

# Ütközés

## Example

In Fig. 1, an 1 DoF oscillator is shown that consists of a beam of mass  $m_2$  and a torsional spring of stiffness  $k_t$ . The beam can rotate about the pin at point A. The system is in the gravitational field, the preload of the torsional spring ensures that the horizontal position of the beam is the equilibrium of the oscillator. The vibration of the steady beam is induced by the impact between the beam and the lumped mass  $m_1$ . Before the impact, the lumped mass free falls from the height  $h$  starting with zero speed.

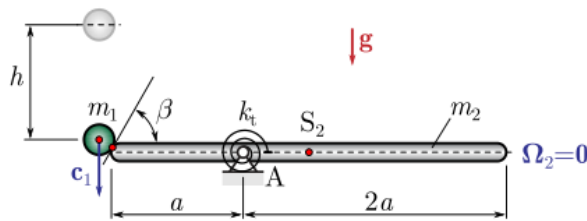


Fig. 1: The investigated system

## Data

$$m_1 = 6 \text{ kg} \quad m_2 = 6 \text{ kg}$$

$$a = 0.3 \text{ m} \quad \beta = 60^\circ$$

$$h = 0.115 \text{ m} \quad e = 1$$

→ tökéletesen rugalmas ütközés

## Task

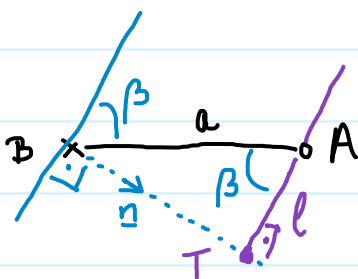
1. Determine the velocity of the lumped mass and the angular velocity of the beam after the impact!

$$m_1 g h = \frac{1}{2} m_1 c_1^2 \Rightarrow c_1 = \sqrt{2gh} = 1,5 \text{ m/s}$$

Allo tengely körüli ütközés - centrális ütközés

$$m_{T2} = \frac{\Theta_A}{\ell^2} \longrightarrow \Theta_A = \Theta_S + \Delta r^2 \cdot m_2 \longrightarrow \begin{cases} \Delta r = 1,5a - a = 0,5a \\ \Theta_S = \frac{1}{12} m_2 (3a)^2 \end{cases}$$

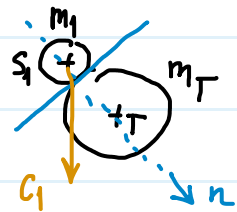
$$\Theta_A = 0,54 \text{ kgm}^2$$



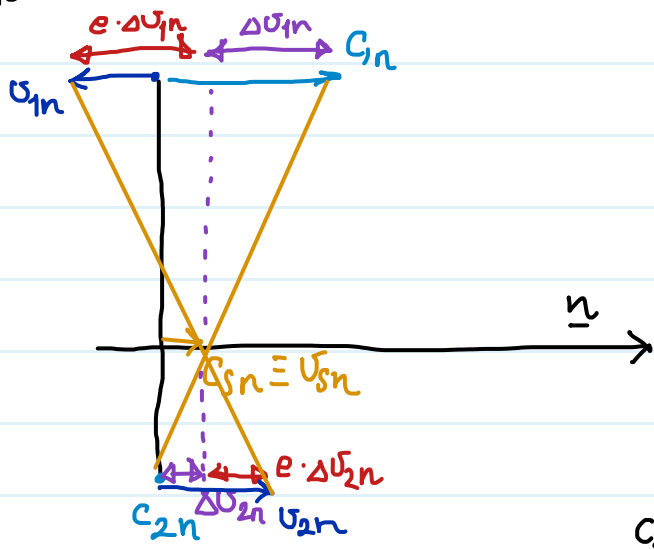
$$\ell = a \cos \beta = 0,15 \text{ m}$$

$$m_{T2} = 24 \text{ kg}$$

$\mu = 0 \Rightarrow$  csak normál irányú sebességkamp. változik!

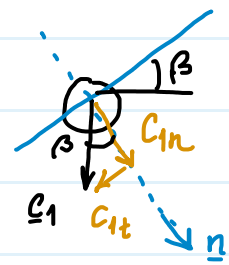


Egyszerűsített Maxwell-ábra



$C_{2n} = 0$

$C_{1n} = C_1 \cos \beta = 0,75 \text{ m/s}$



$$C_{sn} = \frac{C_{2n} \cdot m_{T2} + C_{1n} m_1}{m_{T2} + m_1} = 0,15 \text{ m/s} = v_{sn}$$

$\Delta v_1 = C_{1n} - C_{sn} = 0,6 \text{ m/s}$

$\Delta v_2 = C_{2n} - C_{sn} = -0,15 \text{ m/s}$

$v_{1n} = v_{sn} - e \Delta v_1 = -0,45 \text{ m/s}$

$v_{2n} = v_{sn} - e \Delta v_2 = 0,15 - (-0,15) = 0,3 \text{ m/s}$

ell.: 
$$v_{sn} = \frac{-0,45 \cdot m_1 + 0,3 \cdot m_{T2}}{m_1 + m_{T2}} = 0,15 \text{ m/s} \checkmark$$

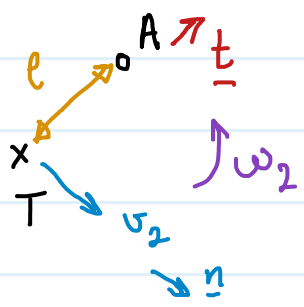
$v_{1t} = C_{1t}$

$v_{2t} = C_{2t} = 0 \Rightarrow v_2 = 0,3 \text{ m/s} \leftarrow T \text{ norm sebessége}$

$C_{1t} = C_1 \sin \beta = 1,3 \text{ m/s} \Rightarrow v_1 = \sqrt{v_{1n}^2 + v_{1t}^2} = 1,375 \text{ m/s}$

$v_n = 0 \text{ m/s}$

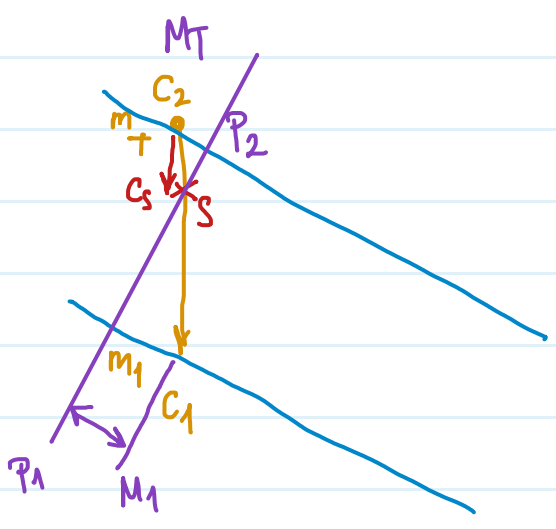
$\omega_2 = \frac{v_2}{l} = 2 \frac{\text{rad}}{\text{s}}$



$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \omega_2 \end{pmatrix} \times \begin{pmatrix} 0 \\ l \\ 0 \end{pmatrix} + \begin{pmatrix} v_2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -\omega_2 l + v_2 \\ 0 \\ 0 \end{pmatrix}$$

$$v_n = v_2 + \omega_2 \times r_{AT}$$
  
 $[n, t, z]$

# Maxwell data



## 1 DoF csillapítatlan rendszer

In Fig. 1, a 1 DoF oscillator is shown that consists of the mass  $m_1$ , the springs of stiffness  $k_1$  and  $k_2$ . As it can be seen one of the springs connects to a cantilever beam, which length is  $l$ , its cross section is characterized by  $a$  and  $b$ , the elastic modulus of its material refers to  $E$ . The mass of the beam is negligible. The displacement of the mass  $m_1$  is described by the generalized coordinate  $x$ . The vibration of the system is induced by the impact between the mass  $m_1$  and lumped mass  $m_2$ .

