Ütközés

Example

In Fig. 1, an 1 DoF oscillator is shown that consists of a beam of mass m_2 and a torsional spring of stiffness k_t. The beam can rotate about the pin at point A. The system is in the gravitational field, the preload of the torsional spring ensures that the horizontal position of the beam is the equilibrium of the oscillator. The vibration of the steady beam is induced by the impact between the beam and the lumped mass m_1. Before the impact, the lumped mass free falls from the height h starting with zero speed.

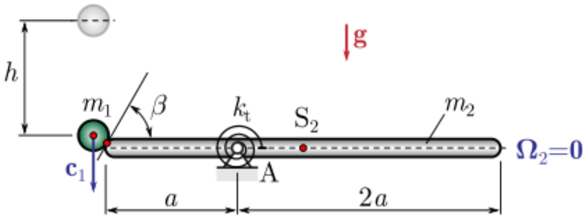


Fig. 1: The investigated system

Data

$m_1 = 6\,\mathrm{kg}$ $m_2 = 6\,\mathrm{kg}$
$a = 0.3\,\mathrm{m}$ $\beta = 60\,°$
$h = 0.115\,\mathrm{m}$ $e = 1$ → tökéletesen rugalmas ütközés

Task

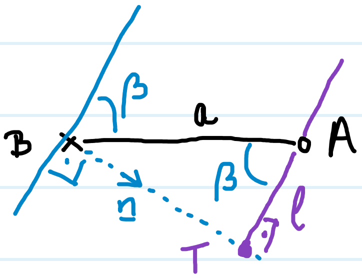
1. Determine the velocity of the lumped mass and the angular velocity of the beam after the impact!

$$m_1 g h = \frac{1}{2} m_1 c_1^2 \Rightarrow c_1 = \sqrt{2gh} = 1{,}5\ \mathrm{m/s}$$

Álló tengely körüli ütközés - centrikus ütközés

$$m_{T_2} = \frac{\Theta_A}{\ell^2} \longrightarrow \Theta_A = \Theta_{S_2} + \Delta r^2 \cdot m_2 \rightarrow \begin{cases} \Delta r = 1{,}5a - a = 0{,}5a \\ \Theta_{S_2} = \frac{1}{12} m_2 (3a)^2 \end{cases}$$

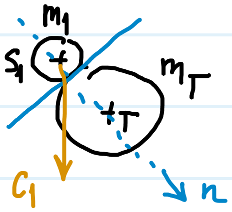
$$\Theta_A = 0{,}54\ \mathrm{kgm^2}$$



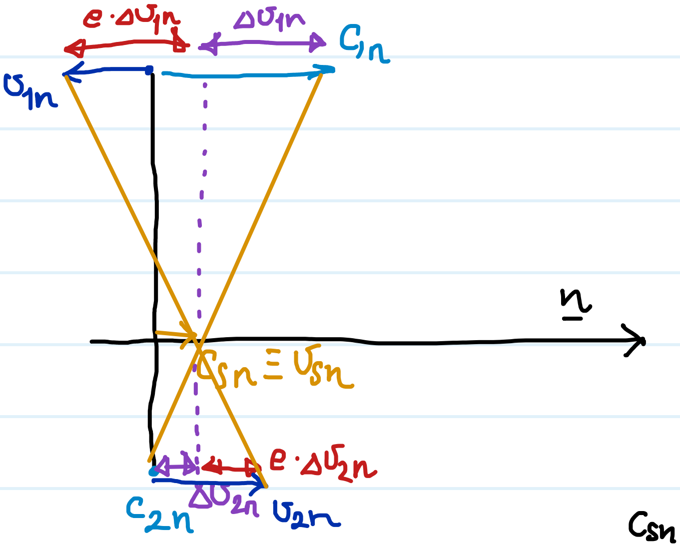
$$\ell = a \cos\beta = 0{,}15\,\mathrm{m}$$

$$m_{T_2} = 24\,\mathrm{kg}$$

$\mu=0 \Rightarrow$ csak normál irányú sebességkomp. változik!

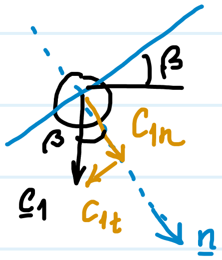


Egyszerűsített Maxwell-ábra



$c_{2n}=0$

$c_{1n} = c_1 \cos\beta = 0{,}75 \ m/s$



$$c_{Sn} = \frac{c_{2n}\cdot m_{T2} + c_{1n} m_1}{m_{T2}+m_1} = 0{,}15 \ m/s = v_{Sn}$$

$\Delta v_1 = c_{1n} - c_{Sn} = 0{,}6 \ m/s$

$\Delta v_2 = c_{2n} - c_{Sn} = -0{,}15 \ m/s$

$v_{1n} = v_{Sn} - e\Delta v_1 = -0{,}45 \ m/s$

$v_{2n} = v_{Sn} - e\Delta v_2 = 0{,}15 - (-0{,}15) = 0{,}3 \ m/s$

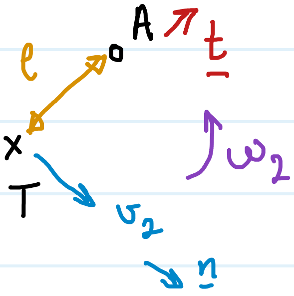
ell.: $$v_{Sn} = \frac{-0{,}45\cdot m_1 + 0{,}3\cdot m_{T2}}{m_1+m_{T2}} = 0{,}15 \ m/s \ ✓$$

$v_{1t} = c_{1t}$ $v_{2t} = c_{2t} = 0 \Rightarrow v_2 = 0{,}3 \ m/s \leftarrow$ T pont sebessége

$c_{1t} = c_1 \sin\beta = 1{,}3 \ m/s \Rightarrow v_1 = \sqrt{v_{1n}^2 + v_{1t}^2} = 1{,}375 \ m/s$

$v_A = 0 \ m/s$

$$\omega_2 = \frac{v_2}{\ell} = 2 \ \frac{rad}{s}$$

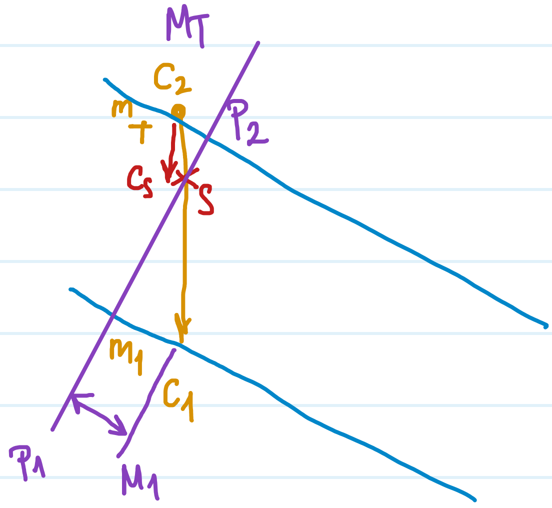


$$\left(\begin{bmatrix}0\\0\\0\end{bmatrix} = \begin{bmatrix}0\\0\\\omega_2\end{bmatrix} \times \begin{bmatrix}0\\\ell\\0\end{bmatrix} + \begin{bmatrix}v_2\\0\\0\end{bmatrix} = \begin{bmatrix}-\omega_2\ell + v_2\\0\\0\end{bmatrix}\right)$$

$\underline{v}_A = \underline{v}_2 + \underline{\omega}_2 \times \underline{r}_{AT}$

$[\underline{n},\underline{t},\underline{z}]$

Maxwell dbra



1 DoF csillapítatlan rendszer

In Fig. 1, a 1 DoF oscillator is shown that consists of the mass m_1, the springs of stiffness k_1 and k_2. As it can be seen one of the springs connects to a cantilever beam, which length is l, its cross section is characterized by a and b, the elastic modulus of its material refers to E. The mass of the beam is negligible. The displacement of the mass m_1 is described by the generalized coordinate x. The vibration of the system is induced by the impact between the mass m_1 and lumped mass m_2.

Data

$a = 0.006\,\mathrm{m}$	$b = 0.025\,\mathrm{m}$
$l = 0.5\,\mathrm{m}$	$E = 200\,\mathrm{GPa}$
$m_1 = 5\,\mathrm{kg}$	$m_2 = 1\,\mathrm{kg}$
$k_1 = 100\,\mathrm{N/m}$	$k_2 = 50\,\mathrm{N/m}$
$c_1 = 0\,\mathrm{m/s}$	$c_2 = 0.6\,\mathrm{m/s}$
$e = 0.5$	

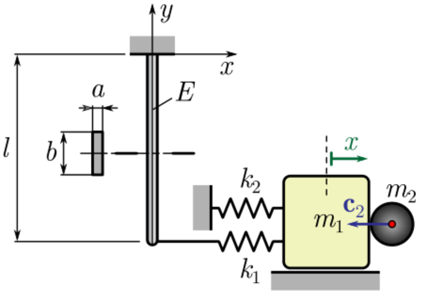


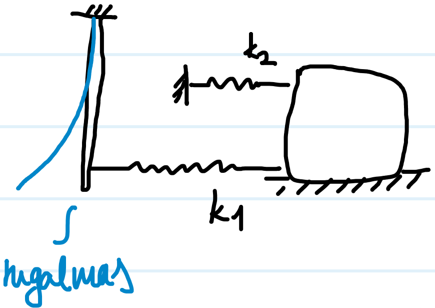
Fig. 1: The investigated system

Tasks

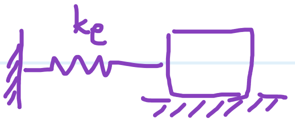
1. Determine the natural angular frequency of the system!
2. Calculate the maximal displacement, velocity and acceleration of the oscillation that is generated by the impact! Sketch the time histories of the displacement, velocity and acceleration!

a)

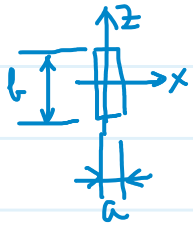
Eredő rugómerevség



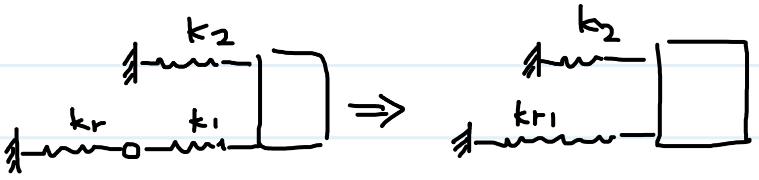
$\Rightarrow$



Castigliano: $\frac{Fl^3}{3I_zE} = f \Rightarrow f = \frac{F}{k_r} \Rightarrow k_r = \frac{3I_zE}{l^3}$



$I_z = \frac{a^3 b}{12}$



Soros rugó: erő ugyanannyi: $F = k_r \xi_r = k_1 \xi_1 = k_{r1}(\xi_1 + \xi_r) = k_{r1}\xi$

$\xi = \xi_r + \xi_1$

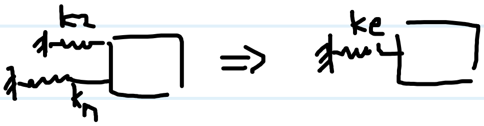
$\underbrace{k_r \xi_r} = \underbrace{k_1(\xi - \xi_r)} = k_{r1}\xi$

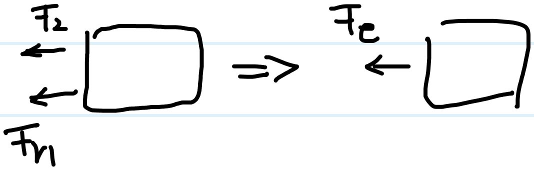
$\xi_r(k_r + k_1) = k_1 \xi$

$\xi_r = k_1\xi / (k_1 + k_r)$

$$k_r \xi_r = k_{r1}\xi \Rightarrow \frac{k_1 k_r}{k_1+k_r}\xi = k_{r1}\xi \Rightarrow k_{r1} = \frac{k_1 k_r}{k_1+k_r}$$

Párhuzamos rugó: elmozdulás ugyanannyi

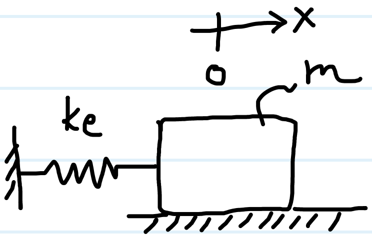




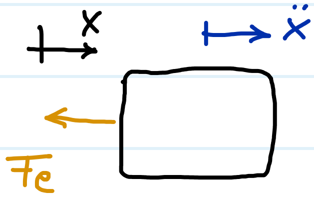
$$F_e = F_{r1} + F_2$$

$$k_e \eta = k_{r1}\eta + k_2\eta$$

$$k_e = k_{r1} + k_2 = k_2 + \frac{k_1 k_r}{k_1+k_r} = 145{,}575\ N/m$$



SZTA



Dn.alapt. $x: \quad m_1\ddot{x} = -F_e = -k_e x$

$$F_e = k_e x$$

mozgásegyenlet: $\ddot{x} + \frac{k_e}{m_1}x = 0$

referenciaalak: $\ddot{x} + \omega_n^2 x = 0$

$$\omega_n = \sqrt{\frac{k_e}{m_1}} = 5{,}39\ \frac{rad}{s}$$

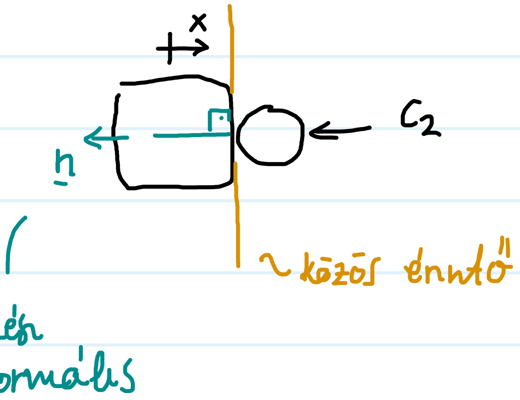
$$f_n = \frac{\omega_n}{2\pi} = 0{,}858\ Hz \qquad (T_n = 1/f_n = 1{,}166\ s)$$

b, $\ddot{x} + \omega_n^2 x = 0$ $x_0 = 0\,m$

$x(0) = x_0$ $v_0 \leftarrow$ ütközés utáni sebesség

$\dot{x}(0) = v_0$

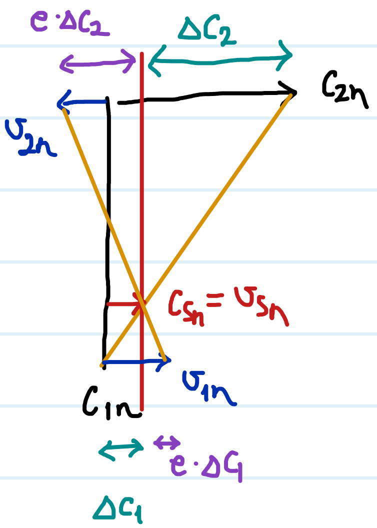
centrikus ütközés



$\underline{n} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$c_{2n} = c_2$

$c_{1n} = 0$



$$c_{Sn} = \frac{m_1 c_{1n} + m_2 c_{2n}}{m_1 + m_2} = 0{,}1 \; m/s$$

$\Delta c_2 = c_{Sn} - c_{2n} = -0{,}5 \; m/s$

$v_{2n} = c_{Sn} + e \cdot \Delta c_2 = -0{,}15 \; m/s$

$\Delta c_1 = c_{Sn} - c_{1n} = 0{,}1 \; m/s$

$v_{1n} = c_{Sn} + e \cdot \Delta c_1 = 0{,}15 \; m/s$

$v_0 = -0{,}15 \; m/s$ (mert $\overset{\leftarrow}{\underline{n}}$ DE! $\vec{x}$)

Próbafüggvény $x(t) = A e^{\lambda t} \longrightarrow \ddot{x}(t) = A \lambda^2 e^{\lambda t}$

