is easy to see that the equilibrium is asymptotically stable for $p = 1$, when (4.16) is an ordinary differential equation. As in *Theorem 3.31*, it can be shown that for p values sufficiently close to 1, all the characteristic roots are to the left of the line $\lambda = -M(<< 0)$ on the complex plane, except two of them which are in the $(1 >>)\rho$-neighbourhoods of the two characteristic roots in the case of $p = 1$. Taking into account the continuous dependence of the characteristic roots on p, we find that $\operatorname{Re}\lambda < 0$ for all the zeros of D. $\triangle$

This theorem means that the equilibrium is stable for any delay if $(1 \geq)p > 0.543$, that is when the weight of the present is great enough in comparison with that of the past. This result is clearly shown by the stability chart of *Fig. 4.1.* The result of *Theorem 4.8* is also presented there when $p = 1/2$, $\tau_{cr} = \pi$. However, the most interesting part of the chart is the bend $0.52 < p < 0.543$. In this interval, there is not only one critical value with respect to the delay τ. The equilibrium can be stabilized by great time lags, in spite of the fact that it loses its stability at a small delay. Moreover, as p tends to the less root of $16p^2 - 40p + 17 = 0$, the number of the stable and unstable intervals of the delay τ tends to infinity.

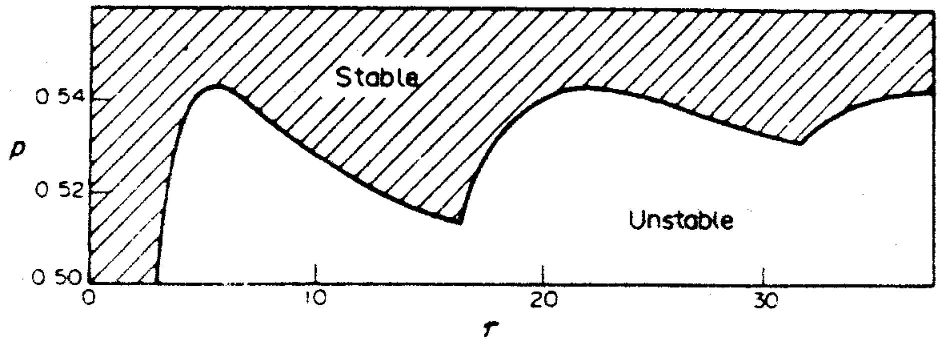

Fig. 4.1. Stability chart of RDDE (4.16)

It would, of course, be very important to know something about the existence of periodic solutions. As outlined in *Section 2.5*, the existence of Hopf bifurcation can be proved quite easily. As a matter of fact, it has been shown at the ends of the proofs of *Theorems 4.6, 4.7* and *4.8*, that two complex conjugate zeros of the characteristic function cross the imaginary axis from left to the right as the length of the delay is increased through its critical value. If it can also be proved that they cross the imaginary axis with non-zero velocity, then the Hopf Bifurcation Theorem applies. In the case of the kernel (4.15), there may exist characteristic roots which come back again to the negative half of the complex plane, as the delay is increased further on, and a sequence of Hopf bifurcations may arise. However, from application viewpoint, it is not satisfactory to know about the existence of periodic solutions only. We want to study their stability as well, that is to determine whether the Hopf bifurcation is supercritical (stable limit cycle exists around the unstable equilibrium) or subcritical (unstable limit cycle exists around the stable equilibrium). Moreover, an analytic approximation of the periodic solution may also be very useful.

The literature presents a lot of results related to the existence of Hopf bifurcations in first and second order systems (see, for instance, the examples and references in [26]). Freedman and Rao [18] have also proved the existence of Hopf bifurcation in Freedman's predator prey model. But there are much less examples for the investigation of the super- or subcriticality of the bifurcation, since this requires a long and tedious calculation. For first order scalar RFDEs, such analyses can be found in references [27,30,46,50], and for a second order RDDE, there are numerical results in [43]. We shall present this investigation for the two-dimensional non-linear system of the Lotka-Volterra-MacDonald model (4.8).

We recall, first, that Farkas et al. [15,16] have proved the existence of a supercritical Hopf bifurcation in (4.8) when $\eta(\theta) = e^{\theta/\tau}$. Szabó [58] has given the necessary and sufficient condition of supercriticality:

$$K\kappa > \gamma\left(\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{\epsilon}{\gamma}\frac{\gamma+\epsilon}{\gamma+\epsilon/2}}\right).$$

Since (4.8) can be transformed into a three-dimensional system of ordinary differential equations in this case, the long algorithm of the calculation can be carried out explicitly, keeping all the parameters of the system. In the infinite dimensional case, we shall fix all the parameters except the delay, in order to have an explicit calculation.

We shall need the following

Lemma 4.10. Let us consider the two-dimensional scalar system given in its Poincaré normal form:

$$\dot{y}_1 = \beta y_2 + f_{20}y_1^2 + f_{11}y_1y_2 + f_{02}y_2^2 + f_{30}y_1^3 + f_{21}y_1^2y_2 + f_{12}y_1y_2^2 + f_{03}y_2^3\,, \qquad (4.17a)$$

$$\dot{y}_2 = -\beta y_1 + g_{20}y_1^2 + g_{11}y_1y_2 + g_{02}y_2^2 + g_{30}y_1^3 + g_{21}y_1^2y_2 + g_{12}y_1y_2^2 + g_{03}y_2^3 , \quad (4.17b)$$

where β is positive. The zero solution is asymptotically stable (unstable) if $\Delta < 0$ ($\Delta > 0$), where

$$\Delta = \frac{1}{8}\Big(\frac{1}{\beta}\big((f_{20}+f_{02})(-f_{11}+g_{20}-g_{02}) + (g_{20}+g_{02})(f_{20}-f_{02}+g_{11})\big)$$
$$+(3f_{30}+f_{12}+g_{21}+3g_{03})\Big) .$$

Proof. See [1] or [27]. $\triangle$

Theorem 4.11. Consider the RFDE (4.8) with the kernel given by (4.15), where $p = 1/2$ and the parameters are fixed as in (4.14). After a time transformation $t = \tau\tilde{t}$, (4.8) assumes the form

$$\begin{pmatrix} \dot{x}_1(\tilde{t}) \\ \dot{x}_2(\tilde{t}) \end{pmatrix} = \begin{pmatrix} -\frac{\tau}{2} & -\tau \\ \frac{\tau}{4} & 0 \end{pmatrix} \begin{pmatrix} x_1(\tilde{t}) \\ x_2(\tilde{t}) \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \frac{\tau}{4} & 0 \end{pmatrix} \begin{pmatrix} x_1(\tilde{t}-1) \\ x_2(\tilde{t}-1) \end{pmatrix}$$
$$+ \begin{pmatrix} -\frac{\tau}{2}x_1^2(\tilde{t}) - \tau x_1(\tilde{t})x_2(\tilde{t}) \\ \frac{\tau}{2}x_1(\tilde{t})x_2(\tilde{t}) + \frac{\tau}{2}x_1(\tilde{t}-1)x_2(\tilde{t}) \end{pmatrix} . \quad (4.18)$$

The zero solution is asymptotically stable if $(0 \leq)\tau \leq \tau_{cr}$ and it is unstable if $\tau > \tau_{cr}$, where $\tau_{cr} = \pi$. For any $\tau > \tau_{cr}$, where $\tau - \tau_{cr}$ is sufficiently small, there exists an orbitally asymptotically stable periodic solution of (4.18) around the unstable zero solution, i.e. the Hopf bifurcation is supercritical at $\tau = \tau_{cr}$.

Proof. As a matter of fact, it is the algorithm published in [27] that we have taken as a basis in this proof.

Theorem 4.8 implies that the characteristic function

$$D(\lambda) = \lambda^2 + \frac{\tau}{2}\lambda + \frac{\tau^2}{4} + \frac{\tau^2}{4}e^{-\lambda}$$

of (4.17) is stable (unstable) if $\tau < \tau_{cr}$ ($\tau > \tau_{cr}$) where $\tau_{cr} = \pi$. If $\tau = \tau_{cr}$, then D has two pure imaginary zeros

$$\lambda_{1,2} = \pm i\beta , \quad \beta = \frac{\pi}{2}$$

and all the other characteristic roots have negative real parts. It is easy to see that the zero λ_1 is a smooth function of the delay τ, and it is uniquely determined by $D(\lambda_1(\tau),\tau) \equiv 0$ and $\lambda_1(\pi) = i\pi/2$. Since

$$\operatorname{Re}\frac{d\lambda_1(\pi)}{d\tau} = \frac{\pi}{\pi^2+8\pi+20} > 0\,,$$

the complex conjugate characteristic roots $\lambda_{1,2}$ cross the imaginary axis with a positive velocity as τ is increased through τ_{cr}. Thus, the Hopf bifurcation theorem implies the existence of closed orbits for some τ close to τ_{cr}.

In order to investigate the stability of these orbits, the stability of the zero solution has to be examined in the critical case $\tau = \tau_{cr}$, when the non-linear terms are relevant in the local behaviour of the solutions. Our aim is to reduce this investigation for an ordinary differential equation like (4.17), and to apply *Lemma 4.10*. This reduction requires a long and tedious calculation as follows.

Let us transform (4.18) with the critical parameter $\tau = \pi$ into the operator differential equation

$$\dot{x}_t = Ax_t + F(x_t) \tag{4.19}$$

where $x = \mathrm{col}(x_1, x_2)$, x_t is defined as in *Definition 1.1*, $x_t \in \mathbf{B}$, where the length of the delay is finite ($h = 1$) and the dimension $n = 2$. The linear operator $A : \mathbf{B} \to \mathbf{B}$ and the non-linear one $F : \mathbf{B} \to \mathbf{B}$ are defined by

$$A\phi(\theta) = \begin{cases} \frac{d}{d\theta}\phi(\theta)\,, & \theta \in [-1,0) \\ L\phi(0) + M\phi(-1)\,, & \theta = 0 \end{cases}\,,$$

where $\phi \in \mathbf{B}$ and the matrices L and M are

$$L = \begin{pmatrix} -\frac{\pi}{2} & -\pi \\ \frac{\pi}{4} & 0 \end{pmatrix}, \quad M = \begin{pmatrix} 0 & 0 \\ \frac{\pi}{4} & 0 \end{pmatrix},$$

and

$$F(\phi)(\theta) = \begin{cases} 0\,, & \theta \in [-1,0) \\ \begin{pmatrix} -\frac{\pi}{2}\phi_1^2(0) - \pi\phi_1(0)\phi_2(0) \\ \frac{\pi}{2}\phi_1(0)\phi_2(0) + \frac{\pi}{2}\phi_1(-1)\phi_2(0) \end{pmatrix}, & \theta = 0 \end{cases}\,,$$

where $\phi = \mathrm{col}(\phi_1, \phi_2)$. The operator differential equation can be transformed into its Poincaré normal form with the help of the "eigenvectors" of the operator A. The right-hand side eigenvectors $s_1, s_2 \in \mathbf{B}$ belonging to the eigenvalues $\lambda_{1,2} = \pm i\pi/2$ are calculated as the solution of the four-dimensional first order boundary value problem

$$As_1 = -\frac{\pi}{2}s_2\,,$$

$$As_2 = \frac{\pi}{2}s_1\,.$$

After the substitution of A, the calculation results

$$s_1(\theta) = \begin{pmatrix} -2 \\ 1 \end{pmatrix} \cos(\frac{\pi}{2}\theta) + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \sin(\frac{\pi}{2}\theta)\,,$$

$$s_2(\theta) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cos(\frac{\pi}{2}\theta) + \begin{pmatrix} -2 \\ 1 \end{pmatrix} \sin(\frac{\pi}{2}\theta)\,.$$