### Data

$a = 0.006 \text{ m}$	$b = 0.025 \text{ m}$
$l = 0.5 \text{ m}$	$E = 200 \text{ GPa}$
$m_1 = 5 \text{ kg}$	$m_2 = 1 \text{ kg}$
$k_1 = 100 \text{ N/m}$	$k_2 = 50 \text{ N/m}$
$c_1 = 0 \text{ m/s}$	$c_2 = 0.6 \text{ m/s}$
$e = 0.5$	

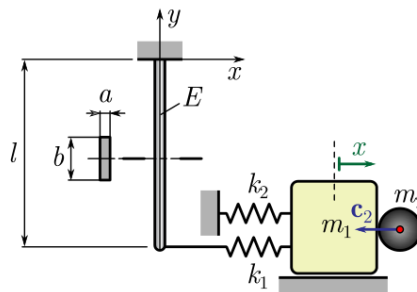


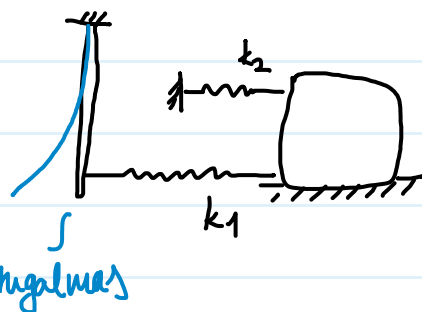
Fig. 1: The investigated system

### Tasks

1. Determine the natural angular frequency of the system!
2. Calculate the maximal displacement, velocity and acceleration of the oscillation that is generated by the impact! Sketch the time histories of the displacement, velocity and acceleration!

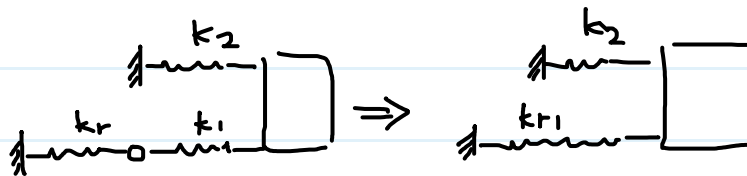
a)

Eredő rugóerősség



Castigliano:  $\frac{F l^3}{3 \frac{1}{2} E} = f \Rightarrow f = \frac{F}{k_r} \Rightarrow k_r = \frac{3 \frac{1}{2} E}{l^3}$

$$\frac{1}{2} = \frac{a^3 b}{12}$$



Soros rugó: eredő rugóerősségi:  $F = k_r \xi_r = k_1 \xi_1 = k_{r1} (\xi_1 + \xi_r) = k_{r1} \xi$

$$\xi = \xi_r + \xi_1$$

$$k_r \xi_r = k_1 (\xi - \xi_r) = k_{r1} \xi$$

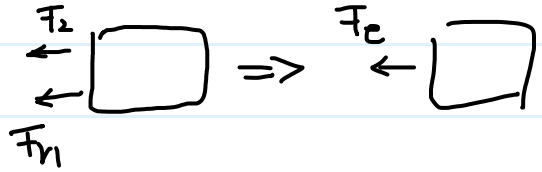
$$\xi_r (k_r + k_1) = k_1 \xi$$

$$\xi_r = k_1 \xi / (k_1 + k_r)$$

$$k_r \xi_r = k_{r1} \xi \Rightarrow \frac{k_1 k_r}{k_1 + k_r} \xi = k_{r1} \xi \Rightarrow k_{r1} = \frac{k_1 k_r}{k_1 + k_r}$$

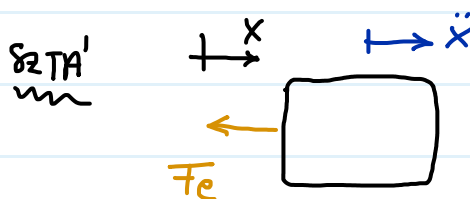
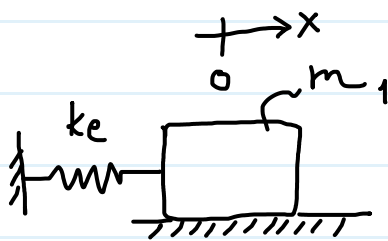
Parhuzamos rugó: elmozdulás ugyanahányi   $\Rightarrow$  

$$F_e = F_{r1} + F_2$$



$$k_e \eta = k_{r1} \eta + k_2 \eta$$

$$k_e = k_{r1} + k_2 = k_2 + \frac{k_1 k_r}{k_1 + k_r} = 145,575 \text{ N/m}$$



Din. alapt.

$$x: m_1 \ddot{x} = -F_e = -k_e x$$

$$F_e = k_e x \quad \uparrow$$

mozgásegyenlet:  $\ddot{x} + \frac{k_e}{m_1} x = 0$

referenciaalak:  $\ddot{x} + \omega_n^2 x = 0$

$$\omega_n = \sqrt{\frac{k_e}{m_1}} = 5,39 \frac{\text{rad}}{\text{s}}$$

$$f_n = \frac{\omega_n}{2\pi} = 0,858 \text{ Hz}$$

$$(T_n = 1/f_n = 1,166 \text{ s})$$

$\theta_1 \quad \ddot{x} + \omega_n^2 x = 0$

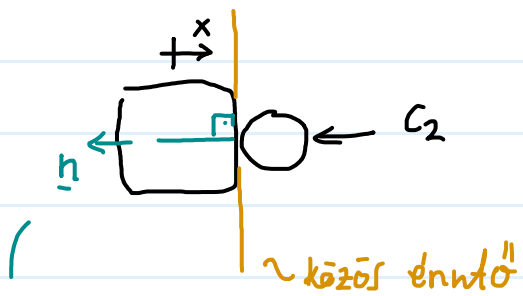
$x_0 = 0 \text{ m}$

$x(0) = x_0$

$v_0 \leftarrow$  ütközés utáni sebesség

$\dot{x}(0) = v_0$

centrikus ütközés



$n = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$C_{2n} = C_2$

$C_{1n} = 0$

ütközés normális

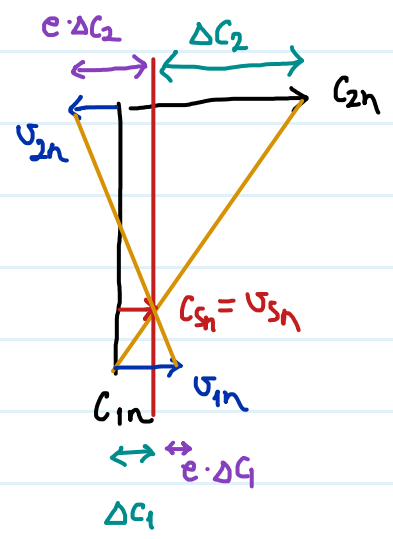
$C_{sn} = \frac{m_1 C_{1n} + m_2 C_{2n}}{m_1 + m_2} = 0,1 \text{ m/s}$