$A \lambda^2 e^{\lambda t} + \omega_n^2 A e^{\lambda t} = 0 \Rightarrow$ karakterisztikus egyenlet: $\lambda^2 + \omega_n^2 = 0$

$$\lambda^2 = -\omega_n^2 = -\frac{k_e}{m_1}$$

$$\lambda = \pm \omega_n \sqrt{-1} = \pm \sqrt{-\frac{k_e}{m_1}} \in \mathbb{C}$$

mozgástörvény: $x(t) = C_1 \cos(\omega_n t) + C_2 \sin(\omega_n \cdot t)$

$x(0) = C_1 = 0$

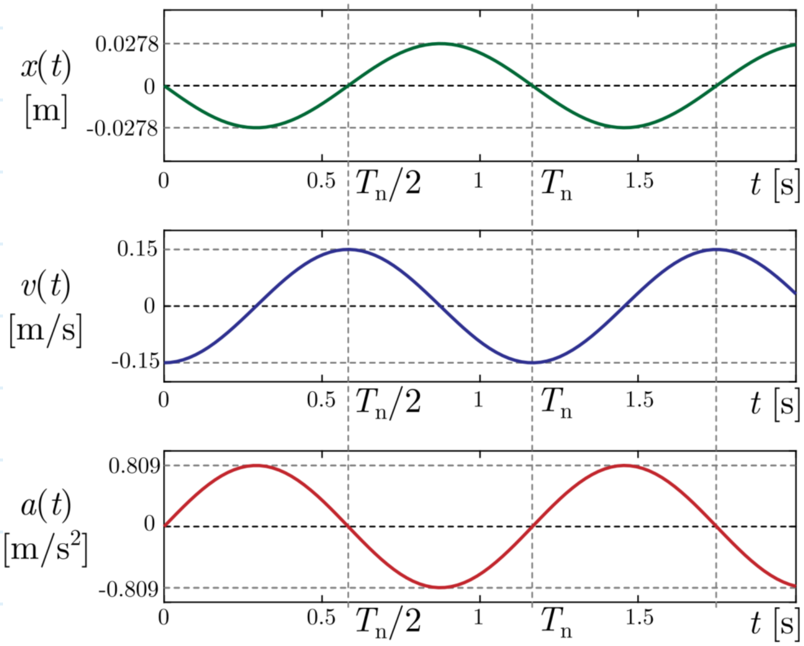
$\dot{x}(t) = -C_1 \omega_n \sin(\omega_n t) + C_2 \omega_n \cos(\omega_n t)$

$\dot{x}(0) = C_2 \omega_n = v_0 \Rightarrow C_2 = v_0/\omega_n = -0{,}0278\,m$

$x(t) = -0{,}0278 \sin(5{,}39 \cdot t)$ $\dot{x}(t) = -0{,}15 \cos(5{,}39t)$

$\ddot{x}(t) = 0{,}809 \sin(5{,}39t)$

$x_{max} = 0{,}0278 m$ $v_{max} = 0{,}15\,m/s$ $a_{max} = 0{,}809\,m/s^2$



1 DoF lengő kar

The swinging arm shown in Fig. 1 consists of two mass-less rods and three lumped masses m_1, m_2 and m_3. The swinging arm can only rotate about the joint A. To describe the motion of the swinging arm, the angle φ, measured from the horizontal axis, is used as generalized coordinate. The arm is in the gravitational field. The equilibrium position is located at $\varphi = 0$, which is ensured by the preload in the spring of stiffness k.

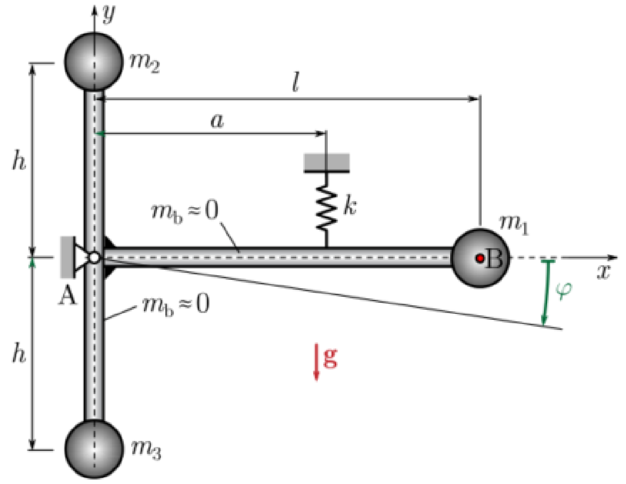


Fig. 1: The mechanical model of the swinging arm

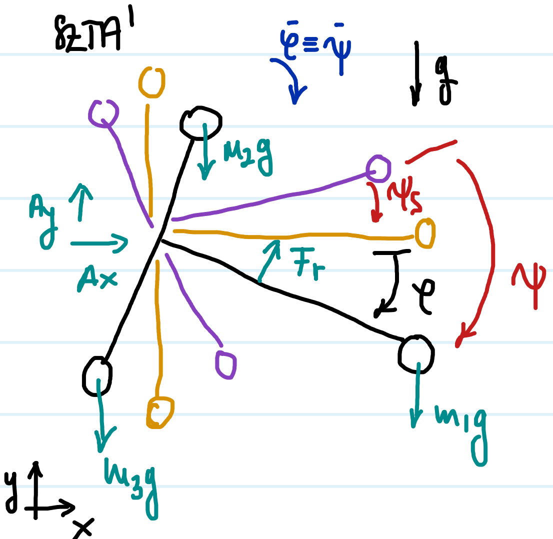
Data

$m_1 = 2\,\mathrm{kg}$	$l = 1\,\mathrm{m}$
$m_2 = 4\,\mathrm{kg}$	$h = 0.5\,\mathrm{m}$
$m_3 = 3\,\mathrm{kg}$	$a = 0.6\,\mathrm{m}$
$k = 10^4\,\mathrm{N/m}$	$g = 9.81\,\mathrm{m/s^2}$

Tasks

1. Derive the equation of motion for small oscillation around the equilibrium!
2. Calculate the natural angular frequency, natural frequency and the period of oscillation! ($\omega_n = 30.96$ rad/s, $f_n = 4.93$ Hz, $T_n = 0.203$ s)
3. Determine the law of motion for the initial condition given by the vertical position $y_B(t = 0) = -0.01$ m and velocity $v_{By}(t = 0) = -1$ m/s of the point B! ($C_1 = 0.01$ rad, $C_2 = 0.033$ rad)
4. Calculate the maximum force in the spring for the given initial condition! ($F_{r,\max} = 235.56$ N)

1.



- egyensúly
- rugó megnyúlás nélkül
- kitérített

kis kitérés!

$$F_r(\varphi) \cong F_{rs} + ka\varphi$$

$$M_r(\varphi) \cong M_{rs} + ka^2\varphi = F_{rs}a + ka^2\varphi$$

Dn. alapt.: z: $-\Theta_A \ddot{\varphi} = -m_1 g l \cos\varphi + M_r(\varphi) - m_2 g h \sin\varphi + m_3 g h \sin\varphi$

$$\Rightarrow -\Theta_A \ddot{\varphi} = -m_1 g \varphi + F_{rs} a + ka^2\varphi - m_2 g h \varphi + m_3 g h \varphi$$

egyensúlynál $\varphi=0\,;\ \underbrace{\dot\varphi=0\,;\ \ddot\varphi=0}_{\text{egyensúly feltétele}}$ $\Rightarrow$ $0=-m_1 g l + F_{rs}\, a$ $\Rightarrow$ $F_{rs} = \frac{m_1 g l}{a}$

így választottuk meg!

Tehát a mozg. egy.: $\Theta_A \ddot\varphi + \varphi\left[ka^2 + m_3 g h - m_2 g h\right] = 0$

$$\Theta_A = m_1 \cdot l^2 + h^2 (m_2 + m_3)$$

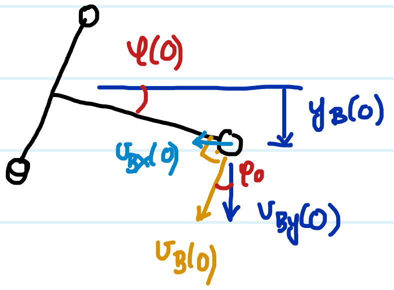
$$\Rightarrow \ddot\varphi\left(m_1 l^2 + h^2(m_2+m_3)\right) + \varphi\left[ka^2 + m_3 g h - m_2 g h\right] = 0$$

2. Ref. alak: $\ddot\varphi + \omega_n^2 \varphi = 0$

$$\Rightarrow \omega_n = \sqrt{\frac{ka^2 + gh(m_3-m_2)}{m_1 l^2 + h^2(m_2+m_3)}} = 30{,}96\ \mathrm{rad/s}$$

$$f_n = \frac{\omega_n}{2\pi} = 4{,}928\ 1/\mathrm{s} \qquad T_n = 0{,}2029\ \mathrm{s}$$

3. $\left.\begin{aligned} y_B(0) &= -0{,}01\,\mathrm{m} \\ v_{By}(0) &= -1\,\mathrm{m/s} \end{aligned}\right\} \Rightarrow$



B pont körpályán mozog!
⇓
Sebesség tangenciális a pályára!

$$y_B(0) = -l\sin(\varphi(0))$$

$$\varphi(0) = \arcsin\left[-\frac{y_B(0)}{l}\right] = 0{,}573° = 0{,}01\ \mathrm{rad}$$

$$v_B(0) = \frac{-v_{By}(0)}{\cos(\varphi(0))} = 1\ \mathrm{m/s}$$

$\Rightarrow$ lineáris közelítés jó

hisz $y_B(0) \approx -l\varphi(0)$

$-v_{By}(0) \approx v_B(0)$

$$\left.\dot\varphi(0) = \frac{v_B(0)}{l} = 1\ \mathrm{rad/s}\ \right\} \oplus$$

Próbafgv.: $\varphi(t) = A e^{\lambda t} \rightarrow \ddot{\varphi}(t) = A\lambda^2 e^{\lambda t}$

$$A\lambda^2 e^{\lambda t} + \omega_n A e^{\lambda t} = 0 \Longrightarrow \lambda^2 + \omega_n^2 = 0 \rightarrow \lambda = \pm \omega_n i$$

$\varphi(t) = C_1 \cos(\omega_n t) + C_2 \sin(\omega_n t)$ $\qquad$ $\varphi(0) = C_1 \cong -y_B(0) = 0{,}01 \text{ rad}$

$\varphi(t) = -C_1 \omega_n \sin(\omega_n t) + C_2 \omega_n \cos(\omega_n t)$ $\qquad$ $\dot{\varphi}(0) = C_2 \omega_n \cong -v_{By}(0)/\ell$

$$\Rightarrow C_2 = \frac{-v_{By}(0)}{\ell \omega_n} = 0{,}0323 \text{ rad}$$

mozg. törv.: $\varphi(t) = 0{,}01 \cos(30{,}96 \cdot t) + 0{,}0323 \sin(30{,}96 t)$

4. $\varphi(t) = \phi \cos(\omega_n t + \delta) = \phi \left[\cos(\omega_n t)\cos(\delta) - \sin(\omega_n t)\sin\delta\right]$

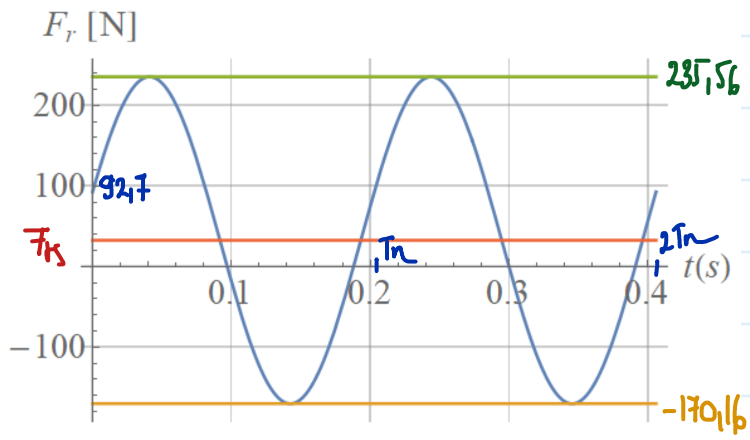
$\varphi(0) = \phi \cos\delta = C_1 \quad (1)$ $\qquad$ $\varphi\left(\frac{\pi}{2\omega_n}\right) = -\phi \sin\delta = C_2 \quad (2)$

$$(1)^2 + (2)^2 \Rightarrow \phi^2 (\cos^2\delta + \sin^2\delta) = C_1^2 + C_2^2$$

$$\Downarrow$$

$$\phi = \sqrt{C_1^2 + C_2^2} = 0{,}03381 \text{ rad}$$

$$F_r(\varphi) = F_{rs} + ka\varphi \Rightarrow F_{rmax} = F_r(\phi) = \frac{m_1 g \ell}{a} + ka\phi = 235{,}56 \text{ N}$$



$$F_r(0) = \frac{m_1 g \ell}{a} + ka\,\varphi(0) = 92{,}7 \text{ N}$$

$$F_{rmin} = F_r(-\phi) = -170{,}16 \text{ N}$$

Vasúti ütköző

In Fig. 1, a railroad car of mass m crashes into a buffer stop with the initial velocity v_0. The buffer stop is assumed to be immovable. In order to consider the elasticity and the energy dissipation of the buffer stop during the impact, we $\overset{\text{use}}{\text{us}}$ the simplified mechanical model shown in the figure. While the wagon is touching the buffer, the spring stiffness and damping coefficients of the buffers stops can be combined into the equivalent stiffness k and equivalent damping factor $2c$.

$\hat{c} = 2c = 2\ 10^5\ \frac{Ns}{m}$

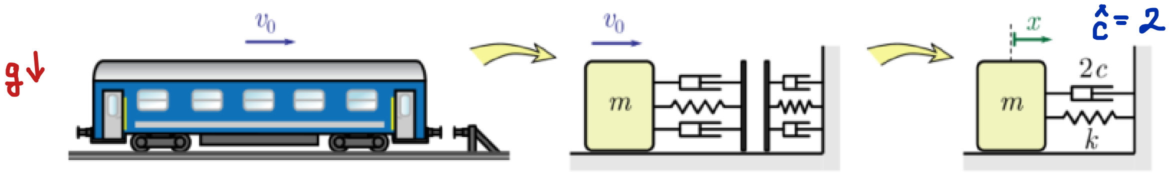


Fig. 1: Mechanical modeling of the impact as a damped oscillator

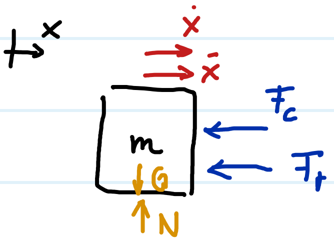
Data

$m = 5 \cdot 10^4$ kg $\quad v_0 = 1$ m/s
$k = 10^6$ N/m $\quad c = 10^5$ Ns/m

Tasks

1. Calculate the maximum spring force arising during the impact! ($F_{r,\max} = 128.55$ kN)

SZTÁ



$F_c = \hat{c} \cdot \dot{x}$

$F_r = kx$

Din. alapt

$$m\ddot{x} = -F_c - F_r = -\hat{c}\dot{x} - kx$$

$$m\ddot{x} + \hat{c}\dot{x} + kx = 0$$

Ref. alak.

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0 \longrightarrow 2\zeta\omega_n = \frac{\hat{c}}{m}\ ; \quad \omega_n^2 = \frac{k}{m}$$

relatív csillapítási tényező

$\omega_n = 4{,}47\ \text{rad}/s$

$\zeta = \frac{\hat{c}}{2}\frac{1}{\omega_n m} = 0{,}45$

csillapított rendszer sajátkörfrekvenciája: $\omega_d = \omega_n\sqrt{1-\zeta^2} = 4\ \text{rad}/s$

$f_D = \frac{\omega_d}{2\pi} = 0{,}64\ Hz$ $\qquad T_D = \frac{1}{f} = 1{,}57\ s$

mivel az ütköző felület tömege 0 $\Rightarrow$ $\dot{x}(0) = v_0$

valamint $x(0) = 0\ m$

próbafgv:

$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$ $\qquad x(t) = Ae^{\lambda t}$; $\dot{x}(t) = A\lambda e^{\lambda t}$; $\ddot{x}(t) = A\lambda^2 e^{\lambda t}$

$A\lambda^2 e^{\lambda t} + 2\zeta\omega_n A\lambda e^{\lambda t} + \omega_n^2 A e^{\lambda t} = 0$

$\lambda^2 + \underbrace{2\zeta\omega_n}_{=b}\lambda + \omega_n^2 = 0 \longrightarrow \lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4\omega_n^2}}{2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$

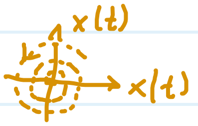
kar. egy.