The left-hand side eigenvectors $n_1, n_2 : [0,1] \to \mathbf{R}^2$ of A, belonging to the same eigenvalues, are determined from a similar boundary value problem as above:

$$A^\star n_1 = \frac{\pi}{2}n_2\,,$$

$$A^\star n_2 = -\frac{\pi}{2}n_1\,.$$

In these equations, $A^\star$ denotes the adjoint operator given by

$$A^\star\psi(\sigma) = \begin{cases} -\frac{d}{d\sigma}\psi(\sigma)\,, & \sigma \in (0,1] \\ L^\star\psi(0) + M^\star\psi(1)\,, & \sigma = 0 \end{cases}\,,$$

where $L^\star$ and $M^\star$ are, of course, the transposed matrices, and $\psi : [0,1] \to \mathbf{R}^2$ is continuous. The solution

$$n_1(\sigma) = \frac{1}{\pi^2 + 8\pi + 20}\left(\begin{pmatrix} -8 - 2\pi \\ 8 \end{pmatrix} \cos(\frac{\pi}{2}\sigma) + \begin{pmatrix} -4 \\ -16 - 4\pi \end{pmatrix} \sin(\frac{\pi}{2}\sigma)\right)\,,$$

$$n_2(\sigma) = \frac{1}{\pi^2 + 8\pi + 20}\left(\begin{pmatrix} 4 \\ 16 + 4\pi \end{pmatrix} \cos(\frac{\pi}{2}\sigma) + \begin{pmatrix} -8 - 2\pi \\ 8 \end{pmatrix} \sin(\frac{\pi}{2}\sigma)\right)$$

has been calculated taking into account the conditions of orthonormality:

$$< n_1, s_1 >= 1, \quad < n_1, s_2 >= 0,$$

$$< n_2, s_1 >= 0, \quad < n_2, s_2 >= 1,$$

where the scalar product $< .,. >$ defined in [25] gets the special form

$$< \psi, \phi >= \psi^*(0)\phi(0) + \int_{\xi=-1}^{0} \psi^*(\xi+1)M\phi(\xi)d\xi .$$

Let us introduce the new variables

$$y_1 =< n_1, x_t > ,$$

$$y_2 =< n_2, x_t > ,$$

$$w = x_t - y_1 s_1 - y_2 s_2 ,$$

where $y_1, y_2 : [t_0, \infty) \to \mathbf{R}$ and $w : [t_0, \infty) \to \mathbf{B}$. The time dependent scalars y_1 and y_2 are the coordinates of x_t in the directions of s_1 and s_2, respectively, while the other "parts" of x_t are contained by the infinite dimensional w. Since

$$\dot{y}_1 =< n_1, \dot{x}_t >=< n_1, Ax_t + F(x_t) >$$

$$=< A^* n_1, x_t > + < n_1, F(x_t) >=< \frac{\pi}{2} n_2, x_t > + n_1^*(0)F(x_t)(0) ,$$

$$\dot{y}_2 =< -\frac{\pi}{2} n_1, x_t > + n_2^*(0)F(x_t)(0) ,$$

$$\dot{w} = Ax_t + F(x_t) - \dot{y}_1 s_1 - \dot{y}_2 s_2 ,$$

the Poincaré normal form of the operator differential equation (4.19) is as follows:

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \\ \dot{w} \end{pmatrix} = \begin{pmatrix} 0 & \frac{\pi}{2} & 0 \\ -\frac{\pi}{2} & 0 & 0 \\ 0 & 0 & A \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ w \end{pmatrix}$$

$$+ \begin{pmatrix} n_1^*(0)F(w + y_1 s_1 + y_2 s_2)(0) \\ n_2^*(0)F(w + y_1 s_1 + y_2 s_2)(0) \\ F(w + y_1 s_1 + y_2 s_2) - \sum_{j=1}^{2} n_j^*(0)F(w + y_1 s_1 + y_2 s_2)(0)s_j \end{pmatrix} ,$$

where the actual expressions of A, F, n_1, n_2, s_1 and s_2 must be substituted into the right-hand side of the equation. In the infinite dimensional state space, there is a centre manifold which is tangent to the (y_1, y_2) plane at the origin, and which is locally invariant and attractive. Thus, its equation can be approximated in the form

$$w(\theta) = \frac{1}{2}(h_{11}(\theta)y_1^2 + 2h_{12}(\theta)y_1y_2 + h_{22}(\theta)y_2^2) \tag{4.20}$$

when the third and higher order terms are neglected. The unknowns $h_{11}, h_{12}, h_{22} \in \mathbf{B}$ can be calculated from a six-dimensional first order boundary value problem which is established from the right-hand side of the third equation of the Poincaré normal form and from the time-derivative of (4.20):

$$\dot{w} = -\frac{\pi}{2}h_{12}y_1^2 + \frac{\pi}{2}(h_{11} - h_{22})y_1y_2 + \frac{\pi}{2}h_{12}y_2^2 + \text{h.o.t.}$$

The solution of this problem is as follows:

$$h_{11}(\theta) = \begin{pmatrix} 0 \\ -2 \end{pmatrix} \sin(\pi\theta),$$

$$h_{12}(\theta) = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \cos(\pi\theta) + \frac{8}{\pi^2 + 8\pi + 20}\left(\begin{pmatrix} 4 \\ -6-\pi \end{pmatrix}\cos(\frac{\pi}{2}\theta) + \begin{pmatrix} 8+2\pi \\ -2-\pi \end{pmatrix}\sin(\frac{\pi}{2}\theta)\right),$$

$$h_{22}(\theta) = \begin{pmatrix} 0 \\ 2 \end{pmatrix} \sin(\pi\theta).$$

Let us use these functions in the approximative equation (4.20) of the equation of the centre manifold. If it is substituted into the first two equations of the Poincaré normal form, and the terms of at least fourth degree are omitted, then we get the two-dimensional system of ordinary differential equations which describes the flow on the centre manifold locally. This system has the same form as (4.17) having the coefficients

$$\beta = \frac{\pi}{2}, \quad f_{20} = -f_{02} = -\frac{8\pi}{\pi^2 + 8\pi + 20}, \quad f_{11} = -\frac{4\pi(\pi + 4)}{\pi^2 + 8\pi + 20},$$

$$f_{30} = 0, \quad f_{12} = \frac{\pi}{\pi^2 + 8\pi + 20}\left(16 - \frac{64(\pi + 4)}{\pi^2 + 8\pi + 20}\right),$$

$$g_{20} = -g_{02} = -\frac{4\pi(\pi+4)}{\pi^2+8\pi+20}, \quad g_{11} = \frac{8\pi}{\pi^2+8\pi+20},$$

$$g_{21} = \frac{\pi}{\pi^2+8\pi+20}\left(-16-8\pi+\frac{64(\pi+4)}{\pi^2+8\pi+20}\right), \quad g_{03} = 0.$$

The coefficients f_{21}, f_{03}, g_{30} and g_{12} will later be of no importance. If Δ is calculated in *Lemma 4.10* with these data, we get

$$\Delta = -\frac{\pi^2}{\pi^2+8\pi+20}.$$

This is negative, and *Lemma 4.10* implies the asymptotic stability of the zero solution of the operator differential equation. This also means that the zero solution of (4.18) is asymptotically stable in the critical case $\tau = \tau_{cr}$. Thus, the Hopf bifurcation is supercritical and the proof of *Theorem 4.11* is completed. Δ

Remark 4.12. According to [1], the approximate frequency β and the amplitude a of the orbitally asymptotically stable periodic solution of (4.18) can be given by

$$\beta \approx \frac{\pi}{2} \quad \text{and}$$

$$a \approx \sqrt{-\frac{1}{\Delta}\operatorname{Re}\frac{d\lambda_1(\tau_{cr})}{d\tau}(\tau-\tau_{cr})} = \sqrt{\frac{\tau-\pi}{\pi}}.$$

Although this result is very simple, the calculation of Δ in the formula of a is very difficult, as it has been shown in the proof of *Theorem 4.11.* Δ

The finite dimensional analysis in [16] and the infinite dimensional one in *Theorem 4.11* give the same qualitative result for the RFDE (4.8). However, if we consider the kernel (4.15) in (4.8) and investigate the non-linear system

$$\begin{pmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{pmatrix} = \begin{pmatrix}-\frac{1}{2} & -1\\ \frac{p}{2} & 0\end{pmatrix}\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix} + \begin{pmatrix}0 & 0\\ \frac{1-p}{2} & 0\end{pmatrix}\begin{pmatrix}x_1(t-\tau)\\ x_2(t-\tau)\end{pmatrix}$$
$$+\begin{pmatrix}-\frac{1}{2}x_1^2(t) - x_1(t)x_2(t)\\ px_1(t)x_2(t) + (1-p)x_1(t-\tau)x_2(t)\end{pmatrix}, \tag{4.21}$$

then there may be a lot of bifurcation points with respect to the delay τ, as is shown in *Fig. 4.1.* With the help of a computer program, we have repeated the long calculation presented in the proof of *Theorem 4.11*, and the bifurcations were analysed at $p = 0.535$. *Table 4.1* and *Fig. 4.2* summarize the results. There are five bifurcation points in this case, and there are super-, and subcritical Hopf bifurcations as well.

τ_{cr}	Δ	β	$a/\sqrt{\tau - \tau_{cr}}$
4.265	-0.089	0.442	0.499
8.183	6.283	0.299	0.042
18.48	-0.380	0.442	0.189
29.16	7.781	0.299	0.025
32.69	-0.478	0.442	0.138

Table 4.1

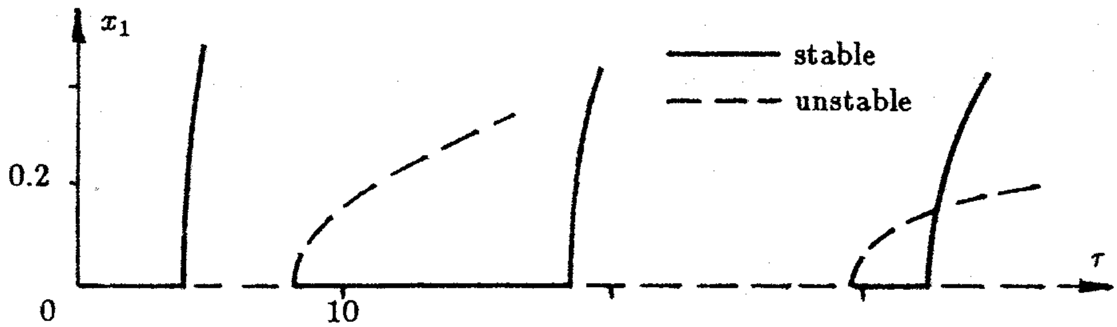

Fig. 4.2. Bifurcation diagram of RDDE (4.21), $p = 0.535$

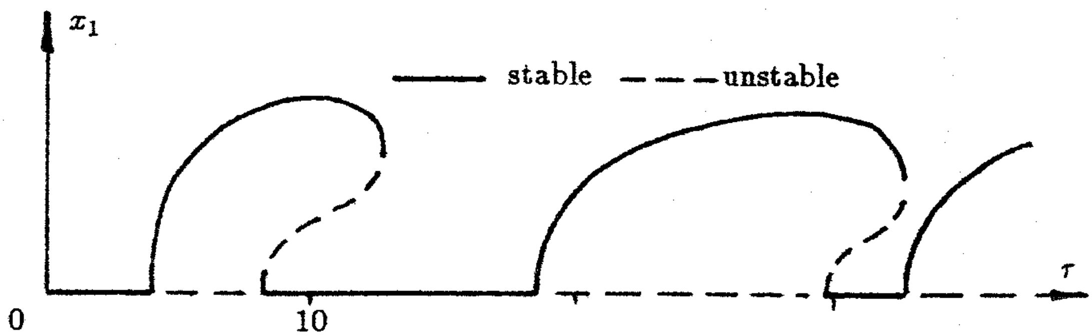

Fig. 4.3. Hypothetic bifurcation diagram of RDDE (4.21), $p = 0.535$

This investigation is, of course, valid locally at the critical delays, and it is very difficult to say anything about the possible connections of the branches shown in *Fig. 4.2*. The trajectories do not cross each other in the infinite dimensional space **B**, but they do so in the plane (x_1, x_2). There are several points along the curve of the stability limit in *Fig. 4.1*, where the Hopf bifurcation is degenerate with codimension 1, that is where $\Delta = 0$ or $\mathrm{Re}\,\lambda_1'(\tau_{cr}) = 0$. In [54], we have given some results for the RFDE (4.8) in this line, using the classification of Golubitsky and Langford [21]. However, it is still only a hypothesis that the global picture of the bifurcation diagram is the one in *Fig. 4.3*.

In the following sections, the analysis of the non-linear RFDE models will not be as deep as in the case of the predator-prey model (4.8) of Lotka-Volterra-MacDonald. However, the same technique can always be applied.