$\Delta C_1 = C_{sn} - C_{1n} = 0,1 \text{ m/s}$

$v_{1n} = C_{sn} + e \cdot \Delta C_1 = 0,15 \text{ m/s}$

$v_0 = -0,15 \text{ m/s}$  (mert  $\leftarrow n$  DE!  $\rightarrow x$ )

$\Delta C_2 = C_{sn} - C_{2n} = -0,5 \text{ m/s}$   
 $v_{2n} = C_{sn} + e \cdot \Delta C_2 = -0,15 \text{ m/s}$



Próba függvény

$x(t) = A e^{\lambda t} \rightarrow \dot{x}(t) = A \lambda^2 e^{\lambda t}$

$A \lambda^2 e^{\lambda t} + \omega_n^2 A e^{\lambda t} = 0 \Rightarrow$  karakterisztikus egyenlet:  $\lambda^2 + \omega_n^2 = 0$

$\lambda^2 = -\omega_n^2 = -\frac{ke}{m_1}$

$\lambda = \pm \omega_n \sqrt{-1} = \pm \sqrt{-\frac{ke}{m_1}} \in \mathbb{C}$

mozgástörvény:  $x(t) = C_1 \cos(\omega_n t) + C_2 \sin(\omega_n t)$

$x(0) = C_1 = 0$

$\dot{x}(t) = -C_1 \omega_n \sin(\omega_n t) + C_2 \omega_n \cos(\omega_n t)$

$$\dot{x}(0) = C_2 \omega_n = v_0 \Rightarrow C_2 = v_0 / \omega_n = -0,0278 \text{ m}$$

$$x(t) = -0,0278 \sin(5,39 \cdot t)$$

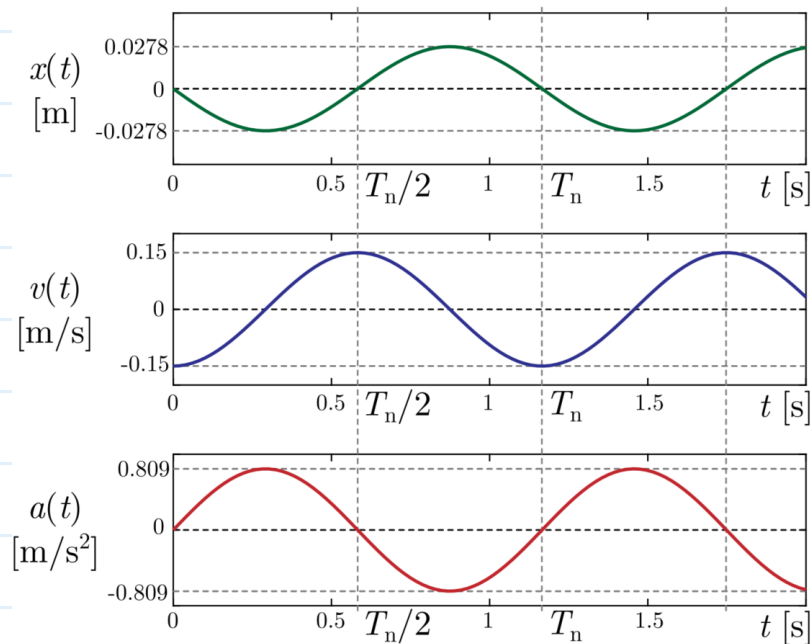
$$\dot{x}(t) = -0,15 \cos(5,39t)$$

$$\ddot{x}(t) = 0,809 \sin(5,39t)$$

$$x_{\max} = 0,0278 \text{ m}$$

$$v_{\max} = 0,15 \text{ m/s}$$

$$a_{\max} = 0,809 \text{ m/s}^2$$



## 1 DoF lengőkar

The swinging arm shown in Fig. 1 consists of two mass-less rods and three lumped masses  $m_1$ ,  $m_2$  and  $m_3$ . The swinging arm can only rotate about the joint A. To describe the motion of the swinging arm, the angle  $\varphi$ , measured from the horizontal axis, is used as generalized coordinate. The arm is in the gravitational field. The equilibrium position is located at  $\varphi = 0$ , which is ensured by the preload in the spring of stiffness  $k$ .

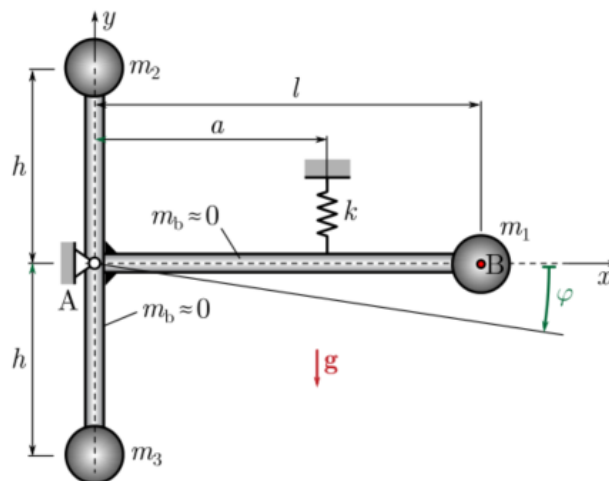


Fig. 1: The mechanical model of the swinging arm

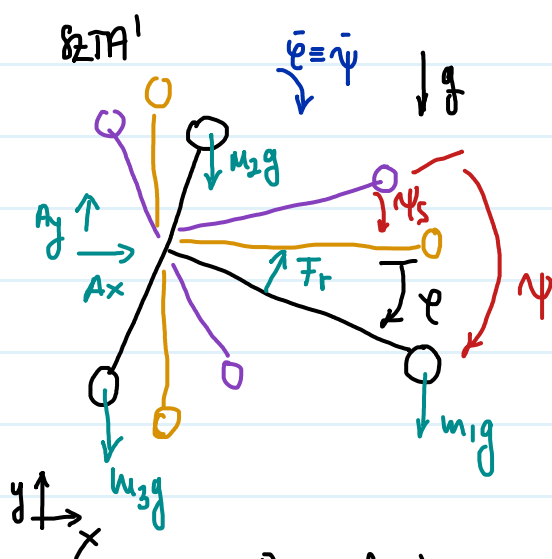
### Data

$m_1 = 2 \text{ kg}$	$l = 1 \text{ m}$
$m_2 = 4 \text{ kg}$	$h = 0.5 \text{ m}$
$m_3 = 3 \text{ kg}$	$a = 0.6 \text{ m}$
$k = 10^4 \text{ N/m}$	$g = 9.81 \text{ m/s}^2$