$b^2 = 4\zeta^2\omega_n^2 \Rightarrow \sqrt{b^2 - 4\omega_n^2} = 2\omega_n\sqrt{\zeta^2 - 1}$

$\lambda_{1,2} = -\zeta\omega_n \pm i\omega_d$

Az általános megoldás: $x(t) = \underbrace{e^{-\zeta\omega_n t}}_{\text{exp. csökk.}}\underbrace{\left[C_1\cos(\omega_d t) + C_2\sin(\omega_d t)\right]}_{\omega_d \text{ sajátkörfrk. oszcillál}}$

$\Rightarrow$ stabil fókusz matematikailag ~



mozgás törvény számítása:

$x(0) = e^0 C_1 = 0 \Rightarrow C_1 = 0$

$\dot{x}(t) = -\underset{\uparrow \atop C_1 = 0}{\zeta}\omega_n e^{-\zeta\omega_n t}\left[C_2\sin(\omega_d t)\right] + e^{-\zeta\omega_n t}\left[C_2\cos(\omega_d t)\cdot\omega_d\right]$

$\dot{x}(0) = e^0\cdot C_2\omega_d = v_0 \Rightarrow C_2 = \frac{v_0}{\omega_d} = 0{,}25\ m$

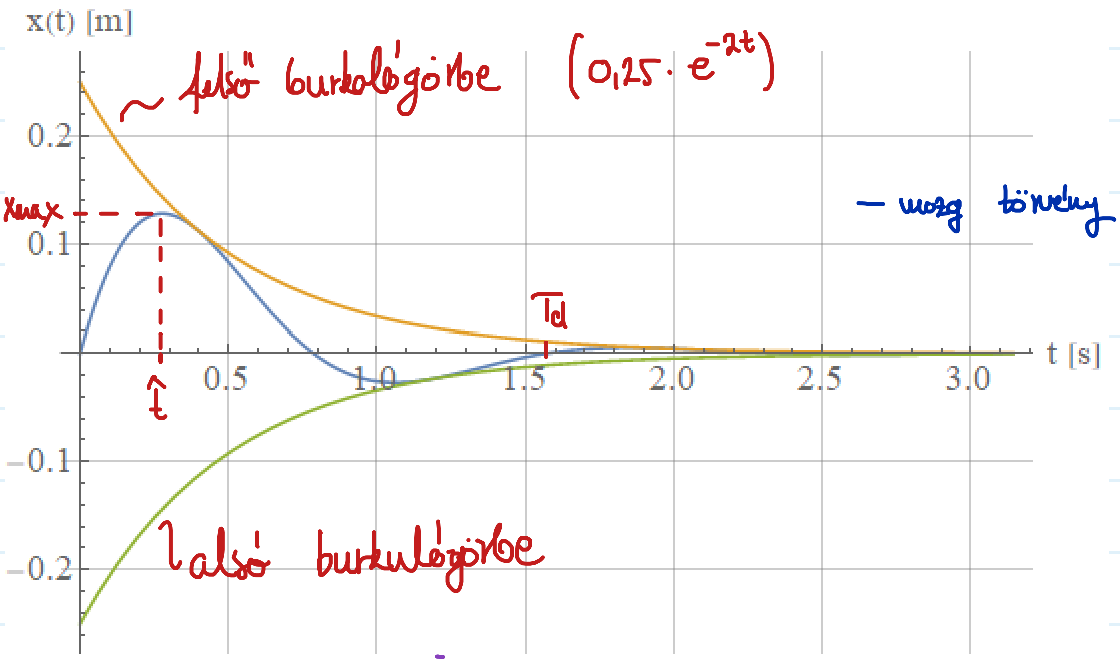
mozg törvény: $x(t) = 0{,}25\cdot e^{-2t}\cdot\sin(4t)$

Fmax a max összenyomódásnál lesz, tehát $\dot{x}(\hat{t}) = 0$ időpontban

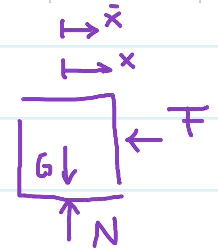
$\dot{x}(t) = -0{,}5\,e^{-2t}\sin(4t) + e^{-2t}\cos(4t) \Rightarrow \dot{x}(\hat{t}) = \underset{\neq 0}{e^{-2\hat{t}}}\underbrace{\left[-0{,}5\sin(4\hat{t}) + \cos(4\hat{t})\right]}_{=0} = 0$

$0{,}5\sin(4\hat{t})=\cos(4\hat{t}) \Rightarrow \operatorname{tg}(4\hat{t})=2 \Rightarrow \hat{t}=\frac{1}{4}\operatorname{arctg}(2)=0{,}2768\ s$

$x_{max}=x(\hat{t})=0{,}1286\ m \longrightarrow F_{max}=k\cdot x_{max}=128{,}55\ kN$



Mi történik elváláskor? $+\ddot{x}$



$m\ddot{x}=-F=-F_c-F_r$

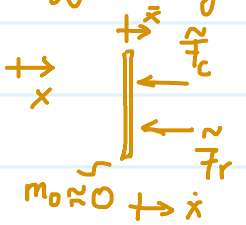
elválás: $F=0 \Rightarrow F_c(\tilde{t})+F_r(\tilde{t})=0 \rightarrow \hat{c}\,\dot{x}(\tilde{t})+kx(\tilde{t})=0$ $\quad\tilde{t}=?$

A csillapítás által disszipált energia: $E^d=T_0'-T_1'=\left(v_0^2-\dot{x}(\tilde{t})^2\right)\frac{1}{2}m=22{,}27\ kJ$

$T_0'=\frac{1}{2}mv_0^2$ (ütközés előtti energia)

$T_1'=\frac{1}{2}m\dot{x}(\tilde{t})^2$ (elváláskor)

Hogy mozog ezután az ütköző?



$m_0\ddot{x}\approx 0=-\tilde{F}_c-\tilde{F}_r=-c\dot{x}-\tilde{k}x \Rightarrow$ kar. egy.: $0=-c\lambda-\tilde{k}$

$\lambda=-\tilde{k}/c$

mozg. törv.: $x(t)=e^{-\tilde{k}t/c}\cdot C_3$

$\Rightarrow$ exponenciálisan tart 0-hoz

1 DoF csillapított lengőkar

In Fig. 1, a swinging arm is shown that consists of two rods with different lengths and masses, and a disk with radius R. The swinging arm can only rotate along joint A, and the rods are connected to the environment through two springs with stiffness k_1 and k_2 and a damper with damping factor c_1. To describe the motion of the swinging arm, the angle φ measured from the horizontal axis is used as generalized coordinate. The structure is in the gravitational field and its equilibrium position is located at $\varphi = 0$. In this equilibrium position, the spring with stiffness k_2 is unloaded.

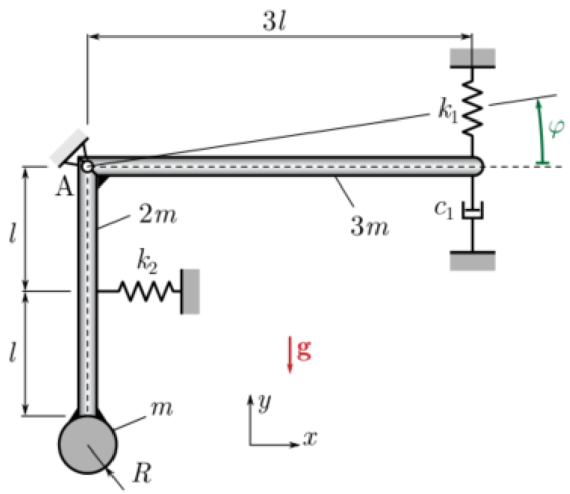


Fig. 1: The mechanical model of the swinging arm

Data

$l = 0.2$ m $\quad R = 0.1$ m $\quad m = 0.12\,\mathrm{kg}$

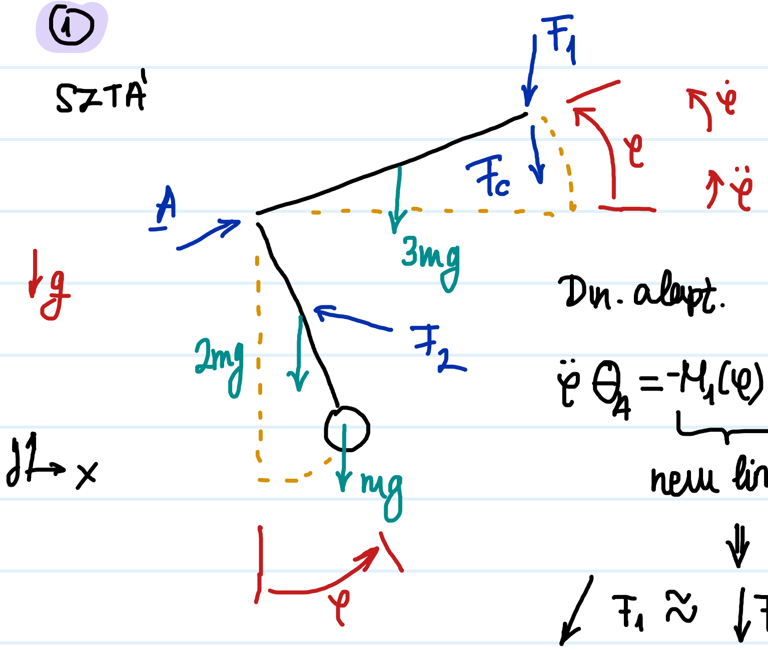
$k_1 = 300\,\mathrm{N/m} \quad k_2 = 10\,\mathrm{N/m} \quad c_1 = 2\,\mathrm{Ns/m}$

Tasks

1. Derive the equation of motion for small oscillations, calculate the natural angular frequencies of the undamped and damped system, and the damping ratio! ($\omega_n = 35.55$ rad/s, $\omega_d = 35.31$ rad/s, $\zeta = 0.117$ [1])
2. Determine the critical damping factor in order to make the system critically damped! ($c_{1,\mathrm{cr}} = 17.10$ Ns/m)
3. Calculate the maximum force in the spring of stiffness k_1 if the initial conditions are $\varphi(t = 0) = \varphi_0 = 0.01$ rad and $\dot\varphi(t = 0) = 0$ rad/s! ($F_{\mathrm{r1,max}} = 3.009$ N)

①

SZTA

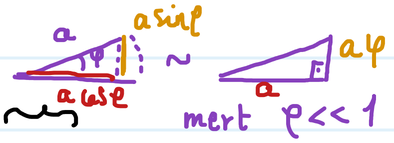


φ = 0 az egyensúlyi helyzet!

Dn. alapt.

$$\ddot\varphi\,\Theta_A = \underbrace{-M_1(\varphi) - M_c(\varphi) - M_2(\varphi)}_{\text{nem lin. kifejezések}} - mg\left[3\frac{3l}{2}\cos\varphi + 2l\sin\varphi + (R+2l)\sin\varphi\right]$$

$\Downarrow$



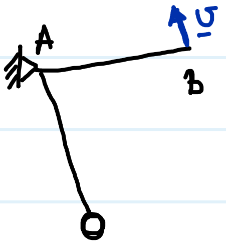
$$\swarrow F_1 \approx \downarrow F_1 \Rightarrow F_1(\varphi) \approx F_{1s} + k_1 3l\varphi$$

kis szögek

$$M_1(\varphi) \approx F_{1s}\cdot 3l + k_1 9l^2\varphi$$

Hasonlóan : $F_2(\varphi) \approx k_2 \ell \varphi \Rightarrow M_2(\varphi) \approx k_2 \ell^2 \varphi$

($F_{S2} = 0$, mert $\varphi = 0$ esetén nincs előfeszítve a feladat szerint!)



redukciós képlet : $\underline{v} = \underline{0} + \underline{\omega} \times \underline{r}_{AB}$

$$\underline{v} = \begin{bmatrix} 0 \\ 0 \\ \dot{\varphi} \end{bmatrix} \times 3\ell \begin{bmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{bmatrix} = \begin{bmatrix} -3\ell\dot{\varphi}\sin\varphi \\ 3\ell\dot{\varphi}\cos\varphi \\ 0 \end{bmatrix} \approx 3\ell \begin{bmatrix} \dot{\varphi}\varphi \\ \dot{\varphi} \\ 0 \end{bmatrix} \approx 3\ell \begin{bmatrix} 0 \\ \dot{\varphi} \\ 0 \end{bmatrix}$$

$\dot{\varphi}\varphi \approx 0$, mert $\varphi \ll 1$; $\dot{\varphi} \ll 1$

$F_c \approx 3\ell\dot{\varphi} \cdot c \rightarrow M_c \approx 9\ell^2 \dot{\varphi} c$

Linearizált mozg. egy. (kis kitérésekre)

$$\Theta_A \ddot{\varphi} = -F_{S1} 3\ell - mg\ell \frac{9}{2} - \varphi\left[k_1 9\ell^2 + k_2\ell^2 + 2mg\ell + mg(2\ell + R)\right] + \dot{\varphi} 9 \ell^2 c$$

$$\Theta_A = \frac{1}{3} 2m\, 4\ell^2 + \frac{1}{3} 3m\, 9\ell^2 + \frac{1}{2} mR^2 + m(2\ell+R)^2 = 0{,}0866 \text{ kgm}^2$$

egyensúly ($\varphi = 0$) esetén: $\dot{\varphi} = 0$ és $\ddot{\varphi} = 0$

$0 = -F_{S1} 3\ell - 4{,}5\, mg\ell \Rightarrow F_{S1} = -1{,}5\, mg$

mozg. egy.: $\Theta_A \dot{\varphi} + \varphi\left[k_1 9\ell^2 + k_2 \ell^2 + mg(4\ell + R)\right] + \dot{\varphi}\, 9\ell^2 c = 0$

ref. alak. $\ddot{\varphi} + 2\zeta\omega_n\dot{\varphi} + \omega_n^2\varphi = 0 \longrightarrow \omega_n = \sqrt{\dfrac{k_1 9\ell^2 + k_2\ell^2 + mg(4\ell+R)}{\Theta_A}} = 35{,}55 \text{ rad/s}$

$$2\zeta\omega_n = \frac{9\ell^2 c}{\Theta_A} \Rightarrow \zeta = \frac{9\ell^2 c}{2\omega_n \Theta_A} = 0{,}1169 \ [1]$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 35{,}31 \text{ rad/s}$$

② krit. csillapítás esetén pont nincs képzetes karakterisztikus gyök

$$\Rightarrow \hat{\omega}_d = 0 \rightarrow \hat{\zeta} = 1 \rightarrow \hat{c} = \frac{\hat{\zeta}\, 2\omega_n \Theta_A}{9\ell^2} = 17{,}1 \text{ Ns/m}$$

③ Próbafgv.: $\varphi(t) = Ae^{\lambda t} \rightarrow \dot{\varphi}(t) = A\lambda e^{\lambda t} \rightarrow \dot{\varphi}(t) = A\lambda^2 e^{\lambda t}$

$$A\lambda^2 e^{\lambda t} + A\lambda e^{\lambda t} \zeta 2\omega_n + Ae^{\lambda t} \omega_n^2 = 0$$

kar. egy.: $\lambda^2 + 2\zeta\omega_n \lambda + \omega_n^2 = 0 \longrightarrow \lambda_{1,2} = -\zeta\omega_n \pm \omega_n \sqrt{\zeta^2 - 1}$

$$\lambda_{1,2} = -\zeta\omega_n \pm i\omega_d$$

$$\varphi(t) = e^{-\zeta\omega_n t}\left[C_1 \cos(\omega_d t) + C_2 \sin(\omega_d t)\right]$$

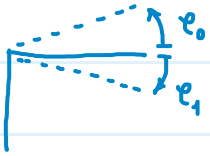
$\varphi(0) = \varphi_0 = e^0 \cdot C_1 = 0{,}01 \text{ rad} \longrightarrow C_1 = 0{,}01 \text{ rad}$

$$\dot{\varphi}(t) = -\zeta\omega_n e^{-\zeta\omega_n t}\left[C_1 \cos(\omega_d t) + C_2 \sin(\omega_d t)\right] + e^{-\zeta\omega_n t}\left[-C_1 \sin(\omega_d t)\omega_d + C_2 \cos(\omega_d t)\omega_d\right]$$