4.2. Biomechanics, man-machine systems

Two examples will be examined in this section. The first one is in connection with a very simple model of a problem of balancing, the second one is related to a model of the manual control of crane handling. Since a human operator is involved in these systems, the delay of the operator's reflexes will have an important role in the mathematical models.

A lot of experimental and theoretical research studies have investigated the different possible ways of the stabilization of inverted pendulums. This problem is interesting either in biology, to explain the self-balancing of the human body, or in robotics, to construct biped robots. From the long series of publications, we mention the references [29,31] only, where the Nyquist plots have been used for stability investigations. We shall give an analytical investigation of the balancing of a stick. The mechanical model of *Fig.*

4.4 is considered, which is the simplest possible model describing the "man-machine" system when somebody places the end of the stick on his fingertip, and tries to move the lowest point of the stick in a way that its upper position should be stable.

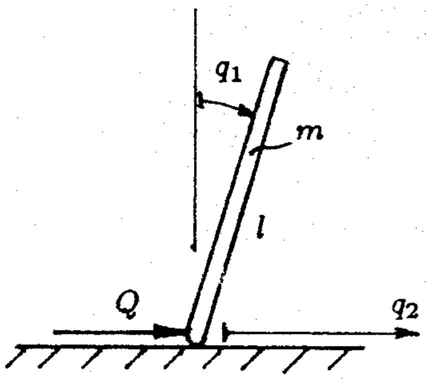

Fig. 4.4. A mechanical model of stick balancing

The actual control force Q is considered in the form

$$Q(t) = b_1\dot{q}_1(t-\tau) + b_0 q_1(t-\tau)\,, \tag{4.22}$$

where τ is the delay of the human reflexes. The positive constant parameters b_1 and b_0 have to be chosen properly. After a short practice, a human being is usually able to find those values of b_1 and b_0 where the upper $q_1 = 0$ position of the stick is stable. Let us determine these values exactly.

The mathematical model of the system has the form

$$\begin{pmatrix} \frac{1}{3}ml^2 & \frac{1}{2}ml\cos q_1(t) \\ \frac{1}{2}ml\cos q_1(t) & m \end{pmatrix} \begin{pmatrix} \ddot{q}_1(t) \\ \ddot{q}_2(t) \end{pmatrix} - \begin{pmatrix} \frac{1}{2}mlg\sin q_1(t) \\ \frac{1}{2}ml\dot{q}_1^2(t)\sin q_1(t) \end{pmatrix} =$$

$$\begin{pmatrix} 0 \\ b_1\dot{q}_1(t-\tau) + b_0 q_1(t-\tau) \end{pmatrix},$$

where m is the mass of the homogeneous rod, l is its length and g stands for the gravitational acceleration. This is a RDDE, where q_2 can easily be eliminated. This yields

$$(4 - 3\cos^2 q_1(t))\ddot{q}_1(t) + 3\dot{q}_1^2(t)\sin q_1(t)\cos q_1(t) - \frac{6g}{l}\sin q_1(t)$$

$$+\frac{6}{ml}\big(b_1\dot{q}_1(t-\tau) + b_0 q_1(t-\tau)\big)\cos q_1(t) = 0\,, \tag{4.23}$$

which is a second order scalar non-linear RDDE. The $q_1 = 0$ solution satisfies, of course, this equation. We are going to investigate this equilibrium. However, there may be other trivial solutions of (4.23). As it will be shown in the following remark, these further trivial solutions of (4.23) do not represent equilibria of the original mechanical system.

Remark 4.13. The trivial solutions of (4.23) in the interval $(-\pi/2, \pi/2)$ are given by

$$b_0 q_1 = mg \tan q_1 .$$

This means that there are 3 solutions if $b_0 > mg$, and there is only the zero solution if $b_0 \leq mg$. If it exists, the non-zero trivial solution $q_1^* \neq 0$ of (4.23) does not represent an equilibrium of the stick, since in this case

$$\ddot{q}_2 = \frac{1}{m} b_0 q_1^* ,$$

that is the stick moves with a constant acceleration, without any rotation. △

If the RDDE (4.23) is linearized at the zero solution with respect to the variation x of q_1, then we obtain

$$\ddot{x}(t) - \frac{6g}{l} x(t) + \frac{6}{ml} b_1 \dot{x}(t-\tau) + \frac{6}{ml} b_0 x(t-\tau) = 0 . \tag{4.24}$$

Theorem 4.14. The trivial solution of the RDDE (4.24) is exponentially asymptotically stable if and only if

$$mg < b_0 < \left(mg + \frac{ml}{6\tau^2}\sigma^2\right) \cos\sigma$$

where σ is the smallest positive zero of the equation

$$b_1 \omega = b_0 \tau \tan \omega$$

in the interval $(0, \pi/2)$.

Proof. The characteristic function assumes the form

$$D(\lambda) = \lambda^2 - \frac{6g}{l}\tau^2 + \frac{6}{ml}\tau b_1 \lambda e^{-\lambda} + \frac{6}{ml}\tau^2 b_0 e^{-\lambda}.$$

Thus,

$$R(\omega) = -\omega^2 - \frac{6g}{l}\tau^2 + \frac{6}{ml}\tau b_1 \omega \sin\omega + \frac{6}{ml}\tau^2 b_0 \cos\omega,$$

$$S(\omega) = \frac{6}{ml}\tau(b_1\omega\cos\omega - \tau b_0 \sin\omega).$$

If $b_0 > mg$ then

$$R(0) = -\frac{6g}{l}\tau^2 + \frac{6}{ml}\tau^2 b_0 > 0.$$

Since σ is defined by $b_1\sigma = b_0\tau\tan\sigma$, $\sigma \in (0,\pi/2)$, it is obvious that $S(\sigma) = 0$, σ is the smallest positive zero of S, and

$$S(\omega) > 0, \quad \text{for} \quad \omega \in (0,\sigma).$$

The condition $b_0 < (mg + ml\sigma^2/(6\tau^2))\cos\sigma$ is equivalent to

$$R(\sigma) < 0.$$

This and $R(0) > 0$ imply that R has zeros ρ_k of uneven number in $(0,\sigma)$. R may have further zeros of even number in (σ,∞), but they are all in intervals where S is negative. Thus,

$$\sum_{k=1}^{r}(-1)^k \operatorname{sgn} S(\rho_k) =$$

$$\sum_{\rho_k\in(0,\sigma)}(-1)^k \operatorname{sgn} S(\rho_k) + \sum_{\rho_k\in(\sigma,\infty)}(-1)^k \operatorname{sgn} S(\rho_k) = -1 + 0 = -1$$

which means that the stability condition (2.22b) in *Theorem 2.19* is fulfilled. On the other hand, it is easy to see that there is at least 1 real positive characteristic root if $b_0 < mg$ (see *Theorem 2.28*), and there are at least two characteristic roots with positive real parts if $R(\sigma) > 0$ (see *Theorem 2.15*). Thus, there is no more stability region, just the one mentioned in the statement. $\triangle$

Fig 4.5 shows the stability chart determined in *Theorem 4.14*. According to *Theorem 2.41*, the zero solution of the non-linear RDDE (4.23) is also asymptotically stable if the point (b_0, b_1) lies in the region of stability. It is obvious that the stability region does not exist always. If the delay τ is too great, or the length l of the stick is too small, the rod cannot be equilibrated.

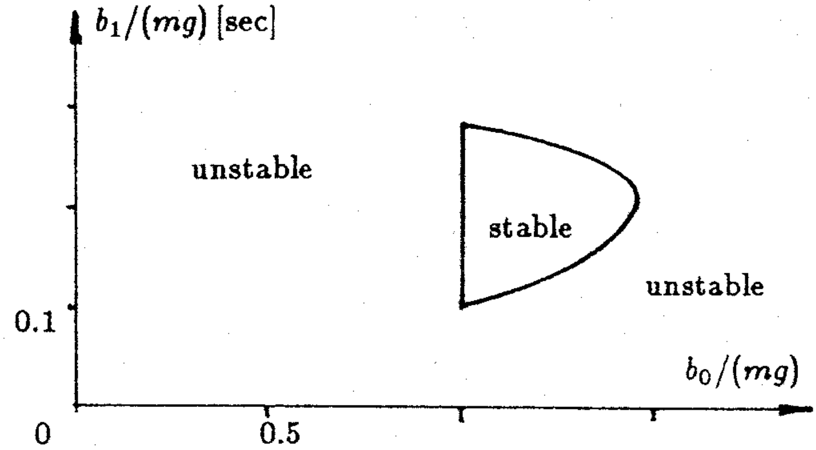

Fig. 4.5. Stability chart for balancing model (4.24), $l = 1\text{m}$, $\tau = 0.1\text{sec}$

Theorem 4.15. The trivial solution of (4.24) is unstable if

$$\tau > \tau_{cr}\,, \quad \tau_{cr} = \sqrt{\frac{l}{3g}}\,. \tag{4.25}$$

If $\tau < \tau_{cr}$ then there always exist positive b_0 and b_1 such that the trivial solution is exponentially asymptotically stable.

Proof. According to *Theorem 4.14*, the upper stability limit with respect to b_0 is expressed by

$$b_{0,max}(\sigma) = \left(mg + \frac{ml}{6\tau^2}\sigma^2\right)\cos\sigma\,, \quad \sigma \in (0, \frac{\pi}{2})\,.$$

If it is less than the lower limit $b_{0,min} = mg$ for all $\sigma \in (0, \pi/2)$, then there is no stability region at all. Since

$$\left.\frac{db_{0,max}}{d\sigma}\right|_{\sigma=0} = 0\,,$$

$b_{0,max} < b_{0,min}$ holds for all $\sigma \in (0, \pi/2)$ if

$$\left.\frac{d^2 b_{0,max}}{d\sigma^2}\right|_{\sigma=0} < 0 .$$

The calculation of the second derivative of $b_{0,max}$ at $\sigma = 0$ results

$$-mg + \frac{ml}{3\tau^2} < 0$$

which gives the critical value τ_{cr} of the delay in the statement. $\triangle$

Theorems 4.14 and *4.15* have interesting physical meaning. In spite of the fact that the model is simplified, and the formula (4.22) of the control force describes only the basic components of the human operator's behaviour, the results are quite reliable even quantitatively. The delay of our reflexes is in the range of 0.1 second. Formula (4.25) means that after a short practice, everybody is able to balance a stick of length l =0.3 metres, when the critical delay is $\tau_{cr} \approx 0.1$ second. Anybody can experience that the longer the stick is, the easier it is to equilibrate it, since τ_{cr} becomes greater. It is impossible to balance short sticks like pencils, etc. Finally, if one is a bit tipsy, the long stick cannot be equilibrated either, because the delay of the reflexes becomes too great. This may cause problems even in the self-balancing of the human body.

The self-balancing of human beings is, of course, a very complicated phenomenon. The body is controlled by us to stabilize it in a position which is physically unstable with a lot of degrees of freedom. However, even a simple inverted pendulum cannot be balanced by means of a single position signal or a single velocity signal. As *Fig. 4.5* shows, there is no stability if either $b_1 = 0$ or $b_0 = 0$ in (4.22). The human brain also has to use both signals, and the ear does provide them. Roughly speaking, the semicircular canals sense the angular velocity, while the attitude is sensed by means of the otolith organs (see [29,52] and the references in them).

Let us, now, investigate the non-linear model of stick balancing when the control force has the simplest non-linear characteristics

$$Q(t) = b_1\dot{q}_1(t-\tau) + b_0 q_1(t-\tau) + b_3 q_1^3(t-\tau)\,. \tag{4.26}$$

In order to make the investigation simpler, let us assume that the delay τ is so small that the terms τ^2 can be neglected in the power series of (4.26), that is

$$Q(t) = -b_1\tau\ddot{q}_1(t) + (b_1 - b_0\tau)\dot{q}_1(t) + b_0 q_1(t) + b_3 q_1^3(t) - 3b_3\tau q_1^2(t)\dot{q}_1(t)\,.$$

The substitution of this control force into the original model (4.23) results in the second order scalar ordinary differential equation

$$(4 - 3\cos^2 q_1 - \frac{6}{ml}\tau b_1 \cos q_1)\ddot{q}_1 + \frac{6}{ml}(b_1 - \tau b_0 - 3\tau b_3 q_1^2)\dot{q}_1 \cos q_1$$

$$+3\dot{q}_1^2 \sin q_1 \cos q_1 - \frac{6g}{l}\sin q_1 + \frac{6}{ml} b_0 q_1 \cos q_1 + \frac{6}{ml} b_3 q_1^3 \cos q_1 = 0\,. \tag{4.27}$$

Theorem 4.16. The zero solution of (4.27) is exponentially asymptotically stable if

$$b_0 > mg\,, \quad b_1 > \tau b_0 \quad \text{and} \quad \tau < \frac{ml}{6b_1}\,.$$

There is a Hopf bifurcation at $b_1 = \tau b_0$, $b_0 > mg$, $6b_1\tau < ml$, which is supercritical (subcritical) if $b_3 < 0$ ($b_3 > 0$).