### Tasks

1. Derive the equation of motion for small oscillation around the equilibrium!
2. Calculate the natural angular frequency, natural frequency and the period of oscillation! ( $\omega_n = 30.96 \text{ rad/s}$ ,  $f_n = 4.93 \text{ Hz}$ ,  $T_n = 0.203 \text{ s}$ )
3. Determine the law of motion for the initial condition given by the vertical position  $y_B(t=0) = -0.01 \text{ m}$  and velocity  $v_{By}(t=0) = -1 \text{ m/s}$  of the point B! ( $C_1 = 0.01 \text{ rad}$ ,  $C_2 = 0.033 \text{ rad}$ )
4. Calculate the maximum force in the spring for the given initial condition! ( $F_{r,\max} = 235.56 \text{ N}$ )

1.



- egyensúly
- rugó megrúgódás nélkül
- kitérítelt

his kitérés!

$$F_r(\varphi) \approx F_{rs} + k a \varphi$$

$$M_r(\varphi) \approx M_{rs} + k a^2 \varphi = F_{rs} a + k a^2 \varphi$$

$$\text{Din. alapt.: } z: -\Theta_A \ddot{\varphi} = -m_1 g l \cos \varphi + M_r(\varphi) - m_2 g h \sin \varphi + m_3 g h \sin \varphi$$

$$\Rightarrow -\Theta_A \ddot{\varphi} = -m_1 g l + F_{rs} a + k a^2 \varphi - m_2 g h \varphi + m_3 g h \varphi$$



egyensúlynál  $\varphi=0; \dot{\varphi}=0; \ddot{\varphi}=0 \Rightarrow 0 = -m_1 g l + F_{Ts} a \Rightarrow F_{Ts} = \frac{m_1 g l}{a}$

így válaszoltuk meg!

egyensúly feltétele

Tehát a mozg. egy.:  $\Theta_A \ddot{\varphi} + \varphi [ka^2 + m_3 g h - m_2 g h] = 0$

$$\Theta_A = m_1 \cdot l^2 + h^2 (m_2 + m_3)$$

$$\Rightarrow \ddot{\varphi} (m_1 l^2 + h^2 (m_2 + m_3)) + \varphi [ka^2 + m_3 g h - m_2 g h] = 0$$

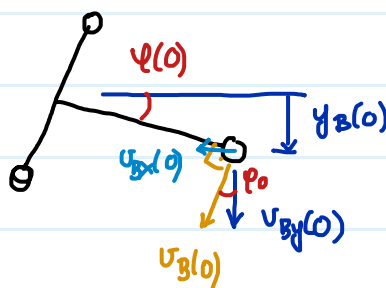
2. Ref. alak:  $\ddot{\varphi} + \omega_n^2 \varphi = 0$

$$\Rightarrow \omega_n = \sqrt{\frac{ka^2 + g h (m_3 - m_2)}{m_1 l^2 + h^2 (m_2 + m_3)}} = 30,96 \text{ rad/s}$$

$$f_n = \frac{\omega_n}{2\pi} = 4,928 \text{ 1/s}$$

$$T_n = 0,2029 \text{ s}$$

3.  $y_B(0) = -0,01 \text{ m}$   
 $v_{By}(0) = -1 \text{ m/s}$



B pont körpályán mozog!

Sebesség tangenciális a pályára!

$$y_B(0) = -l \sin(\varphi(0))$$

$$\varphi(0) = \arcsin \left[ \frac{-y_B(0)}{l} \right] = 0,573^\circ = 0,01 \text{ rad}$$

$$v_y(0) = \frac{-v_{By}(0)}{\cos(\varphi(0))} = 1 \text{ m/s}$$

$\Rightarrow$  közelítés közelítés jö

$$\text{hisz } y_B(0) \approx -l \varphi(0)$$

$$-v_{By}(0) \approx v_y(0)$$

$$\dot{\varphi}(0) = \frac{v_y(0)}{l} = 1 \text{ rad/s} \quad \rightarrow \oplus$$

Freibewegung:  $\varphi(t) = A e^{\lambda t} \rightarrow \ddot{\varphi}(t) = A \lambda^2 e^{\lambda t}$

$$A \lambda^2 e^{\lambda t} + \omega_n A e^{\lambda t} = 0 \Rightarrow \lambda^2 + \omega_n^2 = 0 \rightarrow \lambda = \pm \omega_n i$$

$$\varphi(t) = C_1 \cos(\omega_n t) + C_2 \sin(\omega_n t)$$

$$\varphi(0) = C_1 \stackrel{\approx}{=} -y_B(0) = 0,01 \text{ rad}$$

$$\dot{\varphi}(t) = -C_1 \omega_n \sin(\omega_n t) + C_2 \omega_n \cos(\omega_n t)$$

$$\dot{\varphi}(0) = C_2 \omega_n \stackrel{\approx}{=} -v_B(0) / \ell$$

$$\Rightarrow C_2 = \frac{-v_B(0)}{\ell \omega_n} = 0,0323 \text{ rad}$$

ausg. Lösung:  $\varphi(t) = 0,01 \cos(30,96 \cdot t) + 0,0323 \sin(30,96 t)$

4.  $\varphi(t) = \phi \cos(\omega_n t + \delta) = \phi [\cos(\omega_n t) \cos(\delta) - \sin(\omega_n t) \sin(\delta)]$

$$\varphi(0) = \phi \cos \delta = C_1 \quad (1)$$

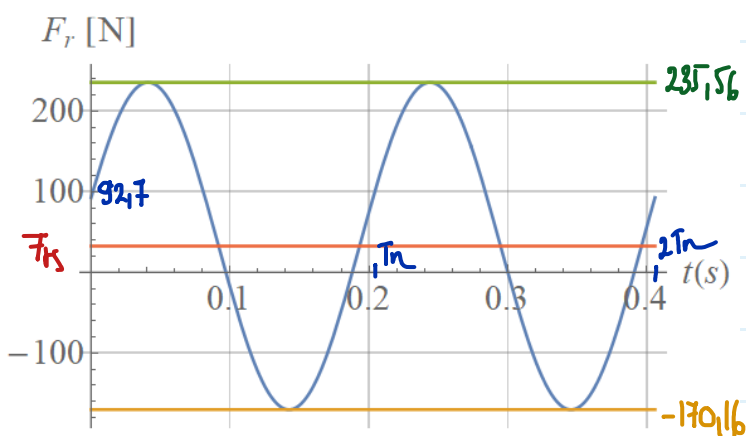
$$\varphi\left(\frac{\pi}{2\omega_n}\right) = -\phi \sin \delta = C_2 \quad (2)$$