$\dot{\varphi}(0) = \dot{\varphi}_0 = -\zeta\omega_n C_1 + C_2\omega_d = 0 \longrightarrow C_2 = \dfrac{\zeta\omega_n C_1}{\omega_d} = 0{,}00118 \text{ rad}$

mivel $\dot{\varphi}_0 = 0 \rightarrow \varphi_0$ szélsőérték!

legyen φ_1 a következő szélsőérték

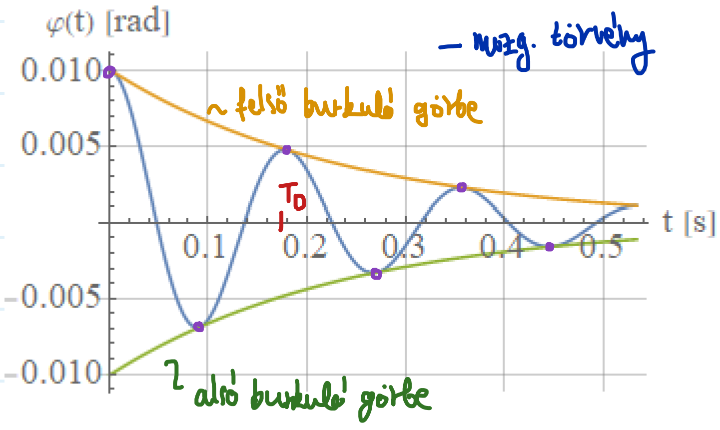


$|\varphi_1| < |\varphi_0|$ csillapítás miatt

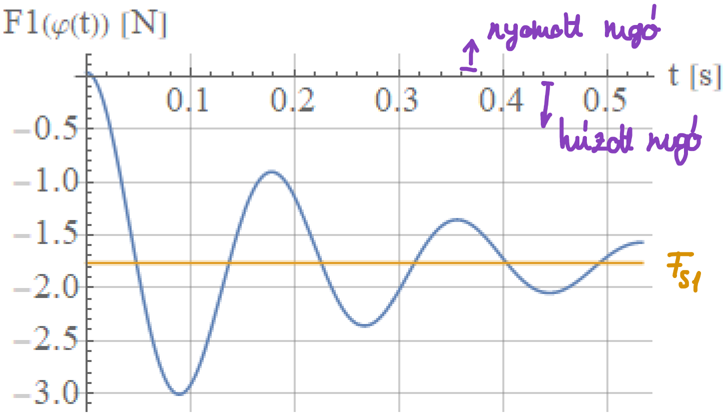
DE! φ_1 irányban előfeszített a rugó ⇒ ellenőrizni kell

$T_D = \dfrac{2\pi}{\omega_D}$ $\qquad \varphi_1 = \varphi(T_D/2) = -0{,}00691 \text{ rad}$

$F_1(\varphi(0)) = 0{,}0342 \text{ N}$ $\qquad F_1(\varphi(T_D/2)) = -3{,}0093 \text{ N} \Rightarrow F_{max} = 3{,}0093 \text{ N}$



o szélső helyzetek



1 DoF gerjesztett csillapított lengőkar

In Fig. 1(a) and (b), a swinging arm is shown that is modelled by a rod with length l and mass m. The swinging arm can only rotate about joint A. Two cases are distinguished: harmonic force excitation (see panel (a)) and harmonic displacement excitation that is applied through a spring with stiffness k_0 (see panel (b)). In both cases, the rod is connected to the environment through a spring with stiffness k (in case (a)) or k_1 (in case (b)) and a damper with damping factor c. To describe the motion of the corresponding 1 DoF swinging arm, the angle φ measured from the horizontal axis is used as generalised coordinate. The structure lies on the horizontal plane and its equilibrium position is located at $\varphi = 0$. In this equilibrium position, the springs with stiffness k (case (a)) and k_1 (case (b)) are unloaded.

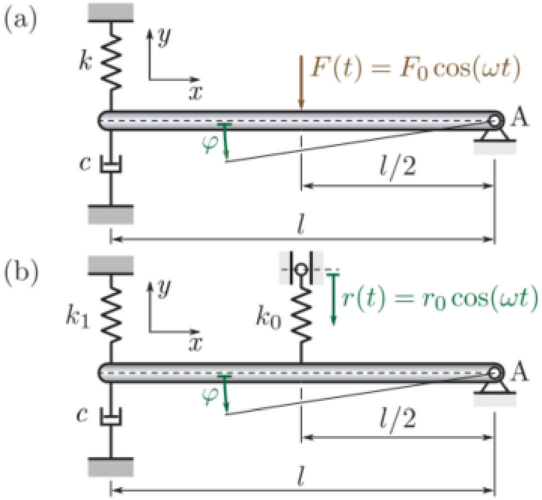
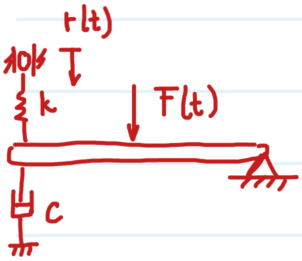


Fig. 1: Mechanical model of a swinging arm. (a) Force excitation. (b) Displacement excitation.

piros értékek arra az esetre, mikor erő ÉS útgerjesztés egyszerre van, az alábbi ábra alapján



Data

$m = 3\,\mathrm{kg}$ $\quad l = 1\,\mathrm{m}$ $\quad \omega = 30\,\mathrm{rad/s}$

$k = 400\,\mathrm{N/m}$ $\quad F_0 = 10\,\mathrm{N}$ 6N $\quad c = 28\,\mathrm{Ns/m}$

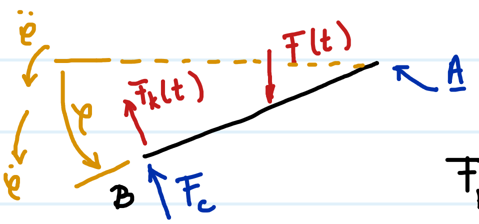
$k_0 = 1000\,\mathrm{N/m}$ $\quad r_0 = 0.01\,\mathrm{m}$ $\quad k_1 = 150\,\mathrm{N/m}$

0,005m

Tasks

1. Derive the equation of motion for both models and calculate the natural angular frequencies of the undamped and damped system, the damping ratio and the static deformation! (For both cases: $\omega_n = 20$ rad/s, $\omega_d = 14.28$ rad/s, $\zeta = 0.7$ [1], $f_0 = 0.0125$ rad)
2. Draw the resonance curve and the phase diagram of the systems!
3. Determine the law of motion $\varphi(t)$ if the initial conditions are $\varphi(t = 0) = \varphi_0 = 0.015$ rad and $\dot\varphi(t = 0) = 0$ rad/s!

1. SZTÁ



Din. alapt.

$$\Theta_A \ddot\varphi = \frac{\ell}{2}\cos\varphi\, F(t) - \ell\cos\varphi\left[F_k(t) + F_c\right]$$

$$F_k(t) = \left(\ell\sin\varphi - r_0\cos(\omega t)\right)k \approx \ell\varphi k - r_0\cos(\omega t)k$$

$$\Theta_A = \frac{1}{3}m\ell^2$$

$$F_c \approx \underbrace{\dot\varphi\ell}_{\approx v_B}\cdot c$$

din. mozg. egy.: $\Theta_A\ddot\varphi + \dot\varphi\ell^2 c + \varphi\ell^2 k = \ell\cos(\omega t)\left[\frac{1}{2}F_0 + r_0 k\right]$

ref. alak: $\ddot\varphi + 2\zeta\omega_n\dot\varphi + \omega_n^2\varphi = f_0\omega_n^2\cos(\omega t)$

$\omega_n = \sqrt{\frac{\ell^2 k}{\Theta_A}} = 20 \frac{rad}{s}$ $2\zeta\omega_n = \frac{\ell^2 c}{\Theta_A} \Rightarrow \zeta = \frac{1}{2}\frac{\ell^2 c}{\omega_n \Theta_A} = 0{,}7\,[-]$

$f_0 \omega_n^2 = \frac{1}{2}\ell F_0 + \ell r_0 k \Longrightarrow f_0 = \ell\,\frac{1/2\,F_0 + r_0 k}{\omega_n^2} = 0{,}0125$ rad

$\omega_d = \omega_n\sqrt{1-\zeta^2} = 14{,}283$ rad/s

② $\varphi(t) = \varphi_h(t) + \varphi_p(t)$

homogén: $\varphi_h(t) = Ae^{\lambda t}$ kar. egy.: $\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0$

kar. gyök: $\lambda_{1,2} = -\zeta\omega_n \pm i\omega_d$

$\varphi_h(t) = e^{-\zeta\omega_n t}\left(C_1\cos(\omega_d t) + C_2\sin(\omega_d t)\right)$

partikuláris: $\varphi_p(t) = \Phi\cos(\omega t + \Delta)$ $\lambda = \frac{\omega}{\omega_n}$

$-\Phi\omega^2\cos(\omega t-\Delta) - 2\zeta\omega_n\Phi\omega\sin(\omega t-\Delta) + \omega_n^2\Phi\cos(\omega t-\Delta) = f_0\omega_n^2\cos(\omega t) \quad \Big| \cdot\omega_n^2$

$\Phi\left[\cos(\omega t-\Delta)\left[1-\lambda^2\right] - \sin(\omega t-\Delta)\,2\zeta\lambda\right] = f_0\cos(\omega t)$; $\tilde{\delta} = \Delta + \vartheta$

megj.: $A\cos\alpha - B\sin\alpha = C \longrightarrow D\left(\frac{A}{D}\cos\alpha - \frac{B}{D}\sin\alpha\right) = C$, ahol $D = \sqrt{A^2+B^2}$

$\frac{A}{D} = \cos\delta$; $\frac{B}{D} = \sin\delta$, hiszen $\sqrt{\cos^2\delta + \sin^2\delta} = \frac{1}{D}\sqrt{A^2+B^2} = \frac{D}{D} = 1$

$D(\cos\alpha\cdot\cos\delta - \sin\alpha\cdot\sin\delta) = D\cos(\alpha+\delta) = C$

$\Phi\sqrt{(1-\lambda^2)^2 + 4\zeta^2\lambda^2}\,\cos(\omega t + \tilde{\delta}) = f_0\cos(\omega t)$; $N = \frac{1}{\sqrt{(1-\lambda^2)^2 + 4\zeta^2\lambda^2}}$

$\Phi\cos(\omega t + \tilde{\delta}) = Nf_0\cos(\omega t)$; $\Phi = Nf_0$; $\tilde{\delta} = 0$

↳ $\Delta = -\vartheta$

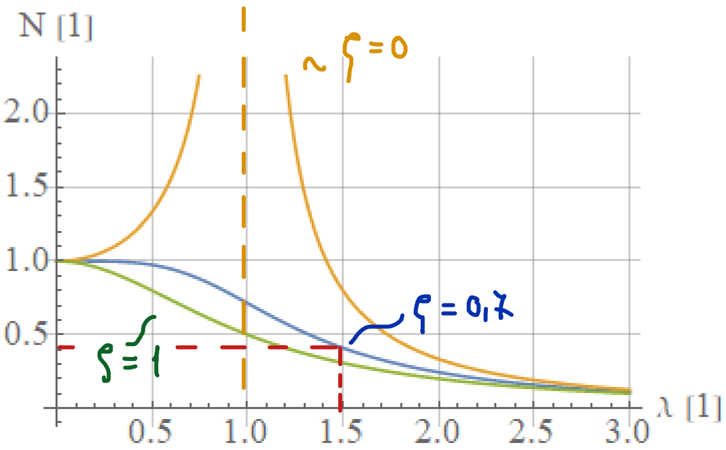
Tehát $N = 0{,}4092\,[-]$ $\Phi = 0{,}00511$ rad $= 0{,}293°$

megj.: $\frac{\sin\delta}{\cos\delta} = \mathrm{tg}\,\delta = \frac{B}{A}$ $\vartheta = \mathrm{arctg}\left(\frac{2\zeta\lambda}{1-\lambda^2}\right) = -1{,}034 + j\pi$ rad

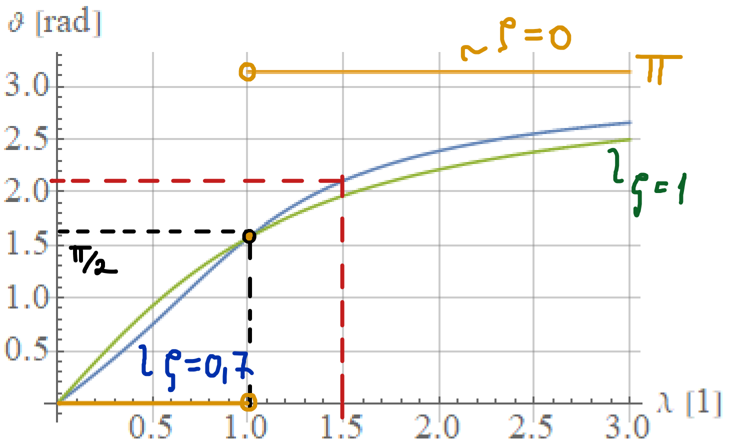
tg π periodikus: $\vartheta \in [0, \pi]$ $\vartheta = 2{,}1077$ rad $= 120{,}763°$

$$\varphi_p(t) = \phi\cos(\omega t - \vartheta)$$

Nagyítás diagram



Fáziskésés diagram



③ $\varphi(t) = e^{-\zeta\omega_n t}\left(C_1\cos(\omega_d t) + C_2\sin(\omega_d\cdot t)\right) + \phi\cos(\omega t - \vartheta)$

$\varphi_0 = \varphi(0) = C_1 + \phi\cos(-\vartheta) \Rightarrow C_1 = \varphi_0 - \phi\cos(-\vartheta) = 0{,}01762 \text{ rad}$

$\dot\varphi(t) = -\zeta\omega_n e^{-\zeta\omega_n t}\left(C_1\cos(\omega_d t) + C_2\sin(\omega_d\cdot t)\right) + e^{-\zeta\omega_n t}\left(-C_1\omega_d\sin(\omega_d t) + C_2\omega_d\cos(\omega_d t)\right) - \phi\omega\sin(\omega t - \vartheta)$

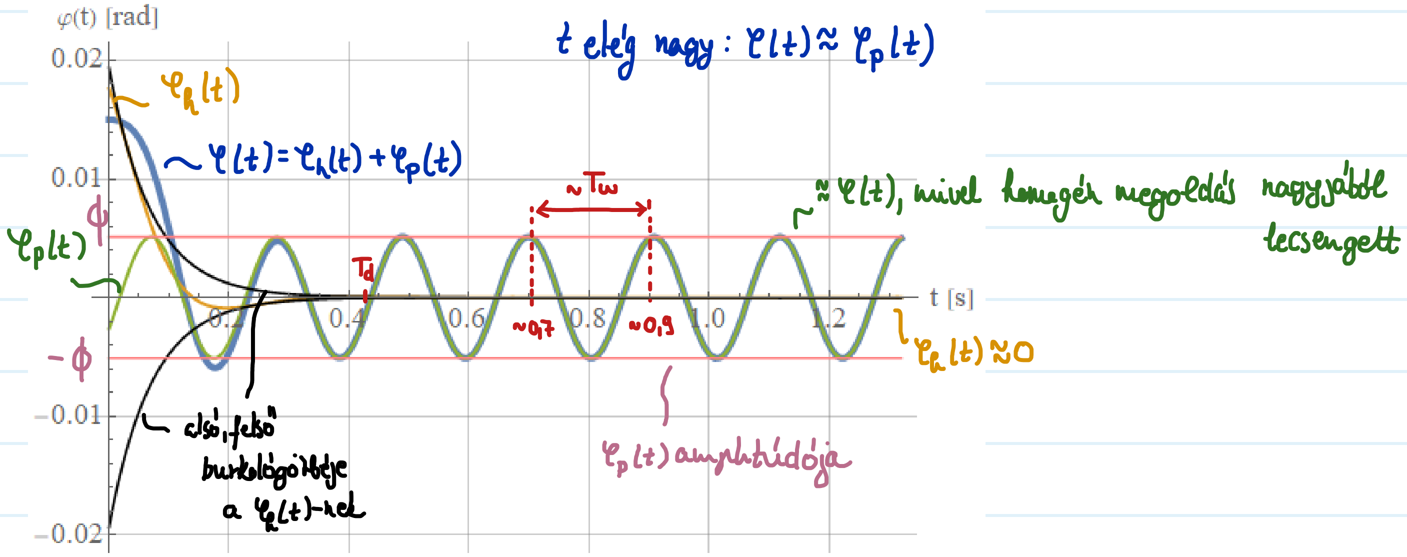
$\dot\varphi_0 = \dot\varphi(0) = -\zeta\omega_n C_1 + C_2\omega_d - \phi\omega\sin(-\vartheta) \Longrightarrow C_2 = \frac{1}{\omega_d}\left[\zeta\omega_n C_1 + \phi\omega\sin(-\vartheta)\right] = 0{,}008036 \text{ rad}$

$T_d = \frac{2\pi}{\omega_d} = 0{,}44 s$ $\qquad$ $T_\omega = \frac{2\pi}{\omega} = 0{,}2094 s$

Mozgás-törvény:

$$\varphi(t) = 0{,}05115\cos(30t - 2{,}1077) + e^{-14t}\left[0{,}01762\cos(14{,}28t) + 0{,}008036\sin(14{,}28t)\right]$$

t elég nagy: $\varphi(t) \approx \varphi_p(t)$



1 DoF negyedjármű modell

In Fig. 1 a quarter car model is shown that can be used to investigate the vertical dynamics of a vehicle. In practice, the stiffness and damping parameters of the suspension system are usually tuned by means of the model in order to achieve the desired road comfort. The mass of the quarter car is denoted by m, while the equivalent stiffness and damping parameters of the suspension and wheel are denoted by k and c, respectively. The road, on which the car is driven with constant longitudinal speed v, is considered to have a sinusoidal shape described by $r(t) = R\sin(\omega t)$, where R is the amplitude of the road disturbances and ω is the excitation frequency characterized by the speed v and the wavelength L. To describe the motion of the quarter car model the vertical displacement $y(t)$ is used as general coordinate. The effect of gravity is neglected and it is assumed that the wheel never lose the contact with the road.