Proof. The variational system of (4.27) at $q_1 = 0$ assumes the form

$$(1 - \frac{6}{ml}\tau b_1)\ddot{x} + \frac{6}{ml}(b_1 - \tau b_0)\dot{x} + \frac{6}{ml}(b_0 - mg)x = 0\,.$$

The application of the Routh-Hurwitz criterion gives the stability conditions of the zero solution. If $b_1 = \tau b_0$, then there are two imaginary characteristic roots

$$\lambda_{1,2} = \pm i\beta\,, \quad \beta = \sqrt{\frac{6(b_0 - mg)}{ml - 6\tau^2 b_0}}\,.$$

Let us omit the fourth and higher degrees of q_1 in (4.27) and introduce the new variables

$$y_1 = q_1\,, \quad y_2 = \frac{\dot{q}_1}{\beta}\,.$$

Then we get the following system at $b_1 = \tau b_0$:

$$\begin{pmatrix} \dot{y}_1 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} 0 & \beta \\ -\beta & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ g_{30}y_1^3 + g_{21}y_1^2 y_2 + g_{12}y_1 y_2^2 \end{pmatrix} \tag{4.28}$$

where

$$g_{21} = \frac{1}{\beta} \frac{18\tau}{ml - 6\tau^2 b_0} b_3 .$$

Thus, in *Lemma 4.10*, we have

$$\Delta = \frac{1}{8} g_{21} .$$

For the parameter domain $b_0 > mg$ and $6b_1\tau < ml$, the sign of Δ is uniquely determined by the sign of b_3. *Lemma 4.10* implies that the Hopf bifurcation is supercritical (subcritical) if b_3 is negative (positive). If $b_3 = 0$ then the Hopf bifurcation is degenerate. $\triangle$

The result of *Theorem 4.16* means that negative b_3 has to be applied in the control (4.26) of the inverted pendulum if we want to have an orbitally asymptotically stable periodic motion around the unstable equilibrium at $b_0 > mg$ and $b_1 < b_0\tau$. Note that (4.28) is not the differential equation of the flow on the centre manifold of the original infinite dimensional system (4.23) with (4.26) at the critical parameters, it may be a good approximation for small delays, though. Strictly speaking, the result of *Theorem 4.16* does not provide any direct conclusion with respect to the non-linear vibrations in the RDDE (4.23) with the non-linear control (4.26). A correct analysis requires an infinite dimensional calculation for (4.23) just as it was in the proof of *Theorem 4.11* for (4.18).

The second example of this section is related to another man-machine system which is described by the block diagram of *Fig. 4.6* [61].

In this system, a human operator manipulates an object by means of a hydraulic servomechanism. We assume that the "machine" part of the system can be described by the simplified transfer function Y given by

$$Y(\lambda) = \frac{1}{J\lambda^2(T^2\lambda^2 + 2\varsigma T\lambda + 1)} ,$$

where T and ς characterise the main properties of the servomechanism, and J is the inertia of the controlled object. As regards the transfer function H of the human being, there are a lot of sophisticated models for it in the literature. A good survey is given in [38], for instance. In order to be able to make analytical stability investigation, we consider the basic component of this transfer function:

$$H(\lambda) = Ke^{-\tau\lambda}(L\lambda + 1)$$

where $K > 0$ is the operator gain, τ is the delay of the operator's reflexes and $L > 0$ is the so-called lead time constant. This transfer function describes exactly the same human response as (4.22) in the stick balancing problem.

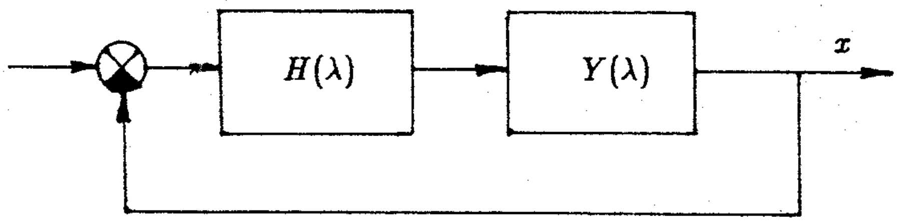

Fig. 4.6. Block diagram of a man-machine system

By applying the inverse Laplace transformation for the overall characteristic function

$$Y_o = \frac{H(\lambda)Y(\lambda)}{1 + H(\lambda)Y(\lambda)},$$

we can reconstruct the differential-difference equation for the scalar state variable x. After the usual time transformation with respect to the delay τ, it assumes the form

$$\frac{d^4}{dt^4}x(t) + 2\frac{\varsigma\tau}{T}\frac{d^3}{dt^3}x(t) + \frac{\tau^2}{T^2}\frac{d^2}{dt^2}x(t) + \frac{K}{J}\frac{L\tau^3}{T^2}\frac{d}{dt}x(t-1) + \frac{K}{J}\frac{\tau^4}{T^2}x(t-1) = 0\,. \quad (4.29)$$

Let us investigate the stability of this linear system in the undamped case, that is when $\varsigma = 0$.

Theorem 4.17. Consider the RDDE (4.29) when $\varsigma = 0$. Its trivial solution is exponentially asymptotically stable if and only if

$$L > \tau \quad \text{and} \quad R(\sigma_{(0)}) < 0$$

and there exists an integer $j \geq 0$ such that

$$R(\sigma_{(2j)}) < 0 \quad \text{and} \quad R(\sigma_{(2j+1)}) > 0\,,$$

where R is given by

$$R(\omega) = \omega^4 - \frac{\tau^2}{T^2}\omega^2 + \frac{K}{J}\frac{L\tau^3}{T^2}\omega \sin\omega + \frac{K}{J}\frac{\tau^4}{T^2}\cos\omega\,,$$

and $\sigma_{(k)}$, $(k = 0, 1, \ldots)$ is the only real root of the equation

$$L\omega = \tau \tan\omega$$

in the interval $(k\pi, k\pi + \pi/2)$.

Proof. Since the characteristic function has the actual form

$$D(\lambda) = \lambda^4 + \frac{\tau^2}{T^2}\lambda^2 + \frac{K}{J}\frac{L\tau^3}{T^2}\lambda e^{-\lambda} + \frac{K}{J}\frac{\tau^4}{T^2}e^{-\lambda}\,,$$

it is easy to see that

$$R(\omega) = \operatorname{Re} D(i\omega)\,,$$
$$S(\omega) = \operatorname{Im} D(i\omega) = \frac{K}{J}\frac{\tau^3}{T^2}(L\omega\cos\omega - \tau\sin\omega)\,.$$

According to the definition of $\sigma_{(k)}$, $(k = 0, 1, \ldots)$, the function S is positive in the intervals $(0, \sigma_{(0)})$ and $(\sigma_{(2j+1)}, \sigma_{(2j+2)})$, and it is negative in $(\sigma_{(2j)}, \sigma_{(2j+1)})$, $(j = 0, 1, \ldots)$. The inequalities in the statement of the theorem mean that the function R changes its sign in $(0, \sigma_{(0)})$ and in one of the intervals $(\sigma_{(2j)}, \sigma_{(2j+1)})$. The positive zeros of R are denoted by $\rho_1 \geq \ldots \geq \rho_r$, where r is even since

$$R(0) > 0 \quad \text{and} \quad \lim_{\omega\to\infty} R(\omega) = +\infty\,.$$

Thus,

$$\sum_{\rho_k \in (0,\sigma_{(0)})} (-1)^k \operatorname{sgn} S(\rho_k) + \sum_{\rho_k \in (\sigma_{(2j)},\sigma_{(2j+1)})} (-1)^k \operatorname{sgn} S(\rho_k) = 1 + 1 = 2\,.$$

Since R has no positive zero outside these intervals, the stability condition (2.22*b*) with $m = 2$

$$\sum_{k=1}^{r} (-1)^k \operatorname{sgn} S(\rho_k) = 2$$

is equivalent to the system of inequalities in the statement, and *Theorem 2.19* implies the stability of the trivial solution. △

The actual expressions of the D-curves

$$R(\omega) = 0\,, \quad S(\omega) = 0\,, \quad \omega \in [0, \infty)$$

can easily be rearranged with respect to L and τ, and *Theorem 4.17* gives the regions of stability bordered by these curves in the parameter plane of the human parameters L and τ in the man-machine system. *Fig. 4.7* shows this stability chart when T, K/J are fixed and $\varsigma = 0$. The shaded domains are the stable ones.

Some Nyquist diagrams of the same system have been presented in [61]. When $\varsigma > 0$, the structure of the stability chart remains the same as that of *Fig. 4.7*, it is much more difficult to plot the D-curves, though. If ς is great enough, the stability domains become joint.

As the stability chart shows, the system cannot be stabilized without a lead compensator (i.e. when $L = 0$). If the operator has a long practice, then he is able to choose the best lead time constant L, and his reflexes may be slower, he does not have to concentrate intensely. However, if the operator's reactions are too fast (τ is about 0.1 second) and L is too great, then the system is unstable. This case can be thought as human fluster. As the disjoint domains of stability clearly prove, it may be better to have a greater delay in the system to achieve stability.

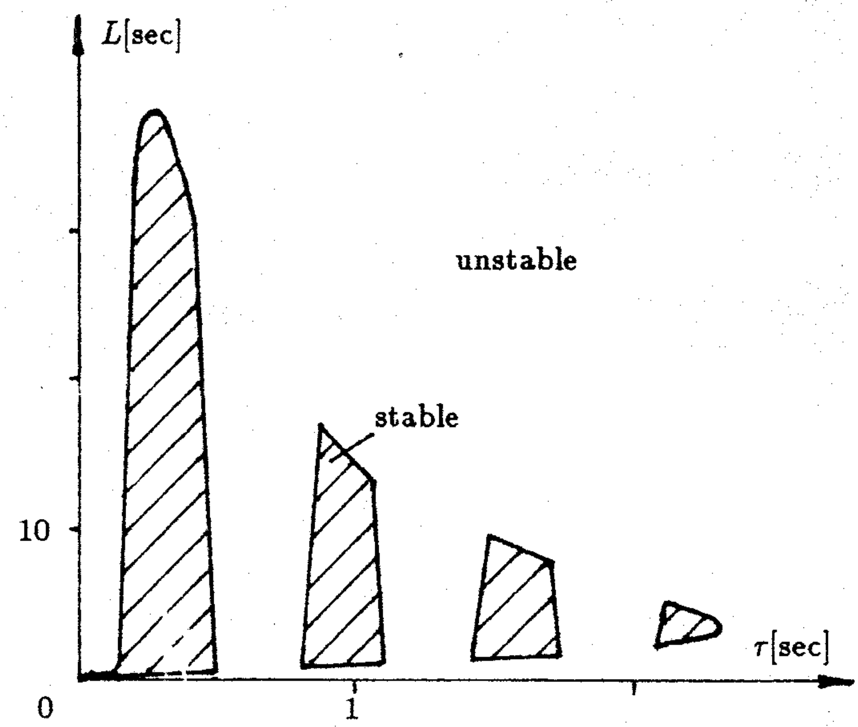

Fig. 4.7. Stability chart of man-machine model (4.29),

$T = 0.1\text{sec}$, $K/J = 0.1\text{sec}$, $\varsigma = 0$

Although the models of man-machine systems in this section were definitely simplified, the analytical investigation of these models and that of their results, the stability charts, have valuable information about the roles of the actual parameters in the system.

4.3. Robotics

There are a lot of stability problems in robotics which cannot be explained if the delays in the robot systems are ignored. Delays may occur in the control system of the robot, in the information transmission as well as in the mechanical part of the robot.