$$(1)^2 + (2)^2 \Rightarrow \phi^2 (\cos^2 \delta + \sin^2 \delta) = C_1^2 + C_2^2$$

$$\Downarrow$$

$$\phi = \sqrt{C_1^2 + C_2^2} = 0,03381 \text{ rad}$$

$$F_r(\varphi) = F_{r,s} + k_a \varphi \Rightarrow F_{r,\max} = F_r(\phi) = \frac{m_1 g \ell}{a} + k_a \phi = 235,56 \text{ N}$$



$$F_r(0) = \frac{m_1 g \ell}{a} + k_a \varphi(0) = 92,7 \text{ N}$$

$$F_{r,\min} = F_r(-\phi) = -170,16 \text{ N}$$

## Vasúti ütközés

In Fig. 1, a railroad car of mass  $m$  crashes into a buffer stop with the initial velocity  $v_0$ . The buffer stop is assumed to be immovable. In order to consider the elasticity and the energy dissipation of the buffer stop during the impact, we <sup>use</sup> the simplified mechanical model shown in the figure. While the wagon is touching the buffer, the spring stiffness and damping coefficients of the buffers stops can be combined into the equivalent stiffness  $k$  and equivalent damping factor  $2c$ .

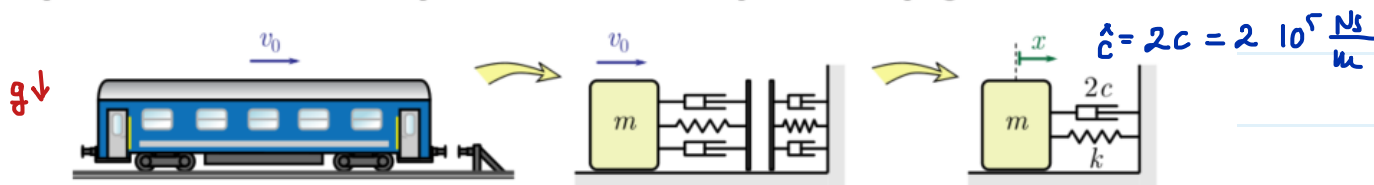


Fig. 1: Mechanical modeling of the impact as a damped oscillator

### Data

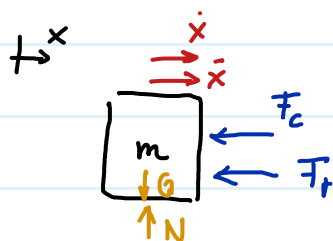
$$m = 5 \cdot 10^4 \text{ kg} \quad v_0 = 1 \text{ m/s}$$

$$k = 10^6 \text{ N/m} \quad c = 10^5 \text{ Ns/m}$$

### Tasks

1. Calculate the maximum spring force arising during the impact! ( $F_{r,\max} = 128.55 \text{ kN}$ )

SZTA!



$$F_c = \hat{c} \cdot \dot{x}$$

$$F_r = kx$$

Din. alapt

$$m\ddot{x} = -F_c - F_r = -\hat{c}\dot{x} - kx$$

$$m\ddot{x} + \hat{c}\dot{x} + kx = 0$$

Ref. alak.

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2x = 0 \rightarrow 2\zeta\omega_n = \frac{\hat{c}}{m}; \quad \omega_n^2 = \frac{k}{m}$$

relatív csillapítás  
tényező!

$$\omega_n = 4,47 \text{ rad/s}$$

$$\zeta = \frac{\hat{c}}{2} \frac{1}{\omega_n m} = 0,45$$

csillapított rendszer sajátfrekvenciája:  $\omega_H = \omega_n \sqrt{1 - \zeta^2} = 4 \text{ rad/s}$

$$f_0 = \frac{\omega_d}{2\pi} = 0,64 \text{ Hz}$$

$$T_D = \frac{1}{f} = 1,57 \text{ s}$$

mivel az ütköző felület tömege 0  $\Rightarrow \dot{x}(0) = v_0$

valamint  $x(0) = 0 \text{ m}$

próbaérv:

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2x = 0$$

$$x(t) = Ae^{\lambda t}; \dot{x}(t) = A\lambda e^{\lambda t}; \ddot{x}(t) = A\lambda^2 e^{\lambda t}$$

$$A\lambda^2 e^{\lambda t} + 2\zeta\omega_n A\lambda e^{\lambda t} + \omega_n^2 A e^{\lambda t} = 0$$

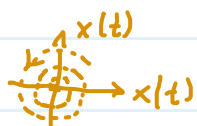
$$\lambda^2 + \underbrace{2\zeta\omega_n}_{=b}\lambda + \omega_n^2 = 0 \longrightarrow \lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4\omega_n^2}}{2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

kar. egy.

$$b^2 = 4\zeta^2\omega_n^2 \Rightarrow \sqrt{b^2 - 4\omega_n^2} = 2\omega_n\sqrt{\zeta^2 - 1}$$

$$\lambda_{1,2} = -\zeta\omega_n \pm i\omega_d$$

Az általános megoldás:  $x(t) = \underbrace{e^{-\zeta\omega_n t}}_{\text{exp. csökt.}} \left[ \underbrace{C_1 \cos(\omega_d t) + C_2 \sin(\omega_d t)}_{\omega_d \text{ szögkarfrik. oszcillál}} \right]$

$\Rightarrow$  stabil fókusz matematikailag  $\sim$  

mozgás törvény redukálása:

$$x(0) = e^0 C_1 = 0 \Rightarrow C_1 = 0$$

$$\dot{x}(t) = -\zeta\omega_n e^{-\zeta\omega_n t} [C_2 \sin(\omega_d t)] + e^{-\zeta\omega_n t} [C_2 \cos(\omega_d t) \cdot \omega_d]$$