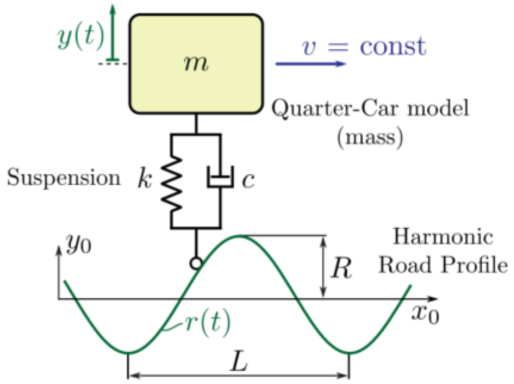


Fig. 1: Mechanical model of the quarter-car model.

Data

$m = 300\,\text{kg}$ $\quad k = 2\cdot 10^5\,\text{N/m}$ $\quad c = 9300\,\text{Ns/m}$ $\quad v = 36\,\text{km/h} = 10\,\text{m/s}$

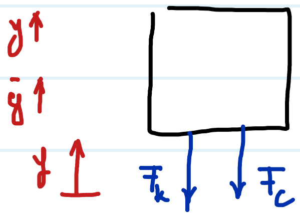
$R = 0.04\,\text{m}$ $\quad L = 1.2\,\text{m}$

$r(t) = R\sin(\omega t)$

Tasks

1. Derive the equation of motion for the quarter car model!
2. Calculate the natural angular frequency of the damped an the undamped system, the damping ratio, the frequency ratio and the static deformation! ($\omega_\text{n} = 25.82\,\text{rad/s}$, $\omega_\text{d} = 14.28\,\text{rad/s}$, $\zeta = 0.6$, $\lambda = 2.03$, $f_0 = 0.105\,\text{m}$)
3. Determine the stationary motion $y_\text{p}(t) = Y\sin(\omega t + \delta - \vartheta)$of the vehicle! ($Y = 0.0266\,\text{m}$, $\delta = 1.181\,\text{rad}$, $\vartheta = 2.478\,\text{rad}$)!
4. Determine the maximum damping force $F_{\text{d,max}}$ during the stationary motion! ($F_{\text{d,max}} = 20257\,\text{N}$)

1. SZTÁ



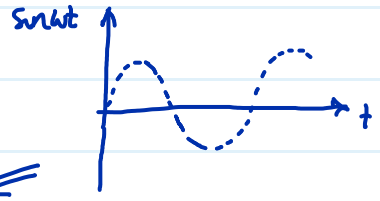
y elmozdulás nyújtja a rugót

Din. alapt.

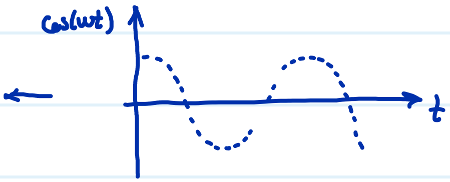
$$m\ddot{y} = -F_k - F_c$$

$$F_k = k\left(y - R\sin(\omega t)\right)$$

$$F_c = c\left(\dot{y} - \dot{r}(t)\right) = c\left(\dot{y} - R\omega\cos(\omega t)\right)$$



↳ útgerjesztés először nyomja a rugót! (ellentétes F_k irányával)



$\dot{r}(t)$ iránya pozitív $\dot{y}$-nal megegyezik

↳ az autó kerékhez képest sebessége $\dot{y} - \dot{r}(t)$

Mozg. egy.. $m\ddot{y} + c\dot{y} + ky = kR\sin(\omega t) + cR\omega\cos(\omega t)$

ahol $v \cdot T_\omega = L \Rightarrow T_\omega = \frac{L}{v} = 0{,}12s \Rightarrow \omega T_\omega = 2\pi \rightarrow \omega = \frac{2\pi}{T_\omega} = 52{,}36$ rad/s

2. Ref. alak: $\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = f_0\omega_n^2\sin(\omega t + \delta)$

$$kR\sin(\omega t) + c\omega R\cos(\omega t) = R\sqrt{k^2+c^2\omega^2}\left[\frac{k}{\sqrt{k^2+c^2\omega^2}}\sin(\omega t) + \frac{c\,\omega}{\sqrt{k^2+c^2\omega^2}}\cos(\omega t)\right] =$$

$$= R\sqrt{k^2+c^2\omega^2}\left[\cos\delta\sin(\omega t) + \sin\delta\cos(\omega t)\right] = R\sqrt{k^2+c^2\omega^2}\sin(\omega t + \delta)$$

$\cos\delta = \frac{k}{\sqrt{k^2+c^2\omega^2}} \longrightarrow \delta = 1{,}181$ rad $\qquad \omega_n = \sqrt{\frac{k}{m}} = 25{,}82 \frac{rad}{s}$

$2\zeta\omega_n = \frac{c}{m} \longrightarrow \zeta = \frac{1}{2}\frac{c}{\omega_n m} = 0{,}6[-]$; $\omega_d = \omega_n\sqrt{1-\zeta^2} = 20{,}65$ rad/s

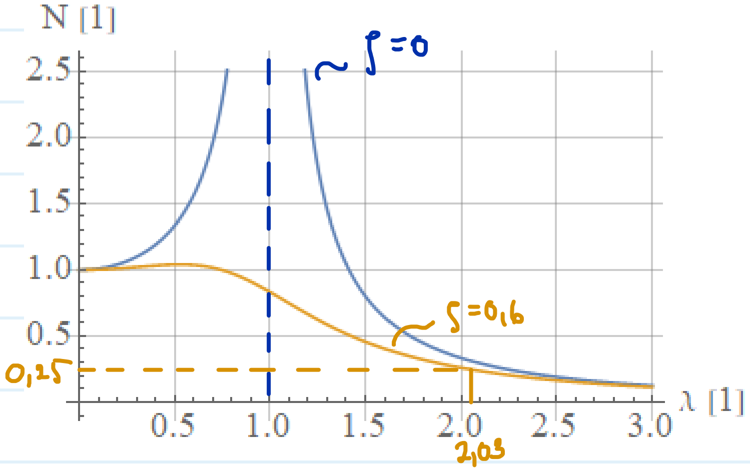
$\lambda = \frac{\omega}{\omega_n} = 2{,}0279\,[-]$, $f_0\,\omega_n^2 = R\sqrt{k^2+c^2\omega^2} \rightarrow f_0 = \frac{R}{m\omega_n^2}\sqrt{k^2+c^2\omega^2} = 0{,}1053\,m$

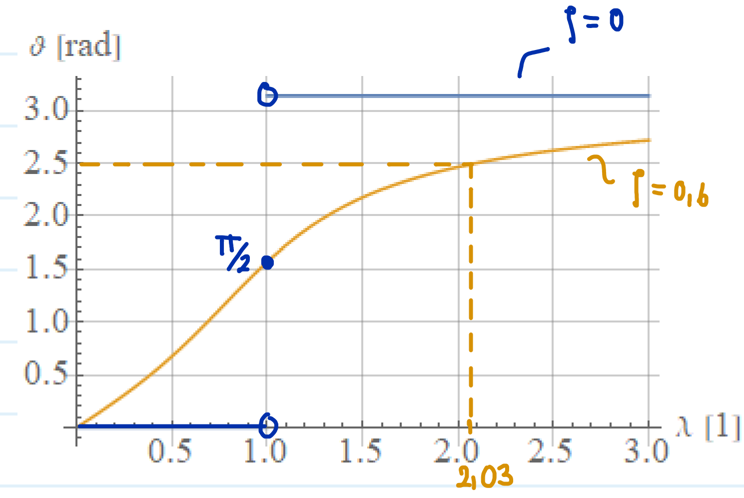
3. $y_p = Y\sin(\omega t + \delta - \vartheta)$

$N = \frac{1}{\sqrt{(1-\lambda^2)^2 + 4\zeta^2\lambda^2}} = 0{,}2531 \qquad Y = N f_0 = 0{,}02664\,m$

$\vartheta = \mathrm{arctg}\left(\frac{2\zeta\lambda}{1-\lambda^2}\right) = -0{,}6638 + k\pi \rightarrow \vartheta = 2{,}4778$ rad

$y_p = 0{,}0266\sin(52{,}36t - 1{,}2967)$





4. $F_c = c\left(\dot{y}(t) - \dot{r}(t)\right)$, ha t nagy $\rightarrow y(t) \approx y_p(t)$

$\dot{y}_p(t) = Y\omega\cos(\omega t + \delta - \vartheta)$

feladat szerint az állandósult megoldásra kell megmondani $F_{c\,max}$ -ot

$$F_c = c\left(Y\omega\,\underbrace{\cos(\omega t + \delta - \vartheta)} - R\omega\cos(\omega t)\right)$$

$\beta = \delta - \vartheta$

$$\cos(\omega t + \beta) = \cos(\omega t)\cos\beta - \sin(\omega t)\sin\beta$$

$$c\omega\left[\cos(\omega t)\left(Y\cos\beta - R\right) - \sin(\omega t)\sin\beta \cdot Y\right]$$

$$A = (Y\cos\beta - R)^2 + Y^2\sin^2\beta = Y^2\underbrace{(\cos^2\beta + \sin^2\beta)}_{=1} - 2YR\cos\beta + R^2$$

$$c\omega\sqrt{A}\left[\frac{Y\cos\beta - R}{\sqrt{A}}\cos\omega t - \frac{Y\sin\beta}{\sqrt{A}}\sin\omega t\right] =$$

$$= c\omega\sqrt{A}\left[\cos\gamma\cos\omega t - \sin\gamma\sin\omega t\right] = c\omega\sqrt{A}\cos(\omega t + \gamma)$$

$$\Rightarrow \quad -c\omega\sqrt{A} \leq F_c \leq c\omega\sqrt{A} \longrightarrow F_{c\,max} = 20{,}3\ \text{kN}$$

1 DoF gerjesztett csillapított lengőkar

In Fig. 1, a welded structure is shown that consists of two rods with different lengths and masses, and a disk with radius R. The structure can only rotate along joint A. The horizontal rod is connected to the environment through a spring with stiffness k_1 and a damper with c_1 damping coefficient, while at point B a harmonic force excitation is applied. On the vertical rod a harmonic displacement excitation is applied through a spring with stiffness k_2. The deflection angle of the structure is measured from the horizontal axis by the general coordinate φ. The structure is in the gravitational field and its equilibrium position is located at $\varphi = 0$, where the spring of stiffness k_1 has static deformation only.

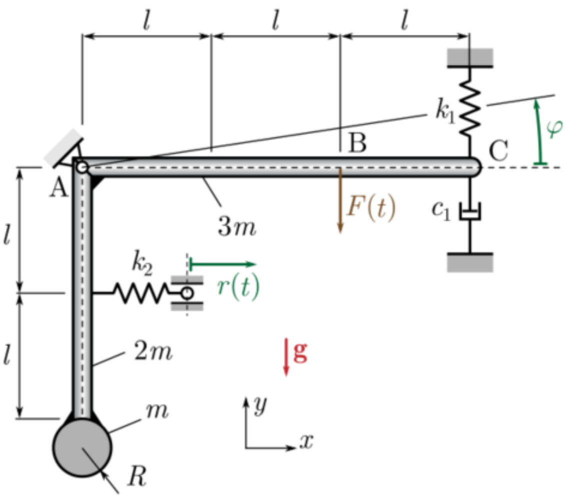


Fig. 1: Mechanical modeling of the structure

Data

$m = 0.12\,\mathrm{kg}$ $l = 0.2$ m $R = 0.1$ m $r(t) = r_0 \sin(\omega t)$ $r_0 = 0.01$ m $\omega = 20\,\mathrm{rad/s}$

$k_1 = 300\,\mathrm{N/m}$ $k_2 = 10\,\mathrm{N/m}$ $c_1 = 2\,\mathrm{Ns/m}$ $F(t) = F_0 \sin(\omega t)$ $F_0 = 2$ N

Tasks

1. Derive the equation of motion for the small vibrations about the equilibrium using the Lagrange equation of the second kind! Calculate the undamped natural angular frequency, the damped natural angular frequency, the damping ratio and the static deformation! ($\omega_n = 35.55$ rad/s, $\omega_d = 35.31$ rad/s, $\zeta = 0.117$, $f_0 = -0.00713$ rad)
2. Calculate the stationary solution of the system! ($\varphi_p(t) = -0.0102 \sin(20t - 0.19)$)

Másodfajú Lagrange-egyenlet

$$\frac{d}{dt}\frac{dT}{d\dot{q}_k} - \frac{dT}{dq_k} + \frac{dD}{d\dot{q}_k} + \frac{dU}{dq_k} = Q_k^* \qquad k = 1,\dots,n \qquad Q_k^* = \sum_i F_i^* \frac{\partial r_i}{\partial q_k}$$

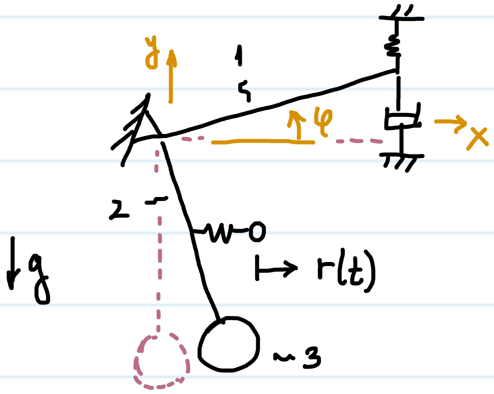
1

$$T = \frac{1}{2} m v_s^2 + \frac{1}{2}\Theta_s \dot{\varphi}^2 = \frac{1}{2}\Theta_A \dot{\varphi}^2 \quad \leftarrow \text{kinetikus energia}$$

itt v_s-t ki kéne fejezni $\dot{\varphi}$ függvényében

$$\Theta_A = \frac{1}{3} 3m (3l)^2 + \frac{1}{3} 2m (2l)^2 + \frac{1}{2} m R^2 + m (2l + R)^2 = 0{,}0866 \text{ kgm}^2$$

$$\frac{dT}{d\varphi} = 0 \quad ; \quad \frac{d}{dt}\frac{dT}{d\dot{\varphi}} = \frac{d}{dt}\left(\Theta_A \dot{\varphi}\right) = \Theta_A \ddot{\varphi}$$



$$\underline{r}_{S1}(\varphi) = \frac{3}{2}\ell \begin{bmatrix} \cos\varphi \\ \sin\varphi \end{bmatrix} ; \quad \underline{r}_{S2}(\varphi) = \ell \begin{bmatrix} \sin\varphi \\ -\cos\varphi \end{bmatrix}$$

$$\underline{r}_{S3}(\varphi) = (2\ell + R) \begin{bmatrix} \sin\varphi \\ -\cos\varphi \end{bmatrix}$$

$$U_g = -mg(2\ell + R)\cos\varphi - 2mg\ell\cos\varphi + 3mg\frac{3}{2}\ell\sin\varphi$$