The master-slave systems present a simple example when the delay arises in the control. As has been discussed in the previous section in connection with man-machine systems, the delay of the human operator's reflex introduces a time lag into the control of master-slave systems. This is more than 0.1 second. The situation is very similar in the case of a manipulator which is on-line controlled by a computer. The sample time of the digital control serves as a basis for the value of the delay, which is usually about 0.01-0.001 second. However, the delayed feed-back is not continuous in digital control, the model of these systems contains a so-called zero order holder as well. This means that a pure discrete mathematical model [57] describes these computer controlled systems better than a RFDE, the latter may be a good approximation, though.

In another important group of problems, the delay occurs in the information transmission. This past-effect becomes crucial in undersea and space teleoperations. The dead time is equal to the time needed by the ultrasonic or electromagnetic wave to cover the distance to and fro between the master and the slave. This delay may exceed 0.1-1.0 second.

If a robot works in a material forming process like milling, rolling, etc., a delayed feed-back can be found in the pure mechanical part of the robot as well. It is inversely proportional to the relative velocity of the tool and the workpiece. This phenomenon will be discussed in detail in the following section.

In this section, two simple mechanical models of robotics will be investigated.

The first model shown in *Fig. 4.8* describes a one degree of freedom, position controlled elastic robot [56]. The position q_2 of the end-effector is detected directly. The control of the robot is assumed to be a kinematical constraint. This means that the actuator, acting on the first body in this model, provides a velocity $\dot{q}_1$ which is determined by the position of the end-effector:

$$\dot{q}_1(t) = -Kq_2(t-\tau)\,,$$

where $K > 0$ is the gain and τ stands for the delay in the control.

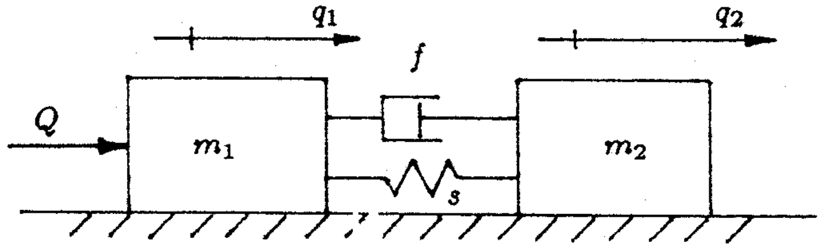

Fig. 4.8. Mechanical model of an elastic robot

By means of the new variable $v = \dot{q}_2$, this mechanical system is covered by the following linear RDDE:

$$\begin{pmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \\ \dot{v}(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ \alpha^2 & -\alpha^2 & -2\kappa\alpha \end{pmatrix} \begin{pmatrix} q_1(t) \\ q_2(t) \\ v(t) \end{pmatrix} + \begin{pmatrix} 0 & -K & 0 \\ 0 & 0 & 0 \\ 0 & -2K\kappa\alpha & 0 \end{pmatrix} \begin{pmatrix} q_1(t-\tau) \\ q_2(t-\tau) \\ v(t-\tau) \end{pmatrix}, \tag{4.30}$$

where $\alpha = \sqrt{s/m_2}$ is the natural frequency of the undamped, uncontrolled system, and $\kappa = f/(2m_2\alpha)$ is the relative damping factor. As the following corollary shows, it is easy to determine those parameters which provide stability for the undamped system.

Corollary 4.18. Consider the RDDE (4.30) with $\kappa = 0$. Its trivial solution is exponentially asymptotically stable if and only if there exists an integer $k \geq 0$ such that

$$\frac{K}{\alpha} < \frac{\pi}{2\alpha\tau} - \left(\frac{\pi}{2\alpha\tau}\right)^3,$$

$$\frac{K}{\alpha} < \frac{1}{\alpha\tau}\left(2k\pi + \frac{\pi}{2}\right) - \frac{1}{(\alpha\tau)^3}\left(2k\pi + \frac{\pi}{2}\right)^3$$

and

$$\frac{K}{\alpha} < -\frac{1}{\alpha\tau}\left(2k\pi + \frac{3\pi}{2}\right) + \frac{1}{(\alpha\tau)^3}\left(2k\pi + \frac{3\pi}{2}\right)^3 .$$

Proof. Since the characteristic function of (4.30) assumes the form

$$D(\lambda) = \lambda^3 + (\alpha\tau)^2\lambda + (\alpha\tau)^3\frac{K}{\alpha}e^{-\lambda} ,$$

the result of *Corollary 3.5* can be applied here directly. If

$$a_1 = (\alpha\tau)^2 \quad \text{and} \quad b = (\alpha\tau)^3\frac{K}{\alpha}$$

are substituted into (3.10) in *Corollary 3.5*, then the stability conditions follow immediately, since all the parameters are positive. $\triangle$

The stability chart for the damped system (4.30) is presented in *Fig. 4.9* in the plane of the dimensionless parameters $(\tau\alpha)$ and (K/α). It has been constructed for a small value of the relative damping $(\kappa = 0.01)$ with the help of *Corollary 4.18* and the D-subdivision method.

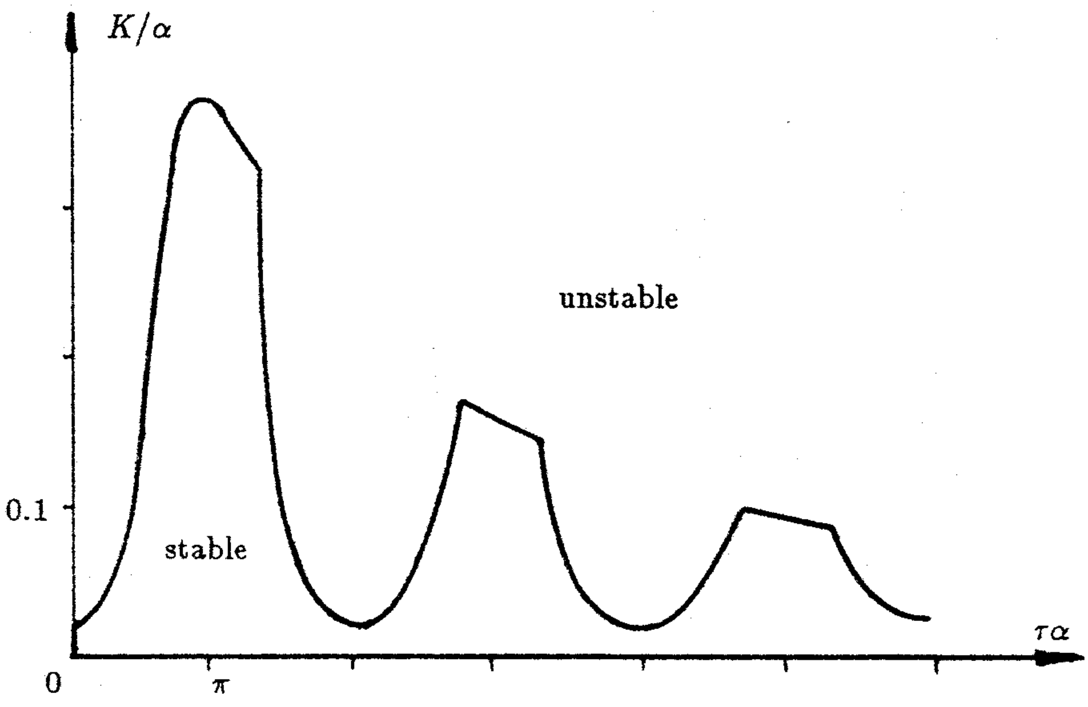

Fig. 4.9. Stability chart for robot model (4.30), $\kappa = 0.01$

In robotics, the application of high gains K is preferred. As the stability chart shows, it is very important to choose a proper ratio of the time delay τ and the oscillation time $2\pi/\alpha$ of the basic vibration mode. It is not true generally for these oscillatory systems that the shortest possible delay is the best.

In the second example of this section, the mathematical model of an experimental master-slave system [63] is examined. This system has been based on a force-reflective manipulator. The linearized equations of motion have the form

$$\begin{pmatrix} m_1 & 0 \\ 0 & m_2 \end{pmatrix} \begin{pmatrix} \ddot{q}_1(t) \\ \ddot{q}_2(t) \end{pmatrix} + \begin{pmatrix} f_1 & 0 \\ 0 & f_2 \end{pmatrix} \begin{pmatrix} \dot{q}_1(t) \\ \dot{q}_2(t) \end{pmatrix} + \begin{pmatrix} K & 0 \\ 0 & K \end{pmatrix} \begin{pmatrix} q_1(t) \\ q_2(t) \end{pmatrix}$$

$$- \begin{pmatrix} 0 & K \\ K & 0 \end{pmatrix} \begin{pmatrix} q_1(t-\tau) \\ q_2(t-\tau) \end{pmatrix} = \begin{pmatrix} Q_1(t) \\ Q_2(t) \end{pmatrix},$$

with subscript 1 for the master and subscript 2 for the slave. Since the actual master-slave system was designed for space applications [63], the delay τ in the information transmission cannot be ignored. $K > 0$ is the gain related to the force reflection.

Let us suppose that an operator with delay ξ of reflexes tries to find the $q_1 = q_2 = 0$ position when there is no load on the slave, i.e.

$$Q_2 \equiv 0 .$$

Since the operator gets the information about the position of the slave with the transmission delay τ (e.g. by means of a camera), the operator's response is delayed by $\tau + \xi$:

$$Q_1(t) = -Lq_2(t - \tau - \xi) ,$$

where $L > 0$ is the operator's gain. Let us assume that the inertia m_1 and the viscous friction f_1 are negligible in master. Then q_1 can be eliminated from the motion equations:

$$m_2\ddot{q}_2(t) + f_2\dot{q}_2(t) + Kq_2(t) - Kq_2(t - 2\tau) + Lq_2(t - 2\tau - \xi) = 0 . \tag{4.31}$$

This RDDE contains two discrete delays: 2τ and $2\tau + \xi$. A high viscous damping f_2 in slave can eliminate the destabilizing effects of the delays. The following statement estimates this value.

Corollary 4.19. The trivial solution of the RDDE (4.31) is exponentially asymptotically stable if

$$f_2 > 2(K + L)\tau + L\xi .$$

Proof. The RDDE (4.31) is a special case of the RDDE (3.29) where $p = 2$ and

$$a_0 = \frac{K}{m_2}, \quad a_1 = \frac{f_2}{m_2}, \quad b_1 = \frac{K}{m_2}, \quad b_2 = -\frac{L}{m_2},$$

$$\tau_1 = 2\tau \quad \text{and} \quad \tau_2 = 2\tau + \xi .$$

Since all the parameters of (4.31) are positive, $a_0 > b_1 + b_2$ and $a_1 > |b_1|\tau_1 + |b_2|\tau_2$ is true and *Theorem 3.23* implies the statement. △

The stability region determined in *Corollary 4.19* is shown in *Fig. 4.10* in the plane of the information delay τ and the operator's reflex delay ξ.

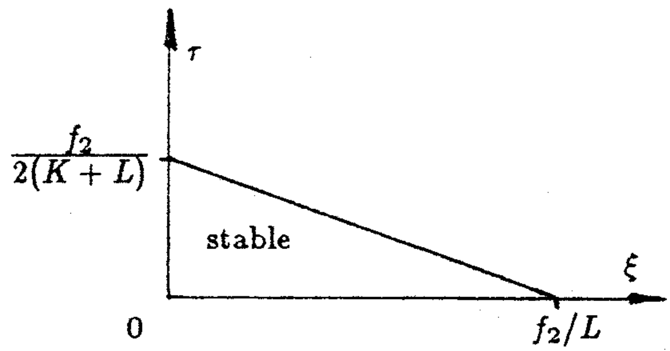

Fig. 4.10. A stability region of master-slave model (4.31)

It is much more difficult to determine an exact stability chart in the plane of the delay parameters when the viscous damping is small in slave, that is when $f_2 \approx 0$. In this case, the RDDE (4.31) has a similar form to that of equation (3.24). Its stability analysis can be carried out in a similar way as in *Theorem 3.13* and in *Corollary 3.14*. *Fig.*

3.13 shows that the stability chart of second order systems with two discrete delays may be very complicated. Really, the stability chart of (4.31) also shows disjoint stability domains either for small or great values of the operator's gain L. As can be seen in *Fig. 4.11*, the operator's delay ξ can be longer and the information delay τ should be shorter if L is great.