$\uparrow$   
 $C_1 = 0$

$$\dot{x}(0) = e^0 \cdot C_2 \omega_d = v_0 \Rightarrow C_2 = \frac{v_0}{\omega_d} = 0,25\omega$$

mozgás törvény:  $x(t) = 0,25 \cdot e^{-2t} \cdot \sin(4t)$

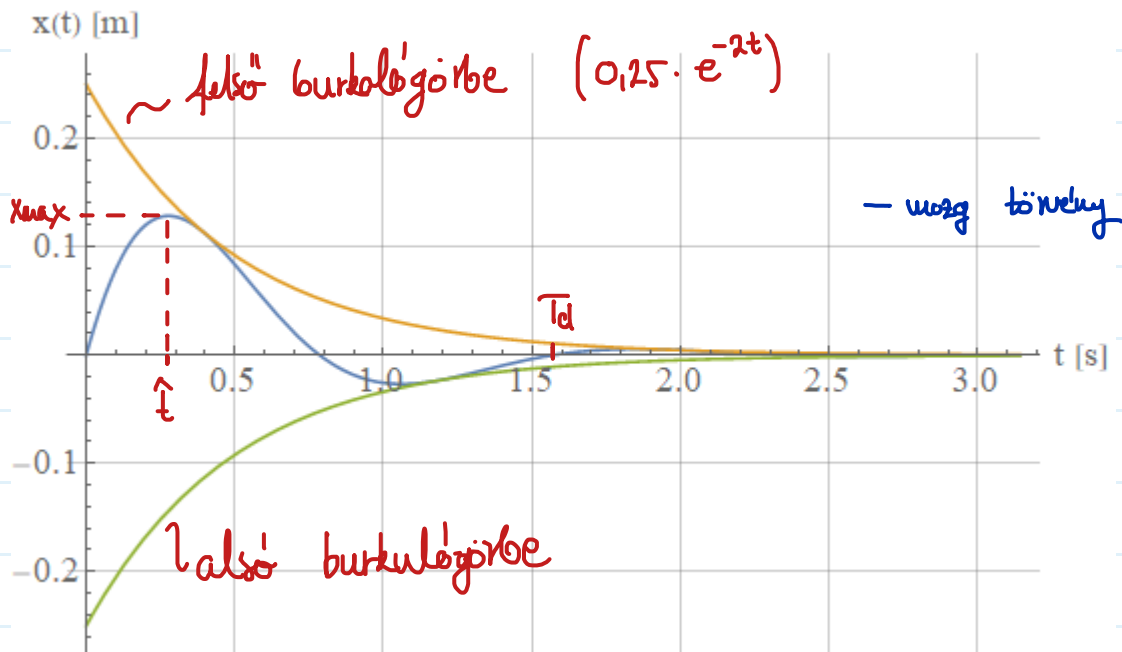
$T_{\text{max}}$  a max üszeregyenértékénél lesz, tehát  $\dot{x}(\hat{t}) = 0$  időpontban

$$\dot{x}(t) = -0,5 e^{-2t} \sin(4t) + e^{-2t} \cos(4t) \Rightarrow \dot{x}(\hat{t}) = e^{-2\hat{t}} \left[ \underbrace{-0,5 \sin(4\hat{t}) + \cos(4\hat{t})}_{=0} \right] = 0$$

$\neq 0$   $\quad \quad \quad = 0$

$$0,5 \sin(4\hat{t}) = \cos(4\hat{t}) \Rightarrow \tan(4\hat{t}) = 2 \Rightarrow \hat{t} = \frac{1}{4} \arctan(2) = 0,2768 \text{ s}$$

$$x_{\max} = x(\hat{t}) = 0,1286 \text{ m} \rightarrow F_{\max} = k \cdot x_{\max} = 128,55 \text{ kN}$$



Me történik elváláskor?  $\vec{x}$

$$m\ddot{x} = -F = -F_c - F_r$$

elválás:  $F = 0 \Rightarrow F_c(\tilde{t}) + F_r(\tilde{t}) = 0 \rightarrow \hat{c} \dot{x}(\tilde{t}) + kx(\tilde{t}) = 0 \quad \tilde{t} = ?$

A csillapítás által disszipált energia:  $E^d = T_0' - T_1' = \left( v_0^2 - \dot{x}(\tilde{t})^2 \right) \frac{1}{2} m = 22,27 \text{ kJ}$

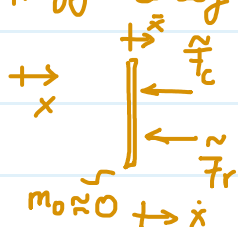
$$T_0' = \frac{1}{2} m v_0^2$$

$$T_1' = \frac{1}{2} m \dot{x}(\tilde{t})^2$$

(ütközés előtti energia)

(elváláskor)

Hogyan mozog ezután az ütköző?



$$m_0 \ddot{x} \approx 0 = -F_c - F_r = -c \dot{x} - kx \Rightarrow \text{kar egy.: } 0 = -c\lambda - k$$

$$\text{mozg. tönv.: } x(t) = e^{-\tilde{k}t/c} \cdot C_3 \quad \lambda = -\tilde{k}/c$$

$\Rightarrow$  exponenciálisan tart 0-hoz

## 1 DoF csillapított lengőkör

In Fig. 1, a swinging arm is shown that consists of two rods with different lengths and masses, and a disk with radius  $R$ . The swinging arm can only rotate along joint A, and the rods are connected to the environment through two springs with stiffness  $k_1$  and  $k_2$  and a damper with damping factor  $c_1$ . To describe the motion of the swinging arm, the angle  $\varphi$  measured from the horizontal axis is used as generalized coordinate. The structure is in the gravitational field and its equilibrium position is located at  $\varphi = 0$ . In this equilibrium position, the spring with stiffness  $k_2$  is unloaded.

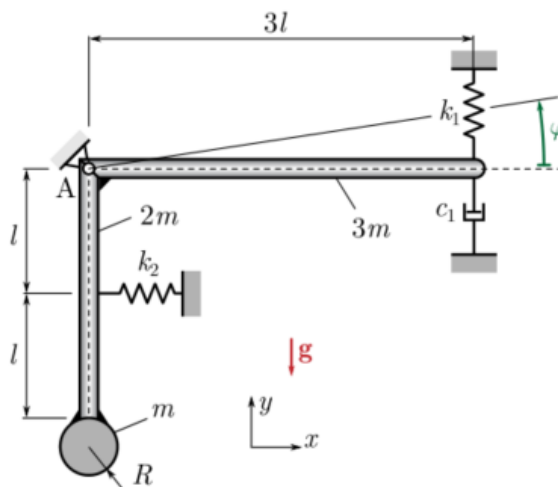


Fig. 1: The mechanical model of the swinging arm

### Data

$$l = 0.2 \text{ m} \quad R = 0.1 \text{ m} \quad m = 0.12 \text{ kg}$$

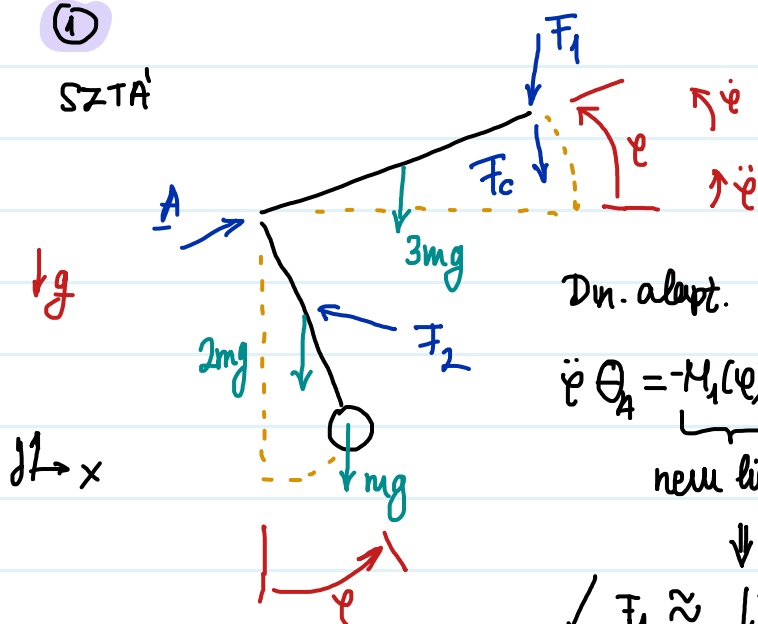
$$k_1 = 300 \text{ N/m} \quad k_2 = 10 \text{ N/m} \quad c_1 = 2 \text{ Ns/m}$$