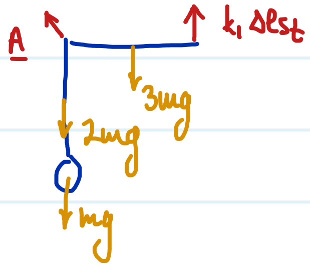
$$U_r = \frac{1}{2}k_1(\Delta\ell_{st} - 3\ell\sin\varphi)^2 + \frac{1}{2}k_2(r(t) - \ell\sin\varphi)^2$$

$$U = U_g + U_r$$

$$\frac{\partial U}{\partial\varphi} = mg(2\ell+R)\sin\varphi + 2mg\ell\sin\varphi + \frac{9}{2}mg\ell\cos\varphi - k_1(\Delta\ell_{st} - 3\ell\sin\varphi)3\ell\cos\varphi - k_2(r(t) - \ell\sin\varphi)\ell\cos\varphi$$

kis elmozdulásokra: $\frac{dU}{d\varphi} = \varphi\left[mg(2\ell + R + 2\ell) + k_1 9\ell^2 + k_2\ell^2\right] + \frac{9}{2}mg\ell - k_1\Delta\ell_{st}3\ell - k_2\ell r(t)$

SZTA' egyensúlynál



$$\Sigma M_z = 0: \quad -3mg\frac{3}{2}\ell + 3\ell k_1\Delta\ell_{st} = 0$$

$$\Delta\ell_{st} = \frac{3mg}{2k_1}$$

$$\frac{dU}{d\varphi} = \varphi\left[mg(4\ell + R) + 9\ell^2 k_1 + \ell^2 k_2\right] - k_2\ell r(t)$$

Rayleigh-féle disszipatív függvény: $D = \frac{1}{2}c\cdot\Delta v^2$

$$\underline{r}_c = 3\ell \begin{bmatrix} \cos\varphi \\ \sin\varphi \end{bmatrix} \rightarrow \underline{v}_c = 3\ell\dot\varphi \begin{bmatrix} -\sin\varphi \\ \cos\varphi \end{bmatrix}$$

$$v_c^2 = (3\ell\dot\varphi)^2 \left[\underbrace{(-\sin\varphi)^2 + \cos\varphi^2}_{=1}\right]$$

$$D = \frac{1}{2}c_1(3\ell\dot\varphi)^2 ; \quad \frac{dD}{d\dot\varphi} = c_1 9\ell^2\dot\varphi$$

külső erők teljesítménye: $P = \underline{F}(t)\cdot\underline{v}_B$, $\quad \underline{F}(t) = \begin{bmatrix} 0 \\ -F(t) \end{bmatrix}$

$$\underline{r}_B = 2\ell \begin{bmatrix} \cos\varphi \\ \sin\varphi \end{bmatrix} \rightarrow \underline{v}_B = 2\ell\dot{\varphi} \begin{bmatrix} -\sin\varphi \\ \cos\varphi \end{bmatrix}$$

$$\frac{\partial r_B}{\partial \varphi} = 2\ell \begin{bmatrix} -\sin\varphi \\ \cos\varphi \end{bmatrix}$$

$$P = -2\ell\dot{\varphi}\cos\varphi\, F(t) \cong -2\ell\dot{\varphi}\, F(t) = Q^* \dot{\varphi}$$

$$Q^* = -2\ell F(t)\cos\varphi$$

$$Q^* \cong -2\ell F(t)$$

$$Q^* = -2\ell F(t)$$

Mozg. egy.:

$$\Theta_A\ddot{\varphi} + g_c\ell^2\dot{\varphi} + \left(mg(4\ell+R) + g\ell^2 k_1 + \ell^2 k_2\right)\varphi = \sin(\omega t)\left[k_2 \ell r_0 - F_0 2\ell\right]$$

Ref. alak: $\ddot{\varphi} + 2\zeta\omega_n\dot{\varphi} + \omega_n^2\varphi = f_0\omega_n^2 \sin(\omega t)$

$$\omega_n = \sqrt{\frac{mg(4\ell+R) + g\ell^2 k_1 + \ell^2 k_2}{\Theta_A}} = 35{,}55\ \frac{\text{rad}}{\text{s}}\ ; \quad \zeta = \frac{1}{2\omega_n}\cdot\frac{g_c\ell^2}{\Theta_A} = 0{,}117\,[-]$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 35{,}31\ \frac{\text{rad}}{\text{s}}\ ; \quad f_0 = \frac{k_2\ell r_0 - F_0 2\ell}{\Theta_A\,\omega_n^2} = -0{,}007126\ \text{rad}$$

2 $\lambda = \dfrac{\omega}{\omega_n} = 0{,}5626\,[-]\ ; \quad N = \dfrac{1}{\sqrt{(1-\lambda^2)^2 + 4\zeta^2\lambda^2}} = 1{,}436\ ; \quad \phi = N f_0 = -0{,}01024\ \text{rad}$

$$\vartheta = \text{arctg}\left(\frac{2\zeta\lambda}{1-\lambda^2}\right) = 0{,}1901\ \text{rad}\ , \quad \varphi_p(t) = \phi\sin(\omega t - \vartheta)$$

$$\varphi_p(t) = -0{,}0102\sin(20t - 0{,}19)$$

Rázóasztal

In Fig. 1, a vibration exciter is shown, which consists of a rigid body with total mass m, four symmetrically rotating eccentric masses with angular velocity ω, eccentricity r, and mass m_0. According to the simplified mechanical model, the moving rigid body is connected to the base with an ideal spring and damper with stiffness k and damping factor c, respectively. To describe the motion of the single-degree-of-freedom vibration exciter, the vertical displacement y measured from the static equilibrium is used as a generalized coordinate. In the equilibrium position ($y = 0$) the spring is preloaded, which results a force acting against the gravity.

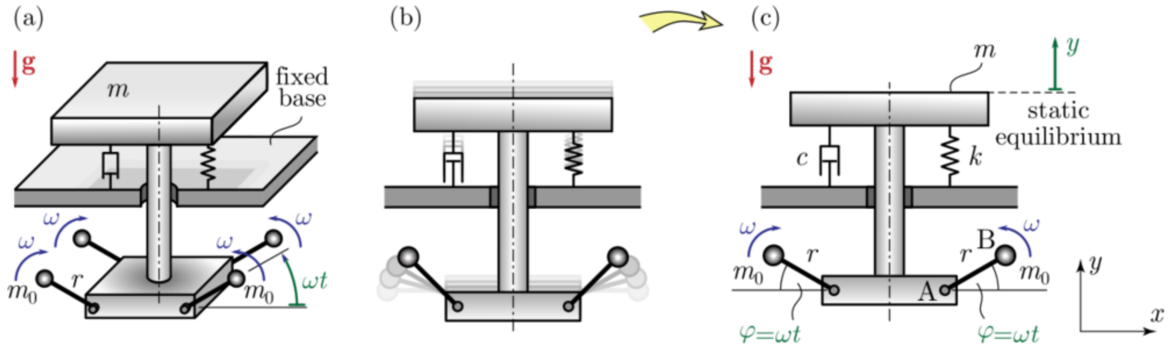


Fig. 1: Mechanical model of a vibration exciter. (a) 3D view. (b) Vibration exciter in operation. (c) Frontal view.

Data

$m = 60\,\mathrm{kg}$ $m_0 = 0.5\,\mathrm{kg}$ (×4 pieces) $r = 0.1\,\mathrm{m}$
$\omega = 75\,\mathrm{rad/s}$ (= const.) $k = 25000\,\mathrm{N/m}$

Tasks

1. Determine the damping factor c for which the relative damping ratio of the system is $\zeta = 0.05$!
2. Determine the amplitude Y of displacement of the stationary solution $y_p(t)$!
3. What is the maximum of the force $F_{b,\max}$, which acts on the fixed base during the stationary motion?

} lásd moodle kidolgozásban

1 Másodfajú Lagrange: $\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k} + \frac{\partial U}{\partial q_k} + \frac{\partial D}{\partial \dot{q}_k} = Q_k^*$

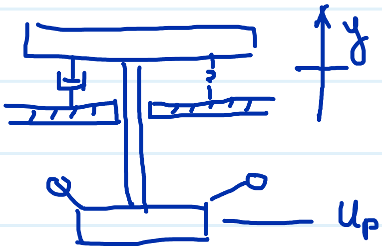
$$T = \frac{1}{2} m \dot{y}^2 + 4 \cdot \frac{1}{2} m_0 v_0^2$$

$$\underline{v}_B = \underline{v}_A + \underline{\omega} \times \underline{r}_{AB} = \begin{bmatrix} 0 \\ \dot{y} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} \times r \begin{bmatrix} \cos(\omega t) \\ \sin(\omega t) \\ 0 \end{bmatrix} = \begin{bmatrix} -r\omega \sin(\omega t) \\ \dot{y} + \omega r \cos(\omega t) \\ 0 \end{bmatrix}$$

$$v_0^2 = \underline{v}_B \cdot \underline{v}_B = \dot{y}^2 + 2\dot{y} r \cos(\omega t) + r^2 \omega^2$$

$$T = \frac{1}{2}m\dot{y}^2 + 2m_0\left(\dot{y}^2 + 2\dot{y}r\omega\cos(\omega t) + r^2\omega^2\right)$$

legyen ott, ahol a rugó terheletlen



bárhol megválasztható ez az y_0 szint

$U_{pot} = 0$ szint

$$U = \underbrace{U_0 + mgy}_{\text{asztal pot energiája}} + 4m_0g\left(r\sin(\omega t) + y - y_0\right) + \frac{1}{2}k(y-y_0)^2$$

$$D = \frac{1}{2}c\dot{y}^2$$

$$\frac{\partial T}{\partial y} = 0 \qquad \frac{\partial T}{\partial \dot{y}} = m\dot{y} + 4m_0\dot{y} + 4m_0 r\omega\cos(\omega t)$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{y}} = \ddot{y}(m + 4m_0) - 4m_0 r\omega^2\sin(\omega t)$$

$$\frac{\partial U}{\partial y} = \underbrace{mg + 4m_0g - ky_0}_{} + ky = ky$$

= 0, statikus erők egyensúlyban vannak

(mozg egyenletben is látszik, ha benn hagyjuk)

$$\frac{\partial D}{\partial \dot{y}} = c\dot{y}$$

Mozg. egy.: $(m+4m_0)\ddot{y} + c\dot{y} + ky + g(m+4m_0) - ky_0 = 4m_0 r\omega^2 \sin(\omega t)$

egyensúlyi helyzetben: $\ddot{y}=0; \dot{y}=0$ és $y=0$ (ilyenkor nincs gerjesztés)

$$\Rightarrow g(m+4m_0) - ky_0 = 0$$

Azaz: $\ddot{y}(m+4m_0) + c\dot{y} + ky = 4m_0 r\omega^2 \sin(\omega t)$

3 $F_b = F_{stat} + F_r(t) + F_c(t)$

ahol $F_{stat} = ky_0 = g(m+4m_0)$

és legyen t kellően nagy:

$F_r(t) \cong y_p(t)\cdot k$ $\quad$ $F_c(t) \cong \dot{y}_p(t)\cdot c$

Többet lásd a moodle kidolgozásban

(hasonló az 1DoF negyedjármű 4. alfeladatához)

2 DoF rendszer

In Fig. 1, a 2 DoF system is shown, which consists of a cart and a pendulum. The cart is modelled by a rigid body that can move in the horizontal direction only, its mass is denoted by m_1 and the pendulum is modelled by a rod of length l and mass m_2. The rod can only rotate about the joint B. Two different types of excitations are applied: harmonic force excitation $F(t) = F_0 \sin(\omega t + \varepsilon)$ at the lower endpoint C of the pendulum and harmonic displacement excitation $r(t) = r_0 \cos(\omega t)$ through a spring connected to the cart of stiffness k. To describe the motion of the corresponding 2 DoF system, the generalised coordinates x_1 and x_2 are introduced. Here, x_1 is the displacement of the centre of gravity of the cart and x_2 is the displacement of the point C of the rod. The structure is in the vertical plane (the gravitational acceleration is g) and its equilibrium position is located at $x_1 = 0$ and $x_2 = 0$.

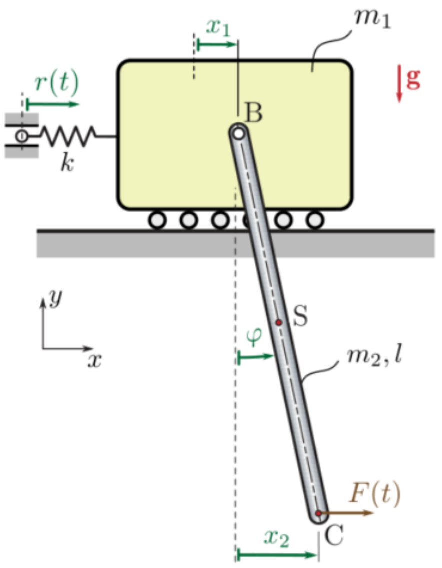


Fig. 1: Mechanical model of the 2 DoF system

most gerjesztés nélkül nézzük!

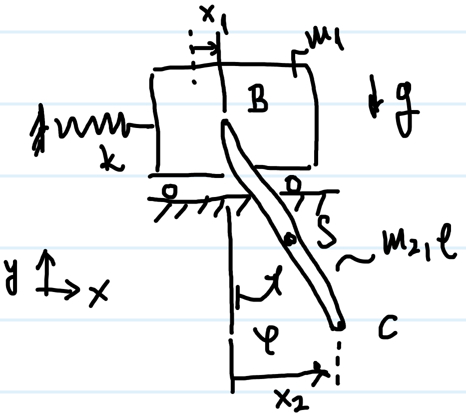
Data

$m_1 = 2\,\mathrm{kg}$	$l = 0.5\,\mathrm{m}$	$\omega = 20\,\mathrm{rad/s}$
$m_2 = 1\,\mathrm{kg}$	$F_0 = 5\,\mathrm{N}$	$\varepsilon = \pi/3$
$k = 100\,\mathrm{N/m}$	$r_0 = 0.01\,\mathrm{m}$	

Tasks

1. Derive the matrix formulation of the equation of motion for the model assuming small oscillations around the equilibrium!
2. Determine the stationary solution of the system! ← nem vizsgáljuk
3. Calculate the maximal excursion of the rod in the stationary solution!

+ ω_n ?
+ lengéskép ?