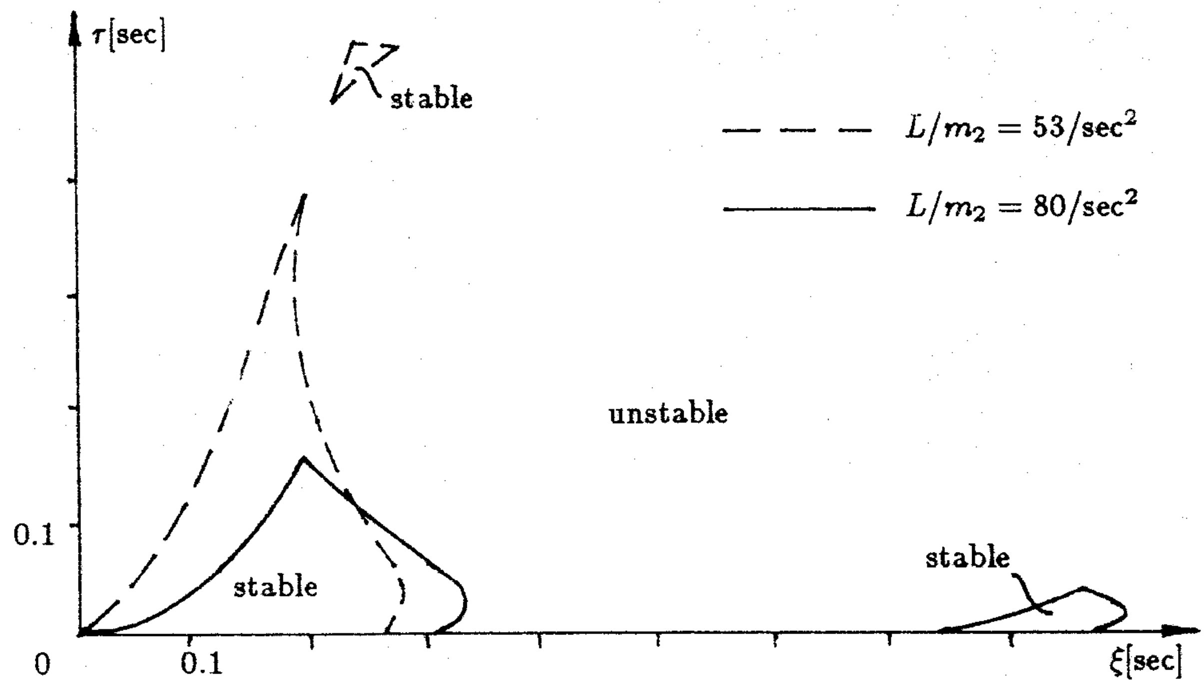

Fig. 4.11. Stability chart of master-slave model (4.31), $f_2 = 0$, $K/m_2 = 100/\text{sec}^2$

In this section, the examples from the field of robotics call the attention that a deep analysis of the delay-effects may be necessary if the stability properties of a robot are not satisfactory.

4.4. Machine tool vibrations

The accuracy of machine tools is highly affected by occurring vibrations. There may be a lot of reasons for vibrations in machine tools. Most of them can be handled with the conventional methods of machine dynamics. However, the self-excited vibrations arising in the cutting process may cause serious difficulties. It is not easy to predict when and why these vibrations appear, and to give a strategy to avoid them only by changing the technological parameters, namely the feed, the depth of cut and the cutting speed.

The most important reason for instability in the cutting process is the so-called regenerative effect. As a matter of fact, it is a past-effect which has a clear physical explanation. Because of some external disturbances, the tool starts a damped vibration relative to the workpiece, and the surface of the workpiece becomes wavy. After a round of the workpiece (or tool) the chip thickness will vary at the tool because of this wavy surface. Thus, the cutting force depends on the actual and delayed values of the relative displacement of the tool and the workpiece where the length of the delay is exactly equal to the time-period τ of the revolution of the workpiece (or tool). This delay is the central idea of the regenerative effect.

In order to represent the open questions in connection with this effect only, we assume orthogonal cutting and simplify the mechanical model of the machine tool according to *Fig. 4.12.* The machine tool is assumed to have one relevant natural frequency $\alpha/(2\pi)$, where $\alpha = \sqrt{s/m}$, and x is the direction of the corresponding oscillation mode. If $\kappa = b/(2m\alpha)$ denotes the relative damping factor, the motion equation of this system has the form

$$\ddot{x}(t) + 2\kappa\alpha\dot{x}(t) + \alpha^2 x(t) = \frac{1}{m}\left(F_x(f(t)) - F_x(f_0)\right), \tag{4.32}$$

where F_x is the x component of the cutting force F. The cutting force depends, of course, on the actual chip thickness f which is changing in time. In the case of steady-state cutting conditions, the chip-thickness is constant and it should have the theoretically required value f_0.

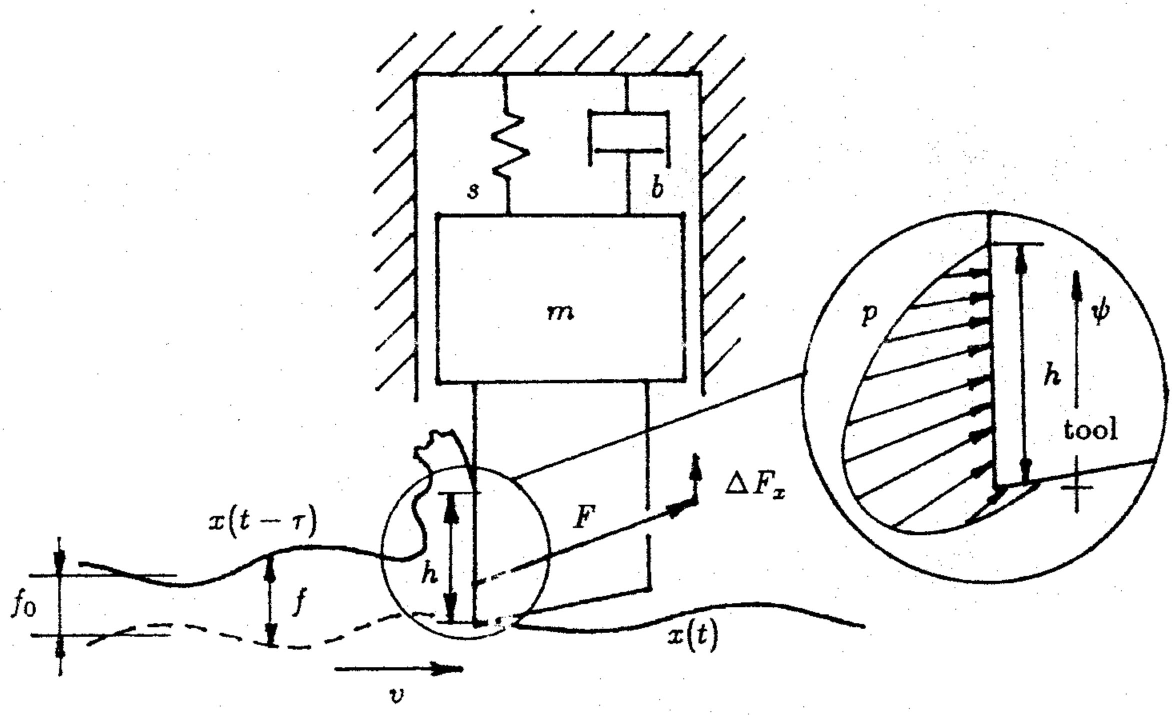

Fig. 4.12. Mechanical model of regenerative machine tool vibration

The cutting force is a strongly non-linear function of the chip thickness. This function is known approximately from experiments in the literature. Its power series at f_0 starts with the terms

$$F_x(f) = F_x(f_0) + \frac{dF_x(f_0)}{df}(f - f_0) + \frac{1}{2}\frac{d^2F_x(f_0)}{df^2}(f - f_0)^2 + \frac{1}{6}\frac{d^3F_x(f_0)}{df^3}(f - f_0)^3 \,. \quad (4.33)$$

In order to get a RDDE with respect to the variable x, we also need the mathematical connection of the actual chip thickness and the tool displacement x. This geometrical connection is clearly shown in *Fig. 4.12*:

$$f(t) = f_0 - x(t) + x(t - \tau)\,, \quad (4.34)$$

which means that the actual chip thickness depends on the present and a previous displacement of the tool. If v stands for the constant cutting speed, d_0 is the diameter of the rotating workpiece and N denotes its angular velocity, then the time delay τ can be expressed as

$$\tau = \frac{d_0 \pi}{v} \quad \text{or} \quad \tau = \frac{2\pi}{N} . \tag{4.35}$$

Substituting (4.33–35) into the motion equation (4.32), we obtain the simplest mathematical model which describes the regenerative machine tool vibrations.

The variational system with respect to the $x = 0$ solution can easily be determined if (4.33) is cut at the linear terms:

$$F_x(f) = F_x(f_0) + k_s(f - f_0) .$$

As (4.34) shows, the linear approximation of the cutting force variation is given by

$$F_x(f(t)) - F_x(f_0) = k_s\big(x(t - \tau) - x(t)\big) , \tag{4.36}$$

where the constant cutting force coefficient k_s is determined by the technological parameters according to

$$k_s = \frac{dF_x(f_0)}{df} .$$

It is, for instance, approximately linearly proportional to the width of cut, it depends inversely on the depth of cut, and it is almost independent from the cutting speed.

The substitution of formulae (4.35) and (4.36) into (4.32) results in the scalar, linear, second order RDDE

$$\ddot{x}(t) + 2\kappa\alpha\dot{x}(t) + (\alpha^2 + \frac{k_s}{m})x(t) = \frac{k_s}{m}x(t - \frac{2\pi}{N}) . \tag{4.37}$$

It has the same form as the RDDE (3.26) where

$$a_1 = 2\kappa\alpha\frac{2\pi}{N} , \quad a_0 = (\alpha^2 + \frac{k_s}{m})\left(\frac{2\pi}{N}\right)^2 , \quad b = \frac{k_s}{m}\left(\frac{2\pi}{N}\right)^2 .$$

The stability chart of (4.37) in *Fig. 4.13* has been determined in the same way as in *Example 3.16* for the equation (3.26). The parameters of the machine tool are fixed, and the stability chart is presented in the plane of the parameters k_s and N. It is easy to transform this chart into the operational space of the technological parameters, the feed, the depth of cut, and the cutting speed. This chart is of great importance in technology design.

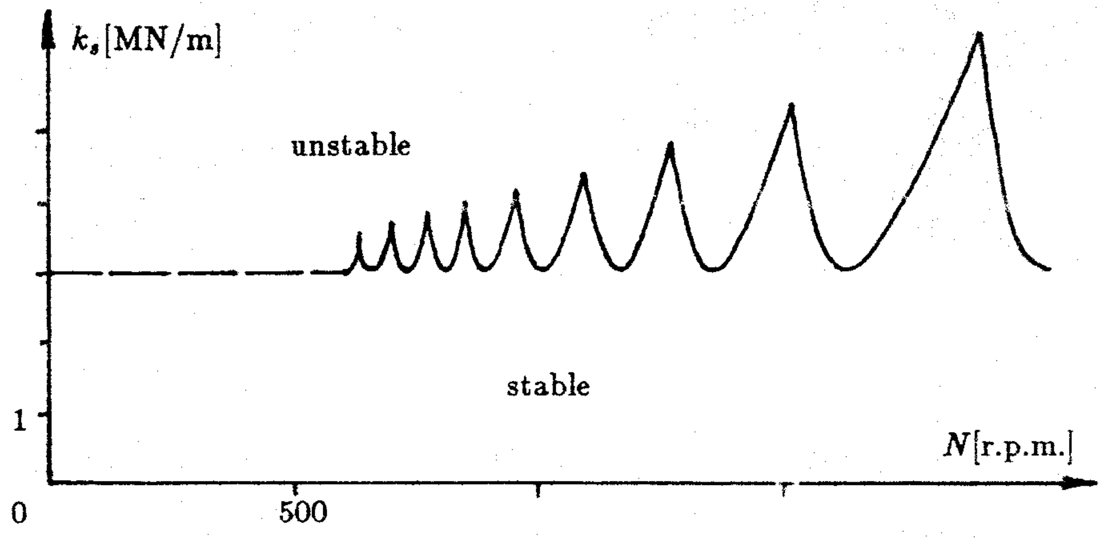

Fig. 4.13. Stability chart of machine tool vibration model (4.37),

$m = 50\,\text{kg}$, $\alpha = 775/\text{sec}$, $\kappa = 0.05$

The mathematical model (4.37) describes well the basic stability properties of regenerative machine tool vibrations. The steady-state cutting process is unstable if the cutting force coefficient k_s is too great. As *Fig. 4.13* shows, there is a domain at $k_s \approx 3-4\,\text{MN/m}$, where the stability of the system depends "periodically" on the cutting speed. This phenomenon is exactly the same as it was in the case of the predator-prey model (4.16) on the stability chart of *Fig. 4.1.* We have to recall that the angular velocity N (or the cutting speed v) is inversely proportional to the time delay τ in (4.35).