### Tasks

1. Derive the equation of motion for small oscillations, calculate the natural angular frequencies of the undamped and damped system, and the damping ratio! ( $\omega_n = 35.55 \text{ rad/s}$ ,  $\omega_d = 35.31 \text{ rad/s}$ ,  $\zeta = 0.117$  [1])
2. Determine the critical damping factor in order to make the system critically damped! ( $c_{1,cr} = 17.10 \text{ Ns/m}$ )
3. Calculate the maximum force in the spring of stiffness  $k_1$  if the initial conditions are  $\varphi(t=0) = \varphi_0 = 0.01 \text{ rad}$  and  $\dot{\varphi}(t=0) = 0 \text{ rad/s}$ ! ( $F_{r1,max} = 3.009 \text{ N}$ )

①

SZTA'



$\varphi = 0$  az egyensúlyi helyzet!

Dn. alapt.

$$\ddot{\Theta}_A = \underbrace{-M_1(\varphi) - M_2(\varphi) - M_3(\varphi)}_{\text{nem lin. kifejezések}} - mg \left[ 3 \frac{3l}{2} \omega \sin \varphi + 2l \cos \varphi + (R+2l) \sin \varphi \right]$$

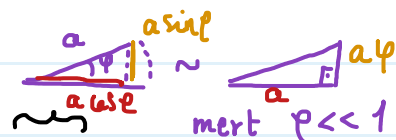
nem lin. kifejezések

↓

$$F_1 \approx \downarrow F_1 \Rightarrow F_1(\varphi) \approx F_{1s} + k_1 3l \varphi$$

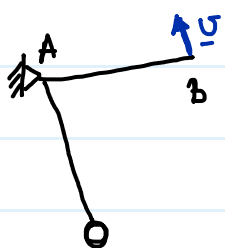
kis szögek

$$M_1(\varphi) \approx F_{1s} \cdot 3l + 4l g e^2 \varphi$$



Hasonlóan:  $F_2(\varphi) \approx k_2 l \varphi \Rightarrow M_2(\varphi) \approx k_2 l^2 \varphi$

$\vec{F}_2 \sim \leftarrow \vec{F}_2$  ( $F_{s2} = 0$ , mert  $\varphi = 0$  esetén nincs előzetes a feladat szerint!)



$\vec{v} \sim \uparrow \vec{v}$  redukciós képlet:  $\vec{v} = \vec{0} + \vec{\omega} \times \vec{r}_{AB}$

$$\vec{v} = \begin{bmatrix} 0 \\ 0 \\ \dot{\varphi} \end{bmatrix} \times 3l \begin{bmatrix} \cos \varphi \\ \sin \varphi \\ 0 \end{bmatrix} = \begin{bmatrix} -3l \dot{\varphi} \sin \varphi \\ 3l \dot{\varphi} \cos \varphi \\ 0 \end{bmatrix} \approx 3l \begin{bmatrix} \dot{\varphi} \\ \dot{\varphi} \\ 0 \end{bmatrix} \approx 3l \begin{bmatrix} 0 \\ \dot{\varphi} \\ 0 \end{bmatrix}$$

$\dot{\varphi} \approx 0$ , mert  $\varphi \ll 1; \dot{\varphi} \ll 1$

$F_c \approx 3l \dot{\varphi} \cdot c \rightarrow M_c \approx gl^2 \dot{\varphi} c$

Lineárizált mozg. egy. (kis kitérésekre)

$$\Theta_A \ddot{\varphi} = -F_{s1} 3l - mgl \frac{3}{2} - \varphi [k_1 gl^2 + k_2 l^2 + 2mgl + mg(2l+R)] + \dot{\varphi} gl^2 c$$

$$\Theta_A = \frac{1}{3} 2m 4l^2 + \frac{1}{3} 3m gl^2 + \frac{1}{2} m R^2 + m(2l+R)^2 = 0,0866 \text{ kgm}^2$$

egyensúly ( $\varphi = 0$ ) esetén:  $\dot{\varphi} = 0$  és  $\ddot{\varphi} = 0$

$$0 = -F_{s1} 3l - 4,5 mgl \Rightarrow F_{s1} = -1,5 mg$$

Mozg. egy.:  $\Theta_A \ddot{\varphi} + \varphi [k_1 gl^2 + k_2 l^2 + mg(4l+R)] + \dot{\varphi} gl^2 c = 0$

rel. alak.  $\ddot{\varphi} + 2\zeta \omega_n \dot{\varphi} + \omega_n^2 \varphi = 0 \rightarrow \omega_n = \sqrt{\frac{k_1 gl^2 + k_2 l^2 + mg(4l+R)}{\Theta_A}} = 35,55 \text{ rad/s}$

$$2\zeta \omega_n = \frac{gl^2 c}{\Theta_A} \Rightarrow \zeta = \frac{gl^2 c}{2\omega_n \Theta_A} = 0,1169 [1]$$

$$\omega_H = \omega_n \sqrt{1 - \zeta^2} = 35,31 \text{ rad/s}$$

② knt. nullapitás esetén pont nincs képzetes karakterisztikus gyök

$$\Rightarrow \hat{\omega}_d = 0 \rightarrow \hat{\zeta} = 1 \rightarrow \hat{c} = \frac{\hat{\zeta}^2 2\omega_n \Theta_A}{gl^2} = 17,1 \text{ Ns/m}$$

③ Próbafügg.:  $\varphi(t) = Ae^{\lambda t} \rightarrow \dot{\varphi}(t) = A\lambda e^{\lambda t} \rightarrow \ddot{\varphi}(t) = A\lambda^2 e^{\lambda t}$

$$A\lambda^2 e^{\lambda t} + A\lambda e^{\lambda t} \zeta 2\omega_n + Ae^{\lambda t} \omega_n^2 = 0$$

korr. egy.:  $\lambda^2 + 2\zeta\omega_n \lambda + \omega_n^2 = 0 \rightarrow \lambda_{1,2} = -\zeta\omega_n \pm \omega_n \sqrt{\zeta^2 - 1}$

$$\lambda_{1,2} = -\zeta\omega_n \pm i\omega_d$$

$$\varphi(t) = e^{-\zeta\omega_n t} [C_1 \cos(\omega_d t) + C_2 \sin(\omega_d t)]$$