általános koordináták. $q = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Másodfajú Lagrange-hoz kell T, U

$$T_1 = \frac{1}{2} m_1 \dot{x}_1^2$$

$$T_2 = \frac{1}{2} m_2 v_{S_2}^2 + \frac{1}{2} \Theta_S \dot{\varphi}^2$$

$$\underline{v}_{S_2} = \underline{v}_B + \underline{\omega} \times \underline{r}_{BS} = \begin{bmatrix} \dot{x}_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\varphi} \end{bmatrix} \times \frac{l}{2} \begin{bmatrix} \sin\varphi \\ -\cos\varphi \\ 0 \end{bmatrix} = \begin{bmatrix} \dot{x}_1 + l/2 \cos\varphi \cdot \dot{\varphi} \\ \dot{\varphi}\, l/2 \sin\varphi \\ 0 \end{bmatrix}$$

$$v_{S_2}^2 = \underline{v}_{S2} \cdot \underline{v}_{S2} = \dot{x}_1^2 + \dot{x}_1 \dot{\varphi} \cos\varphi \cdot l + \dot{\varphi}^2 l^2/4$$

$\Theta_1 = \frac{1}{12} m_2 \ell^2$ $\qquad$ $T_2 = \frac{1}{2} m_2 \left[\dot{x}_1^2 + \dot{x}_1 \dot{\varphi} \ell \cos\varphi + \dot{\varphi}^2 \left(\underbrace{\frac{\ell^2}{4} + \frac{\ell^2}{12}}_{\ell^2/3} \right) \right]$

$$T = T_1 + T_2 = \frac{1}{2} \left[\dot{x}_1^2 (m_1 + m_2) + m_2 \left(\dot{x}_1 \dot{\varphi} \ell \cos\varphi + \dot{\varphi}^2 \ell^2 / 3 \right) \right]$$

x_2 relatív pozíció → $x_2 = \ell \sin\varphi$ → $\dot{x}_2 = \ell \dot{\varphi} \cos\varphi$

kis rezgésekre

$x_2 = \ell \varphi \quad \hookrightarrow \varphi = \frac{x_2}{\ell}$ $\qquad$ $\dot{x}_2 = \ell \dot{\varphi} \quad \hookrightarrow \dot{\varphi} = \frac{\dot{x}_2}{\ell}$

$$T \cong \frac{1}{2} \left[\dot{x}_1^2 (m_1 + m_2) + m_2 \left(\dot{x}_1 \dot{x}_2 \cos(x_2/\ell) + \dot{x}_2^2 / 3 \right) \right]$$

$U_1 = 0$ $\qquad$ $U_2 = -m_2 g \frac{\ell}{2} \cos\varphi \cong -m_2 g \frac{\ell}{2} \cos(x_2/\ell)$

$U_r = \frac{1}{2} k x_1^2$ $\qquad$ $U = U_1 + U_2 + U_r \cong \frac{1}{2} k x_1^2 - m_2 g \frac{\ell}{2} \cos(x_2/\ell)$

Másodfajú Lagrange: $\frac{d}{dt} \frac{\partial T}{\partial \dot{\underline{q}}} - \frac{\partial T}{\partial \underline{q}} + \frac{\partial U}{\partial \underline{q}} + \frac{\partial D}{\partial \dot{\underline{q}}} = \underline{Q}(t)$

Ezt linearizálva: $\underline{\underline{M}} \ddot{\underline{q}} + \underline{\underline{C}} \dot{\underline{q}} + \underline{\underline{K}} \underline{q} = \underline{Q}(t)$ (egyensúlyi helyzet körül)

Azaz $\quad \underline{\underline{M}} = \left[\frac{\partial^2 T}{\partial \dot{q}_i \partial \dot{q}_j} \bigg|_{q_0} \right]$ $\qquad$ $\underline{\underline{K}} = \left[\frac{\partial^2 U}{\partial q_i \partial q_j} \bigg|_{q_0} \right]$

↳ most csak ez a 2 lesz

$\underline{\underline{M}} \ddot{\underline{q}} + \underline{\underline{K}} \underline{q} = \underline{0} \longrightarrow \left(-\underline{\underline{M}} \omega_n^2 + \underline{\underline{K}} \right) \underline{A} = \underline{0} \Rightarrow \det\left(\underline{\underline{K}} - \omega_n^2 \underline{\underline{M}} \right) = 0$

próbafgv.

$\underline{q}_H = \underline{A} e^{i \omega_n t}$

$\left(\underline{\underline{M}}^{-1} \underline{\underline{K}} - \omega_n^2 \underline{\underline{E}} \right) \underline{A} = \underline{0}$

ω_n^2 a sajátérték

illetve a lengéskép ($\underline{A}$) a sajátvektor

$$\frac{\partial T}{\partial \dot{x}_1} = \dot{x}_1 (m_1 + m_2) + \dot{x}_2 m_2 \cos(x_2/l)/2$$

$$\frac{\partial T}{\partial \dot{x}_2} = \frac{1}{2} \dot{x}_1 m_2 \cos(x_2/l) + \dot{x}_2 m_2/3$$

$$\frac{\partial^2 T}{\partial \dot{x}_1 \partial \dot{x}_1} = m_1 + m_2 \longrightarrow \left.\frac{\partial^2 T}{\partial \dot{x}_1^2}\right|_{q=0} = m_1 + m_2$$

$$\frac{\partial^2 T}{\partial \dot{x}_1 \partial \dot{x}_2} = \frac{1}{2} m_2 \cos(x_2/l) \longrightarrow \left.\frac{\partial^2 T}{\partial \dot{x}_1 \partial \dot{x}_2}\right|_{q=0} = \frac{1}{2} m_2$$

$$\frac{\partial^2 T}{\partial \dot{x}_2 \partial \dot{x}_2} = \frac{1}{3} m_2 \longrightarrow \left.\frac{\partial^2 T}{\partial^2 \dot{x}_2^2}\right|_{q=0} = \frac{1}{3} m_2$$

$$\underline{\underline{M}} = \begin{bmatrix} m_1 + m_2 & \frac{1}{2} m_2 \\ \frac{1}{2} m_2 & \frac{1}{3} m_2 \end{bmatrix} = \begin{bmatrix} 3 & 1/2 \\ 1/2 & 1/3 \end{bmatrix} \text{kg}$$

TÖMEGMÁTRIX

$$\frac{\partial U}{\partial x_1} = k x_1 \qquad \frac{\partial U}{\partial x_2} = m_2 g \sin(x_2/l)/2$$

$$\frac{\partial^2 U}{\partial x_1 \partial x_1} = k \longrightarrow \left.\frac{\partial^2 U}{\partial x_1^2}\right|_{q=0} = k$$

$$\frac{\partial^2 U}{\partial x_1 \partial x_2} = 0$$

$$\frac{\partial^2 U}{\partial x_2 \partial x_2} = \frac{m_2 g}{2l} \cos(x_2/l) \longrightarrow \left.\frac{\partial^2 U}{\partial x_2^2}\right|_{q=0} = \frac{m_2 g}{2l}$$

$$\underline{\underline{K}} = \begin{bmatrix} k & 0 \\ 0 & m_2 g/2l \end{bmatrix} = \begin{bmatrix} 100 & 0 \\ 0 & 9{,}81 \end{bmatrix} \text{N/m}$$

MEREVSÉGI MÁTRIX

lin. mozg. egy.: $\begin{bmatrix} 3 & 1/2 \\ 1/2 & 1/3 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 100 & 0 \\ 0 & 9{,}81 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\det(\underline{\underline{K}} - \omega_n^2 \underline{\underline{M}}) = \begin{vmatrix} 100 - 3\omega_n^2 & -\frac{1}{2}\omega_n^2 \\ -\frac{1}{2}\omega_n^2 & 9{,}81 - \frac{1}{3}\omega_n^2 \end{vmatrix} = 0{,}75\omega_n^4 - 62{,}76\omega_n^2 + 9{,}81 = 0$$

$\omega_{n_1}^2 = 20{,}8 \ 1/s^2 \longrightarrow \omega_{n_1} = 4{,}56$ rad/s

$\omega_{n_2}^2 = 62{,}88 \ 1/s^2 \longrightarrow \omega_{n_2} = 7{,}93$ rad/s

1. és 2. sajátfrekvenciák
(sajátérték gyöke)

i. lengéskép (sajátvektor): $(\underline{\underline{K}} - \omega_{n_i}^2 \underline{\underline{M}})\underline{A}_i = \underline{0}$

$\underline{A}_i$ irányt jelöl!

A_{1i} szabadon választható, legyen 1 m

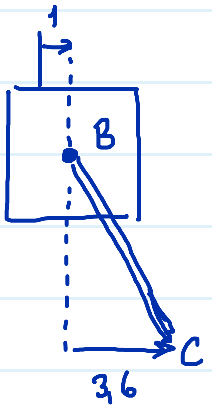
$$\begin{bmatrix} 100 - 3\omega_{n_i}^2 & -\frac{1}{2}\omega_{n_i}^2 \\ -\frac{1}{2}\omega_{n_i}^2 & 9{,}81 - \frac{1}{3}\omega_{n_i}^2 \end{bmatrix} \begin{bmatrix} 1 \\ A_{2i} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$A_{12} = 2 \cdot \frac{100 - \omega_{n_i}^2 \cdot 3}{\omega_{n_i}^2} \longrightarrow A_{12} = 3{,}62 \text{ m} \qquad \longrightarrow A_{22} = -2{,}82 \text{ m}$$

1. lengéskép

$\underline{A}_1 = \begin{bmatrix} 1 \\ 3{,}62 \end{bmatrix}$ m

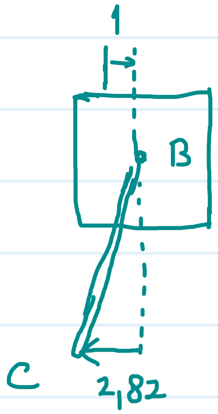
$\omega_{n_1} = 4{,}56 \frac{\text{rad}}{s}$



2. lengéskép

$\underline{A}_2 = \begin{bmatrix} 1 \\ -2{,}82 \end{bmatrix}$ m

$\omega_{n_2} = 7{,}93 \frac{\text{rad}}{s}$



2 DoF gerjesztett rendszer

In Fig. 1, a two-degree-of-freedom system is shown which consists of two rigid bodies: a disk with mass m_1 and radius R and a block with mass m_2. The disk is rolling on a horizontal surface, and its center is connected to the environment through a spring with stiffness k_1. The block is moving vertically in the gravitational field between frictionless linear guideways and it is connected to a spring with stiffness k_2. The other side of this spring is connected to an ideal (inextensible) rope which is fixed to the center of gravity of the disk. The rope is changing direction on an ideal (massless/frictionless) pully, and we assume that due to the pretension of the rope it will remain tensioned during the vibrations.

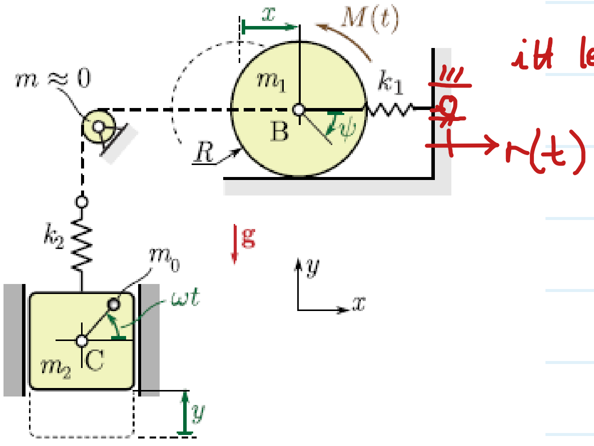


Fig. 1: Mechanical model

itt legyen útgerjesztés

The forced vibration is created by the harmonically varying moment $M(t) = M_0 \cos(\omega t + \varepsilon)$ acting on the disk and the rotating eccentric mass (with angular velocity ω, eccentricity e and mass m_0) connected to the block.

$M_0 \sin(\omega t)$ legyen

In the equilibrium position ($y = 0, \psi = 0$) the springs are preloaded, which results a force acting against the gravitational force.

Data

$m_0 = 0.1\,\mathrm{kg}$	$m_1 = 1\,\mathrm{kg}$	$m_2 = 3\,\mathrm{kg}$	$R = 0.2\,\mathrm{m}$	$e = 0.01\,\mathrm{m}$
$k_1 = 100\,\mathrm{N/m}$	$k_2 = 200\,\mathrm{N/m}$	$M_0 = 3\,\mathrm{Nm}$	$\omega = 30\,\mathrm{rad/s}$	$\varepsilon = \pi/6$

$r(t) = r_0 \sin(\omega t)$

$r_0 = 0{,}01\ m$

Tasks

1. Derive the linear equation of motion with matrix coefficients!
2. Determine the particular part of the law of motion!
3. What is the maximum force in spring k_2 during the stationary vibration?

+4. Determine the natural angular frequencies (ω_{nk}) and the corresponding modeshapes ($\mathbf{A}_k$)!

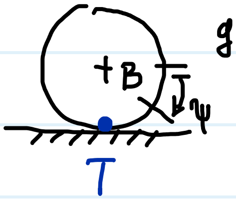
1

$$\underline{q} = \begin{bmatrix} y \\ \psi \end{bmatrix} \qquad \underline{\underline{M}}\,\ddot{\underline{q}} + \underline{\underline{C}}\,\dot{\underline{q}} + \underline{\underline{K}}\,\underline{q} = \underline{Q}^*(t)$$

Másodfajú Lagrange egyensúlyi helyzet körül linearizálással

$$\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k} + \frac{\partial D}{\partial \dot{q}_k} + \frac{\partial U}{\partial q_k} = Q_k^* \quad ; \; k = 1, \ldots, n\right)$$

$$T = \underbrace{\frac{1}{2} m_1 v_B^2 + \frac{1}{2}\Theta_B \omega_1^2}_{\text{kerék}} + \underset{\text{doboz}}{\frac{1}{2} m_2 \dot{y}^2} + \underset{\text{tömegpont}}{\frac{1}{2} m_0 v_0^2}$$



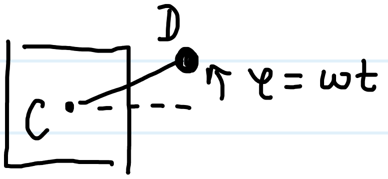
gördül

$\hookrightarrow \underline{v}_T = \underline{0}$

$$\underline{v}_B = \underline{v}_T + \underline{\omega}_1 \times \underline{r}_{TB} = \begin{bmatrix} 0 \\ 0 \\ -\dot{\psi} \end{bmatrix} \times \begin{bmatrix} 0 \\ R \\ 0 \end{bmatrix} = \begin{bmatrix} R\dot{\psi} \\ 0 \\ 0 \end{bmatrix}$$

$$\underline{\omega}_1 = \begin{bmatrix} 0 \\ 0 \\ -\dot{\psi} \end{bmatrix}$$

$$v_B^2 = \underline{v}_B \cdot \underline{v}_B = R^2 \dot{\psi}^2 ; \qquad \Theta_B = \frac{1}{2} m_1 R^2$$



$$\underline{v}_C = \begin{bmatrix} 0 \\ \dot{y} \\ 0 \end{bmatrix} ; \quad \underline{v}_D = \underline{v}_C + \underline{\omega}_3 \times \underline{r}_{CD} ; \quad \underline{r}_{CD} = e \begin{bmatrix} \cos(\omega t) \\ \sin(\omega t) \\ 0 \end{bmatrix}$$

$$\underline{v}_D = \begin{bmatrix} 0 \\ \dot{y} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} \times e \begin{bmatrix} \cos(\omega t) \\ \sin(\omega t) \\ 0 \end{bmatrix} = \begin{bmatrix} -\omega e \sin(\omega t) \\ \dot{y} + \omega e \cos(\omega t) \\ 0 \end{bmatrix}$$

$$v_D^2 = \underline{v}_D \cdot \underline{v}_D = \dot{y}^2 + 2 \omega e \cos(\omega t) \cdot \dot{y} + e^2 \omega^2$$

$$T = \frac{3}{4} m_1 R^2 \dot{\psi}^2 + \frac{1}{2} \left[\dot{y}^2 (m_2 + m_0) + m_0 2 \dot{y} \omega e \cos(\omega t) + m_0 e^2 \omega^2 \right]$$

① $\frac{\partial T}{\partial y} = 0$; ② $\frac{\partial T}{\partial \psi} = 0$; $\frac{\partial T}{\partial \dot{y}} = (m_2 + m_0) \dot{y} + m_0 \omega e \cos(\omega t)$; $\frac{\partial T}{\partial \dot{\psi}} = \frac{3}{2} m_1 R^2 \dot{\psi}$

$\frac{d}{dt} \frac{\partial T}{\partial \dot{y}} = \ddot{y} (m_2 + m_0) - m_0 e \omega^2 \sin(\omega t)$ ① ; $\frac{d}{dt} \frac{\partial T}{\partial \dot{\psi}} = \ddot{\psi} \frac{3}{2} m_1 R^2$ ②

$U_p = 0$ legyen $y = 0$ m egyensúlyi helyzet körül

$$U_1 = m_1 g H ; \quad U_2 = m_2 g y ; \quad U_0 = m_0 g (y + e \sin(\omega t))$$

$$U_{r_1} = \frac{1}{2} k_1 (x - x_{st} - r(t))^2 ; \quad x = R\psi ; \quad x_{st} = R \psi_{st}$$

$$U_{r_2} = \frac{1}{2} k_2 (y - y_{st} - R\psi)^2$$

$$U = g \left[m_1 H + y (m_2 + m_0) + m_0 e \sin(\omega t) \right] + \frac{1}{2} \left[k_1 (R(\psi - \psi_{st}) - r_0 \sin(\omega t))^2 + k_2 (y - y_{st} - R\psi)^2 \right]$$