These basic stability properties given by the stability chart of *Fig. 4.13* have been verified experimentally for the middle range of cutting speeds [60]. However, the experiments show that the real cutting process has much better stability properties at low and high cutting speeds than could be expected on the basis of the mathematical model (4.37).

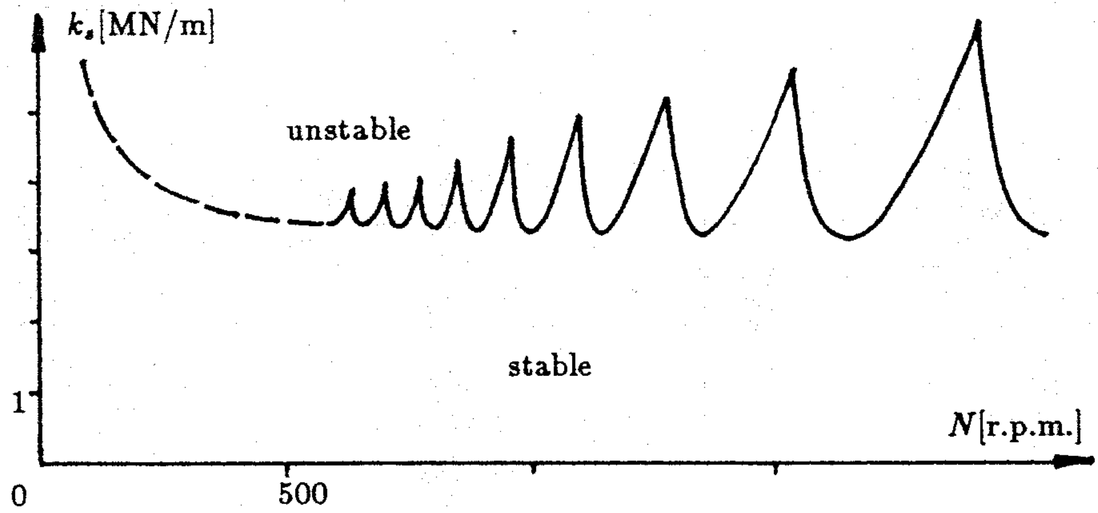

Fig. 4.14. Stability chart of machine tool vibration model (4.39),

$m = 50\,\text{kg}$, $\alpha = 775/\text{sec}$, $\kappa = 0.05$, $C = 0.001$

The so-called dynamic cutting theory of Tobias [60] has introduced an additional damping term into (4.37) at low cutting speeds. This theory considers the cutting force variation in the form

$$F_x(f(t)) - F_x(f_0) = k_d\big(x(t-\tau) - x(t)\big) - Ck_d\tau\dot{x}(t) \qquad (4.38)$$

where

$$k_d = \frac{1}{1+C}k_s$$

is the chip thickness coefficient and C is the penetration rate factor. The physical interpretation of (4.38) is quite complicated and it is different for turning, milling and drilling [60]. If (4.38) is used in the motion equation (4.32), the mathematical model has the form

$$\ddot{x}(t) + \Big(2\kappa\alpha + \frac{C}{1+C}\frac{k_s}{m}\frac{2\pi}{N}\Big)\dot{x}(t) + \Big(\alpha^2 + \frac{1}{1+C}\frac{k_s}{m}\Big)x(t) = \frac{1}{1+C}\frac{k_s}{m}x\Big(t - \frac{2\pi}{N}\Big) \qquad (4.39)$$

which is very similar to the basic model (4.37). The only relevant difference between the two models is the appearance of N^{-1} in the coefficient of $\dot{x}$ which results in a high damping if the angular velocity N (i.e. the cutting speed) is small. The stability chart of this model is presented in *Fig. 4.14*. The penetration rate factor $C = 0.001$ is a typical experimentally determined value for turning.

The comparison of the stability charts of *Figs. 4.13* and *4.14* shows that the second model is more adequate. The mathematical model (4.39) gives greater stability regions at low cutting speeds. However, it still does not explain the experimental fact that there is an improvement of stability properties at high cutting speeds as well. The model given below solves this problem, but this modification of the basic model (4.37) is very different from that of the dynamic cutting theory in (4.38).

The original model (4.37) is based on the regenerative effect which takes into account a "long" time delay τ, the time period of one revolution of the workpiece. But there exists a short regenerative effect as well. The mechanical model in *Fig. 4.12* shows that the cutting force F is the resultant of a distributed force system p on the active face of the tool. In the case of steady-state cutting conditions, its x component is assumed to have the form

$$p_x(f,\psi) = F_x(f)w_x(\psi)\,, \quad \psi \in [0,h]$$

where the function w_x describes the shape of the distributed force system in the direction x, and h is the contact length. We assume that

$$\int_0^h w_x(\psi)d\psi = 1\,.$$

Instead of (4.36) or (4.38), the improved formula of the cutting force variation is given by

$$F_x(f(u)) - F_x(f_0) = k_s \int_0^h w_x(\psi)\big(x(u - d_0\pi - \psi) - x(u-\psi)\big)d\psi \qquad (4.40)$$

where k_s is the same cutting force coefficient as it was in (4.36). Formula (4.40) gives the cutting force variation with respect to the new variable u instead of the time t, where

$$u = vt \qquad (4.41)$$

can be thought of as an arc length. This explains that the long delay is represented by the circumference $d_0\pi$ of the cylindrical workpiece instead of the time τ in accordance with formulae (4.35). If the transformation (4.41) is applied for the motion equation (4.32) as well, and (4.40) is substituted into it, we obtain the scalar RFDE

$$\frac{d^2}{du^2}x(u)+2\kappa\frac{\alpha}{v}\frac{d}{du}x(u)+\left(\frac{\alpha}{v}\right)^2x(u)+\frac{k_s}{mv^2}\int_0^h w_x(\psi)\big(x(u-\psi)-x(u-d_0\pi-\psi)\big)d\psi=0\,.$$

With the help of the dimensionless arc length

$$\tilde{u}=\frac{u}{d_0\pi}\,,$$

this RFDE can be transformed into the form

$$x''(\tilde{u})+2\kappa\alpha\frac{2\pi}{N}x'(\tilde{u})+\alpha^2\left(\frac{2\pi}{N}\right)^2x(\tilde{u})$$

$$+\frac{k_s}{m}\left(\frac{2\pi}{N}\right)^2\int_0^{1/q}W_x(\tilde{\psi})\big(x(\tilde{u}-\tilde{\psi})-x(\tilde{u}-1-\tilde{\psi})\big)d\tilde{\psi}=0\,, \tag{4.42}$$

where $'$ denotes the derivative with respect to $\tilde{u}$. The dimensionless weight function W_x is defined by w_x in the following way:

$$W_x(\tilde{\psi})=d_0\pi w_x(d_0\pi\tilde{\psi})\,,\quad \tilde{\psi}\in[0,\frac{1}{q}]\,, \tag{4.43}$$

where the new parameter

$$q=\frac{d_0\pi}{h}=\frac{\tau}{\xi} \tag{4.44}$$

gives the ratio of the lengths of the long and the short delays in the system. The parameter q will have an important role in the stability investigation. It can be expressed by means of distances: it is the ratio of the circumference $d_0\pi$ of the cylindrical workpiece and the length h of the contact line of the chip and the active face of the tool. But it can also be expressed (see (4.35)) as the ratio of the time period τ of one revolution of the workpiece and the time ξ needed by a particle of the chip to slip along the active face of the tool (i.e. $\xi=h/v$). Roughly speaking, there are two delays in the system, a discrete one related to the long delay, and a continuous one related to the short delay. In order to get a clear picture about them, let us transform the RFDE (4.42) into the standard form

$$x''(\tilde{u}) + 2\kappa\alpha\frac{2\pi}{N}x'(\tilde{u}) + \alpha^2\Big(\frac{2\pi}{N}\Big)^2 x(\tilde{u})$$

$$+\frac{k_s}{m}\Big(\frac{2\pi}{N}\Big)^2\int_{-1/q}^{0} W_x(-\theta)x(\tilde{u}+\theta)d\theta - \frac{k_s}{m}\Big(\frac{2\pi}{N}\Big)^2\int_{-1-1/q}^{-1} W_x(-1-\theta)x(\tilde{u}+\theta)d\theta = 0\,, \tag{4.45}$$

where the long delay is just 1. The stability investigation of the model (4.45) is, of course, much more complicated than those of the RFDEs (4.37) and (4.39). We also have two new parameters: the function w_x and the scalar h which appear as W_x and q in (4.45) according to formulae (4.43) and (4.44).

The RFDE (4.45) is simplified if we consider the exponential weight function w_x given by

$$w_x(\psi) = \frac{1}{h_0}e^{-\psi/h_0}\,, \quad \psi \in [0,\infty)$$

which is a reasonable approximation to describe the distributed force system p_x, it has infinite support ($h=\infty$, $q=0$), though. Formula (4.43) yields

$$W_x(\tilde{\psi}) = \frac{d_0\pi}{h_0}e^{-d_0\pi\tilde{\psi}/h_0} \quad \Rightarrow \quad W_x(-\theta) = q_0 e^{q_0\theta}\,, \quad \theta \in (-\infty, 0]$$

where q_0 has been defined as $d_0\pi/h_0$. The parameter q_0 has the same meaning as q has in (4.44), since h_0 can be thought of as the measure of the short continuous delay. Substituting this kernel into (4.45), and differentiating the equation with respect to $\tilde{u}$, the integrals can be calculated as in the cases of (3.36) in *Theorem 3.31* and (4.8) in *Theorem 4.6*. Thus, the second order scalar RFDE (4.45) can be transformed into a third order scalar RDDE as follows:

$$x'''(\tilde{u}) + \Big(q_0 + 2\kappa\alpha\frac{2\pi}{N}\Big)x''(\tilde{u}) + \Big(2\kappa\alpha\frac{2\pi}{N}q_0 + \alpha^2\big(\frac{2\pi}{N}\big)^2\Big)x'(\tilde{u})$$

$$+\Big(\alpha^2 + \frac{k_s}{m}\Big)\Big(\frac{2\pi}{N}\Big)^2 q_0 x(\tilde{u}) - \frac{k_s}{m}\Big(\frac{2\pi}{N}\Big)^2 q_0 x(\tilde{u}-1) = 0\,. \tag{4.46}$$

This model involves the original model (4.37) as well. If the short delay is negligible, i.e. $q_0 \to \infty$, then (4.46) becomes equivalent to (4.37). If the long delay-effect is negligible, i.e. $q_0 = 0$, then (4.46) gives the ordinary differential equation of a simple damped oscillator. This mathematical model is very much different from the improved model (4.39) of the dynamic cutting theory, but it gives a similar stability chart to that of *Fig. 4.14*. The chart in *Fig. 4.15* has been constructed in the same way as it was done in *Example 3.21* for a third order scalar RDDE. The stability chart shows great regions of stability at low cutting speeds.