$\frac{\partial U}{\partial y} = g(m_2 + m_0) + k_2 (y - y_{st} - R\psi)$ ① ; $\frac{\partial U}{\partial \psi} = k_1 (R(\psi - \psi_{st}) - r_0 \sin(\omega t)) R - k_2 (y - y_{st} - R\psi) R$ ②

$$P_M = \underline{M}(t)\cdot\underline{\omega}_1 = \begin{bmatrix}0\\0\\M_0\sin(\omega t)\end{bmatrix}\cdot\begin{bmatrix}0\\0\\-\dot\psi\end{bmatrix} = -M_0\dot\psi\sin(\omega t) = \hat{\underline{Q}}\cdot\dot{\underline{q}}$$

$$\dot{\underline{q}} = \begin{bmatrix}\dot y\\ \dot\psi\end{bmatrix}$$

$$\frac{\partial P_M}{\partial \dot y} = 0 \quad ① \quad ; \quad \frac{\partial P_M}{\partial \dot\psi} = -M_0\sin(\omega t) \quad ②$$

$$\rightarrow \hat{\underline{Q}} = \begin{bmatrix}0\\ -M_0\sin(\omega t)\end{bmatrix}$$

Tehát

$$\left.\begin{aligned}&\ddot y(m_0+m_2) - m_0 e\omega^2\sin(\omega t) + g(m_2+m_0) + k_2 y - k_2 y_{st} - k_2 R\psi = 0 \quad ①\\ &\ddot\psi\tfrac{3}{2}m_1R^2 + k_1R^2\psi - k_1R^2\psi_{st} - k_1 r_0\sin(\omega t)R - k_2Ry + k_2Ry_{st} + k_2R^2\psi = -M_0\sin(\omega t) \quad ②\end{aligned}\right\}$$

$$\underline{\underline{M}}\,\ddot{\underline{q}} + \underline{\underline{K}}\,\underline{q} = \underline{Q}^*(t)$$

statikus tagok kiejtik egymást

mivel az állapotváltozók $(q, \dot q)$ lineárisan szerepelnek, nem kell linearizálni

$$\underline{\underline{M}} = \begin{bmatrix}m_2+m_0 & 0\\ 0 & \tfrac{3}{2}m_1R^2\end{bmatrix} \quad ; \quad \underline{\underline{K}} = \begin{bmatrix}k_2 & -k_2R\\ -k_2R & R^2(k_1+k_2)\end{bmatrix}$$

statikus tagok: ① : $(m_2+m_0)g = k_2 y_{st}$ ② . $k_1R\psi_{st} = k_2 y_{st}$

időfüggő tagok: (jobboldalra rendezve) $\underline{Q}^*(t) = \begin{bmatrix}m_0 e\omega^2\sin(\omega t)\\ (k_1r_0R - M_0)\sin(\omega t)\end{bmatrix}$

Mozg. egy.: (mátrixos alakja)

$$\begin{bmatrix}m_2+m_0 & 0\\ 0 & \tfrac{3}{2}m_1R^2\end{bmatrix}\begin{bmatrix}\ddot y\\ \ddot\psi\end{bmatrix} + \begin{bmatrix}k_2 & -k_2R\\ -k_2R & R^2(k_1+k_2)\end{bmatrix}\begin{bmatrix}y\\ \psi\end{bmatrix} = \begin{bmatrix}m_0 e\,\omega^2\sin(\omega t)\\ (k_1r_0R - M_0)\sin(\omega t)\end{bmatrix}$$

2

$$\underline{q}_p = \underline{L}\cos(\omega t) + \underline{N}\sin(\omega t) \rightarrow \dot{\underline{q}}_p = -\underline{L}\,\omega\sin(\omega t) + \underline{N}\,\omega\cos(\omega t) = \omega\left[\underline{N}\cos(\omega t) - \underline{L}\sin(\omega t)\right]$$

$$\ddot{\underline{q}}_p = -\underline{L}\,\omega^2\cos(\omega t) - \underline{N}\,\omega^2\sin(\omega t) = -\omega^2\left[\underline{L}\cos(\omega t) + \underline{N}\sin(\omega t)\right]$$

$$\underline{\underline{M}}\,\ddot{\underline{q}}_p + \underline{\underline{K}}\,\underline{q}_p = \underline{Q}^*(t) \rightarrow -\omega^2\underline{\underline{M}}(\underline{L}\cos\omega t + \underline{N}\sin\omega t) + \underline{\underline{K}}(\underline{L}\cos\omega t + \underline{N}\sin\omega t) = \underline{Q}^*(t)$$

$$\underline{L} = \begin{bmatrix}L_1\\ L_2\end{bmatrix} \quad ; \quad \underline{N} = \begin{bmatrix}N_1\\ N_2\end{bmatrix}$$

$$(-\omega^2\underline{\underline{M}} + \underline{\underline{K}})(\underline{L}\cos\omega t + \underline{N}\sin\omega t) = \underline{Q}^*(t)$$

cos és sin függvények együtthatóira igaz az egyenlet

$$(-\omega^2\underline{\underline{M}} + \underline{\underline{K}})\underline{N}\sin(\omega t) = \underline{Q}^*(t) \qquad (-\omega^2\underline{\underline{M}} + \underline{\underline{K}})\underline{L}\cos\omega t = \underline{0}$$

$\uparrow$ $\underline{L} = \underline{0}$

$$\begin{bmatrix} -\omega^2(m_2+m_0)+k_2 & -k_2 R \\ -k_2 R & -\frac{3}{2}\omega^2 m_1 R^2 + R^2(k_1+k_2) \end{bmatrix}\begin{bmatrix} N_1 \\ N_2 \end{bmatrix}\cancel{\sin(\omega t)} = \begin{bmatrix} m_0 e \omega^2 \\ k_1 r_0 R - M_0 \end{bmatrix}\cancel{\sin(\omega t)}$$

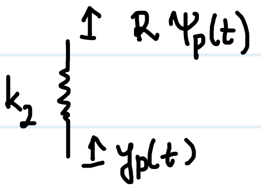
$$\left.\begin{aligned} &(-\omega^2(m_2+m_0)+k_2)N_1 - N_2 k_2 R = m_0 e \omega^2 \\ &-N_1 k_2 R + \left(-\frac{3}{2}\omega^2 m_1 R^2 + R^2(k_1+k_2)\right)N_2 = k_1 r_0 R - M_0 \end{aligned}\right\}$$ 2 egyenlet, 2 ismeretlen

$N_1 = -0{,}0014\,m$; $N_2 = 0{,}068$ rad $\Rightarrow$ $q_p(t) = \begin{bmatrix} -0{,}0014\sin(\omega t) \\ 0{,}068\sin(\omega t) \end{bmatrix} = \begin{bmatrix} y_p(t) \\ \psi_p(t) \end{bmatrix}$

3 statikus tagok : $k_1 R\psi_{st} = k_2 y_{st} = (m_2+m_0)g$

$F_{r2stat} = k_2 y_{st} = (m_2+m_0)g = 30{,}411$ N (nyújtott)

illetve



$\Delta l(t) = R\psi_p(t) - y_p(t) = (RN_2 - N_1)\sin(\omega t)$

$\Delta l(t) = 0{,}015\ m = \Delta l_{max}$

$F_{r2max} = F_{r2stat} + k_2\Delta l_{max} = 33{,}41$ N

4 homogén részből ($q_H = \underline{A}e^{i\omega_n t}$) $\qquad \underline{\underline{M}}\ddot{q} + \underline{\underline{K}}q = \underline{0}$

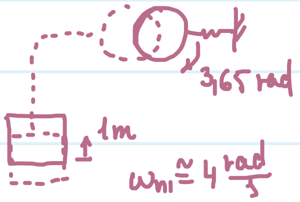
$\det(-\omega_n^2\underline{\underline{M}} + \underline{\underline{K}}) = 0 \longrightarrow \omega_{n1} = 4{,}172$ rad/s ; $\omega_{n2} = 15{,}72$ rad/s

$(-\omega_{ni}^2\underline{\underline{M}} + \underline{\underline{K}})\underline{A}_i = \underline{0}$

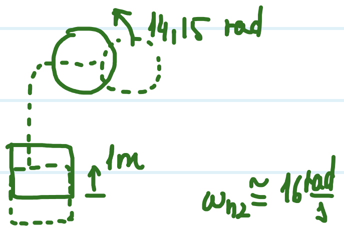
legyen $A_{i1} = 1$

$\hookrightarrow \underline{A}_1 = \begin{bmatrix} 1 \\ 3{,}65 \end{bmatrix}$ [SI] ; $\underline{A}_2 = \begin{bmatrix} 1 \\ -14{,}15 \end{bmatrix}$ [SI]

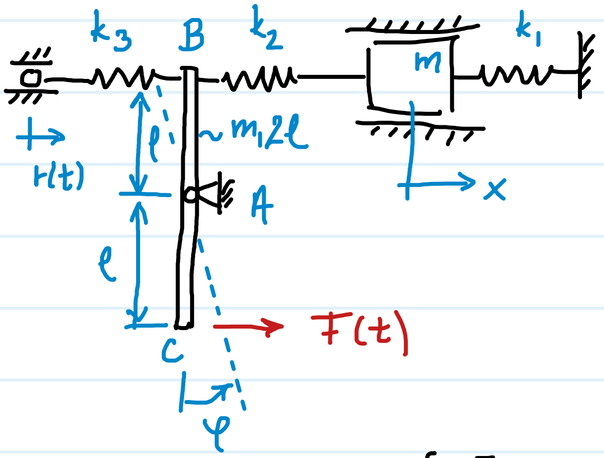
1. lengéskép



2. lengéskép



2DoF gerjesztett rendszer



vízszintes sík

Adatok

$m = 3\,kg$

$\ell = 0{,}5\,m$

$k_1 = k_2 = k_3 = 100\,\frac{N}{m}$

$\omega = 10\,rad/s$

$r(t) = r_0 \sin \omega t$

$F(t) = F_0 \sin \omega t$

$r_0 = 0{,}02\,m$

$F_0 = 12\,N$

$$q = \begin{bmatrix} x \\ \varphi \end{bmatrix}$$

kérdések

a) linearizált mozgásegyenlet?

b) állandósult állapot?

c) k_2-es rugóban ébredő maximális erő? (állandósult állapotban)

a) Másodfajú Lagrange $\frac{d}{dt}\frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q} + \frac{\partial D}{\partial \dot{q}} + \frac{\partial U}{\partial q} = \underline{Q}^*$

$$T = T_1 + T_2 = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} \Theta_A \dot{\varphi}^2 = \frac{1}{2} m \left(\dot{x}^2 + \frac{1}{3} \ell^2 \dot{\varphi}^2 \right)$$

$$\Theta_A = \Theta_S = \frac{1}{12} m 4 \ell^2 = \frac{1}{3} m \ell^2$$

$$\frac{\partial T}{\partial x} = 0, \quad \frac{\partial T}{\partial \varphi} = 0, \quad \frac{\partial T}{\partial \dot{x}} = m\dot{x}; \quad \frac{d}{dt}\frac{\partial T}{\partial \dot{x}} = m\ddot{x}; \quad \frac{\partial T}{\partial \dot{\varphi}} = \frac{1}{3} m \ell^2 \dot{\varphi}$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\varphi}} = \frac{1}{3} m \ell^2 \ddot{\varphi}$$

$$U = U_r + U_g = \frac{1}{2} k_1 x^2 + \frac{1}{2} k_2 (x + \ell \sin\varphi)^2 + \frac{1}{2} k_3 (\ell \sin\varphi + r(t))^2$$

↳ ≡ konstans, mert a testek súlypontjai nem mozdulnak el

$$\frac{\partial U}{\partial x} = k_1 x + k_2 x + k_2 \ell \sin\varphi \cong x(k_1+k_2) + k_2 \ell \varphi$$

$$\frac{\partial U}{\partial \varphi} = k_2(x+\ell\sin\varphi)\ell\cos\varphi + k_3(\ell\sin\varphi + r(t))\ell\cos\varphi \cong k_2\ell x + \ell^2\varphi(k_2+k_3) + k_3\ell r(t)$$

$$D \equiv 0 \rightarrow \frac{\partial D}{\partial \underline{\dot{q}}} = \underline{0}$$

$$P = \underline{F}(t)\cdot\underline{v}_C = \begin{bmatrix} F(t) \\ 0 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} \dot\varphi \ell\cos\varphi \\ \dot\varphi \ell \sin\varphi \\ 0 \end{bmatrix} = F(t)\dot\varphi\ell\cos\varphi \cong \underbrace{F(t)\dot\varphi\ell = \underline{Q}^*(t)\cdot\underline{\dot{q}}}$$

$$\Downarrow$$

$$\underline{v}_C = \underline{v}_A + \underline{\omega}\times\underline{r}_{AC} = \begin{bmatrix} 0 \\ 0 \\ \dot\varphi \end{bmatrix} \times \begin{bmatrix} \ell\sin\varphi \\ -\ell\cos\varphi \\ 0 \end{bmatrix} = \begin{bmatrix} \dot\varphi\ell\cos\varphi \\ \dot\varphi\ell\sin\varphi \\ 0 \end{bmatrix}$$

$$\underline{Q}^*(t) = \begin{bmatrix} 0 \\ F(t)\cdot\ell \end{bmatrix}$$

Tehát

$$\begin{cases} m\ddot{x} + x(k_1+k_2) + k_2\ell\varphi = 0 \\ \frac{1}{3}m\ell^2\ddot\varphi + k_2\ell x + \ell^2\varphi(k_2+k_3) + k_3\ell r(t) = \ell F(t) \end{cases}$$

$$\Rightarrow \underline{\underline{M}}\,\underline{\ddot{q}} + \underline{\underline{C}}\,\underline{\dot{q}} + \underline{\underline{K}}\,\underline{q} = \underline{Q}^*(t)$$

$$\underline{\underline{M}} = \begin{bmatrix} m & 0 \\ 0 & \frac{1}{3}m\ell^2 \end{bmatrix}; \quad \underline{\underline{K}} = \begin{bmatrix} k_1+k_2 & k_2\ell \\ k_2\ell & \ell^2(k_2+k_3) \end{bmatrix}; \quad \underline{Q}^*(t) = \begin{bmatrix} 0 \\ \ell F_0 - k_3\ell r_0 \end{bmatrix}\sin\omega t$$

Azaz

$$\begin{bmatrix} m & 0 \\ 0 & \frac{1}{3}m\ell^2 \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot\varphi \end{bmatrix} + \begin{bmatrix} k_1+k_2 & k_2\ell \\ k_2\ell & \ell^2(k_2+k_3) \end{bmatrix}\begin{bmatrix} x \\ \varphi \end{bmatrix} = \begin{bmatrix} 0 \\ \ell F_0 - k_3\ell r_0 \end{bmatrix}\sin\omega t$$

b) $$\underline{q}_p(t) = \underline{N}\sin(\omega t) + \underline{L}\cos(\omega t) \underset{\uparrow}{=} \underline{N}\sin(\omega t)$$

mivel a gerjesztés sin alakú

$$\underline{\underline{M}}\,\underline{\ddot{q}}_p + \underline{\underline{K}}\,\underline{q}_p = \underline{Q}^*(t)$$

$$(-\omega^2\underline{\underline{M}} + \underline{\underline{K}})\,\underline{N}\sin(\omega t) = \underline{Q}^*(t)$$

$$\begin{bmatrix} -\omega^2 m + k_1 + k_2 & k_2 \ell \\ k_2 \ell & -\frac{\omega^2}{3} m\ell^2 + \ell^2(k_2+k_3) \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \sin\omega t = \begin{bmatrix} 0 \\ \ell F_0 - k_3 \ell r_0 \end{bmatrix} \sin\omega t$$

$$\left.\begin{aligned} k_2 \ell N_2 &= N_1(\omega^2 m - k_1 - k_2) \\ N_1 k_2 \ell &= N_2\left(\frac{\omega^2}{3} m\ell^2 - \ell^2(k_2+k_3)\right) + \ell F_0 - k_3 \ell r_0 \end{aligned}\right\} \begin{matrix} \text{2 egyenlet} \\ \text{2 ismeretlen} \end{matrix}$$

$N_1 = 0{,}05\,m$ $N_2 = 0{,}1\,rad$

$$q_p = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} \sin\omega t = \begin{bmatrix} x_p(t) \\ \varphi_p(t) \end{bmatrix}$$

c) $$F_{r2} = k_2(x_p + \ell \sin\varphi_p) \cong k_2(x_p + \ell\varphi_p) = k_2(N_1 \sin\omega t + \ell N_2 \sin\omega t)$$

$$F_{r2max} = \max(k_2 \sin\omega t(N_1 + \ell N_2)) = k_2(N_1 + \ell N_2) = 10N$$