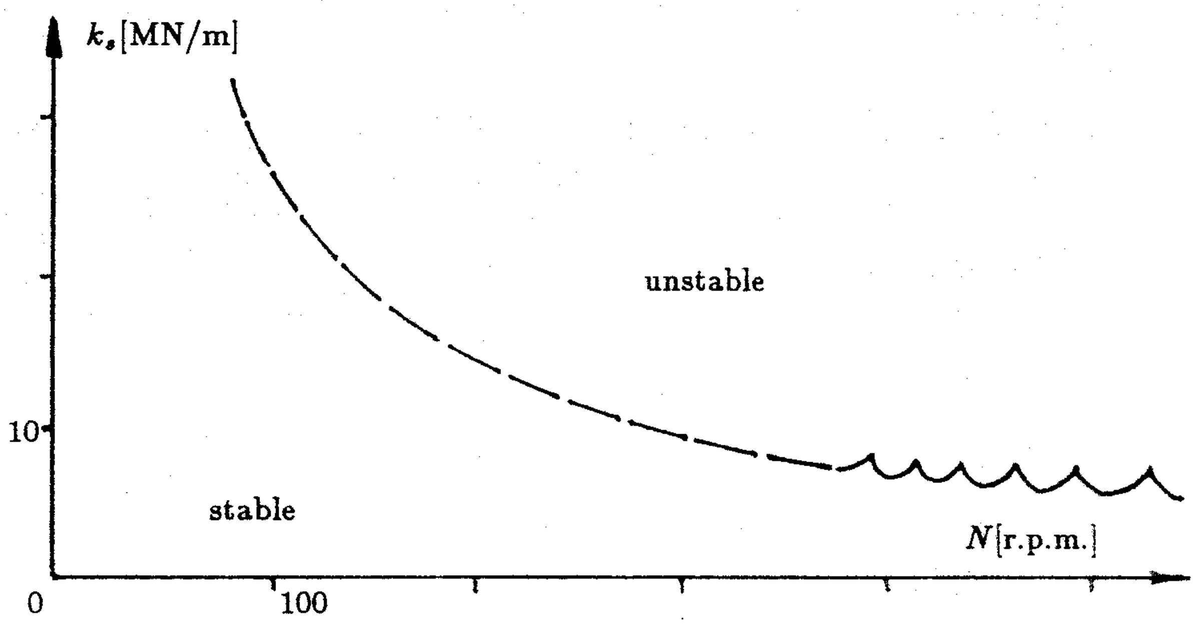

Fig. 4.15. Stability chart of machine tool vibration model (4.46),

$m = 50\,\text{kg}$, $\alpha = 775/\text{sec}$, $\kappa = 0.05$, $q_0 = 25.1$

However, we can obtain even more realistic results if the weight function w_x has a finite support. For example, a good approximation of the force distribution on the tool is given by

$$w_x(\psi) = \frac{1}{h}\big(1 + \cos(\frac{\pi}{h}\psi)\big)\,, \quad \psi \in [0, h]\,,$$

where h is finite. The length h of the contact line of the chip and the tool (see *Fig. 4.12*) may be very small in comparison with the circumference of the workpiece, i.e.

$q = d_0\pi/h$ may be great, but h is not negligible. Formula (4.43) gives the dimensionless weight function

$$W_x(-\theta) = q(1+\cos(q\pi\theta))\,, \quad \theta \in [-\frac{1}{q}, 0]\,.$$

The actual form of the RFDE (4.45) is as follows:

$$x''(\tilde{u}) + 2\kappa\alpha\frac{2\pi}{N}x'(\tilde{u}) + \alpha^2\left(\frac{2\pi}{N}\right)^2 x(\tilde{u})$$

$$+q\frac{k_s}{m}\left(\frac{2\pi}{N}\right)^2\left(\int_{-1/q}^{0}\big(1+\cos(q\pi\theta)\big)x(\tilde{u}+\theta)d\theta - \int_{-1-1/q}^{-1}\big(1+\cos(q\pi(1+\theta))\big)x(\tilde{u}+\theta)d\theta\right) = 0\,. \quad (4.47)$$

Its stability investigation needs a tedious calculation, but it can be carried out in the same way as the RFDE (3.30) was examined in *Theorems 3.24, 3.26* and *3.27*, or as the RFDE (4.8) was analysed in *Theorem 4.7.* The stability chart of (4.47) is shown in *Fig. 4.16* when $q = 100$. This ratio q of the long and short delays may be a typical value for turning. This stability chart gives all the qualitative characteristics experienced in practice. The stability properties of the system improve at low cutting speeds, but they also become better as the cutting speed is increased above 200 r.p.m..

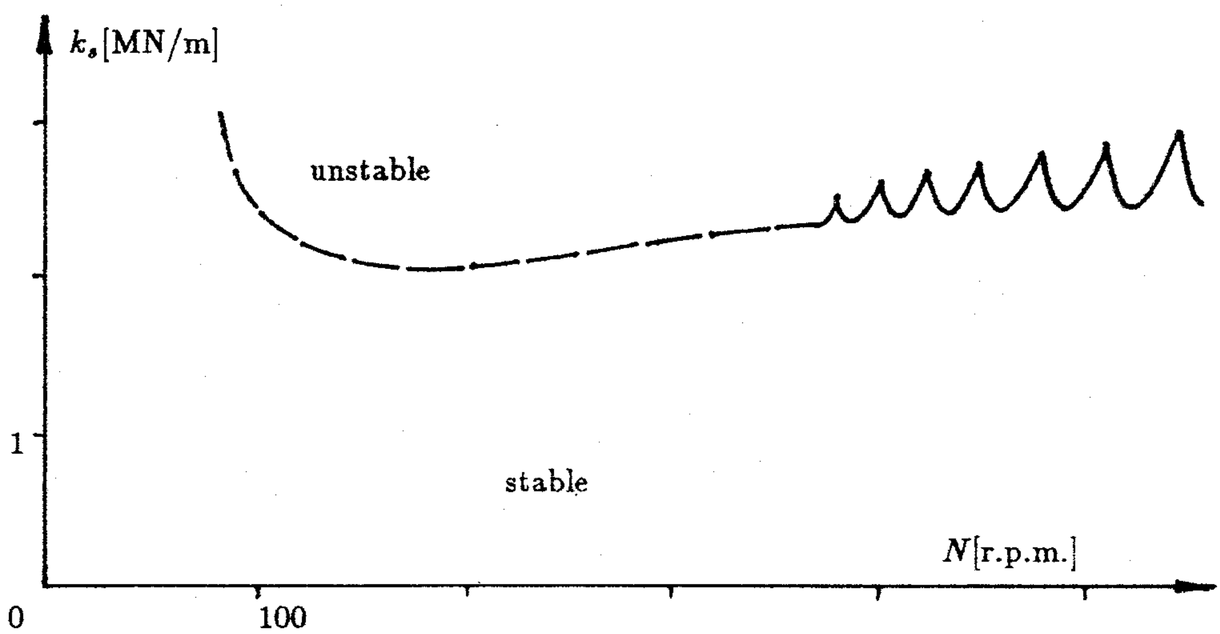

Fig. 4.16. Stability chart of machine tool vibration model (4.47),

$m = 50\,\text{kg}$, $\alpha = 775/\text{sec}$, $\kappa = 0.05$, $q = 100$

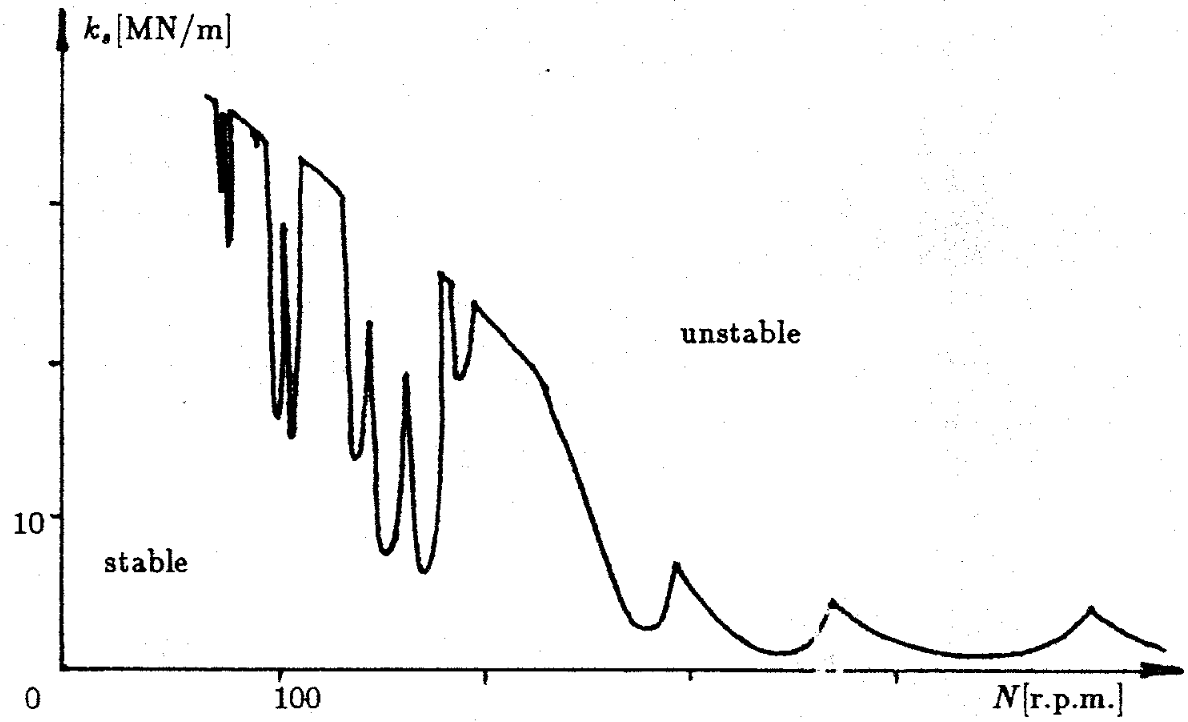

Fig. 4.17. Stability chart of machine tool vibration model (4.48),

$m = 50\,\mathrm{kg}$, $\alpha = 775/\mathrm{sec}$, $\kappa = 0.01$, $q = 4.9$

Fig. 4.17 shows an even more complicated stability chart. In this case, the short regenerative effect is strong, since $q = 4.9$ only, and the weight function w_x is given by

$$w_x(\psi) = \frac{\pi}{2h}\sin\left(\frac{\pi}{h}\psi\right), \quad \psi \in [0, h].$$

Hence,

$$W_x(-\theta) = -\frac{\pi}{2}q\sin(q\pi\theta), \quad \theta \in [-\frac{1}{q}, 0]$$

should be substituted into the RFDE (4.45). This RFDE has the form

$$x''(\tilde{u}) + 2\kappa\alpha\frac{2\pi}{N}x'(\tilde{u}) + \alpha^2\left(\frac{2\pi}{N}\right)^2 x(\tilde{u})$$

$$-q\frac{k_s}{m}\left(\frac{2\pi}{N}\right)^2\frac{\pi}{2}\left(\int_{-1/q}^{0}\sin(q\pi\theta)x(\tilde{u}+\theta)d\theta - \int_{-1-1/q}^{-1}\sin(q\pi(1+\theta))x(\tilde{u}+\theta)d\theta\right) = 0\,. \quad (4.48)$$

It may represent a model of drilling quite well, since q is usually small for drilling, and the maximum value of the distributed cutting force is "behind" the cutting edge (at

$\psi = h/2$ in the case of the actual weight function w_x) because of the negative rake angle at the chisel edge of the drill.

The non-linear machine tool vibrations can be studied if the non-linear cutting force (4.33) is substituted into the motion equation. Athough the equation becomes quite complicated when the non-linear terms are fitted into the RFDE (4.45), it is still easy to prove the existence of Hopf bifurcations at the critical parameters. At the stability limits, on the stability charts of machine tool vibrations, there are two complex conjugate characteristic roots which cross the imaginary axis. The nature of the Hopf bifurcation should be investigated as it was in the case of (4.18) in *Theorem 4.11*. There may be a long series of bifurcation points if the bifurcation parameter is the cutting speed v or the angular velocity N which are inversely proportional to the long delay τ. If we consider the real, experimentally determined coefficients of the non-linear terms in (4.33), then the existence of subcritical Hopf bifurcations can be proved in the mathematical model. Note that the second derivative of the cutting force F_x with respect to the chip thickness f is always negative, while the third derivative is usually positive in practice. The calculations, of course, cannot be carried out by hand, but a computer program like the one used in the investigation of the RFDE (4.21) can follow the long algorithm of the Hopf bifurcation analysis. The experiments in [49] prove the existence of an unstable periodic motion around the stable equilibrium related to the steady-state cutting.

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