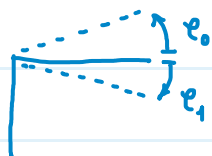
$$\varphi(0) = \varphi_0 = e^0 \cdot C_1 = 0,01 \text{ rad} \rightarrow C_1 = 0,01 \text{ rad}$$

$$\dot{\varphi}(t) = -\zeta\omega_n e^{-\zeta\omega_n t} [C_1 \cos(\omega_d t) + C_2 \sin(\omega_d t)] + e^{-\zeta\omega_n t} [-C_1 \sin(\omega_d t)\omega_d + C_2 \cos(\omega_d t)\omega_d]$$

$$\dot{\varphi}(0) = \dot{\varphi}_0 = -\zeta\omega_n C_1 + C_2 \omega_d = 0 \rightarrow C_2 = \frac{\zeta\omega_n C_1}{\omega_d} = 0,00118 \text{ rad}$$

Mivel  $\dot{\varphi}_0 = 0 \rightarrow \varphi_0$  szélsőérték!

legyen  $\varphi_1$  a következő szélsőérték



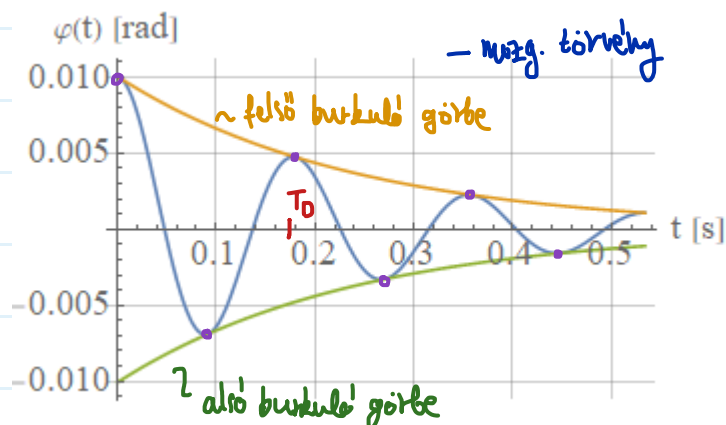
$|\varphi_1| < |\varphi_0|$  csillapítás miatt

DE!  $\varphi_1$  irányban előmozdított a rugó  $\Rightarrow$  ellenőrzem kért

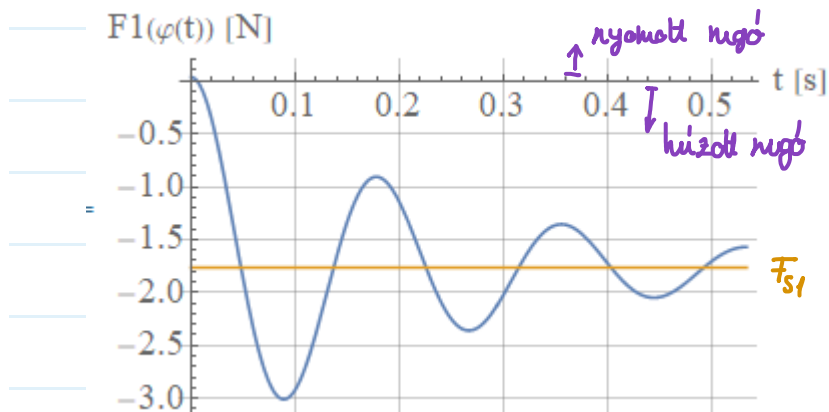
$$T_0 = \frac{2\pi}{\omega_0} \quad \varphi_1 = \varphi(T_0/2) = -0,00691 \text{ rad}$$

$$F_1(\varphi(0)) = 0,0342 \text{ N}$$

$$F_1(\varphi(T_0/2)) = -3,0093 \text{ N} \Rightarrow F_{\max} = 3,0093 \text{ N}$$



• szélső helyzetek





# 1 DoF gerjesztett csillapított lengőkar

In Fig. 1(a) and (b), a swinging arm is shown that is modelled by a rod with length  $l$  and mass  $m$ . The swinging arm can only rotate about joint A. Two cases are distinguished: harmonic force excitation (see panel (a)) and harmonic displacement excitation that is applied through a spring with stiffness  $k_0$  (see panel (b)). In both cases, the rod is connected to the environment through a spring with stiffness  $k$  (in case (a)) or  $k_1$  (in case (b)) and a damper with damping factor  $c$ . To describe the motion of the corresponding 1 DoF swinging arm, the angle  $\varphi$  measured from the horizontal axis is used as generalised coordinate. The structure lies on the horizontal plane and its equilibrium position is located at  $\varphi = 0$ . In this equilibrium position, the springs with stiffness  $k$  (case (a)) and  $k_1$  (case (b)) are unloaded.

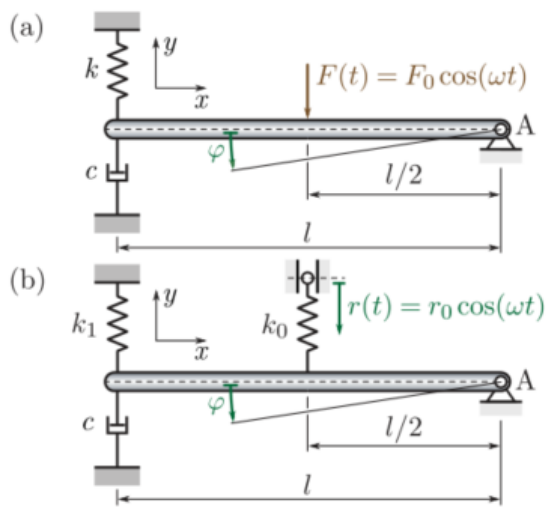


Fig. 1: Mechanical model of a swinging arm. (a) Force excitation. (b) Displacement excitation.

## Data

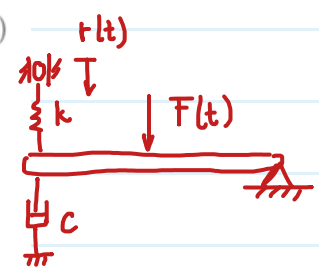
$m = 3 \text{ kg}$	$l = 1 \text{ m}$	$\omega = 30 \text{ rad/s}$
$k = 400 \text{ N/m}$	$F_0 = 10 \text{ N}$ <b>6N</b>	$c = 28 \text{ Ns/m}$
$k_0 = 1000 \text{ N/m}$	$r_0 = 0.01 \text{ m}$	$k_1 = 150 \text{ N/m}$

**0,005m**

## Tasks

- Derive the equation of motion for both models and calculate the natural angular frequencies of the undamped and damped system, the damping ratio and the static deformation! (For both cases:  $\omega_n = 20 \text{ rad/s}$ ,  $\omega_d = 14.28 \text{ rad/s}$ ,  $\zeta = 0.7$  [1],  $f_0 = 0.0125 \text{ rad}$ )
- Draw the resonance curve and the phase diagram of the systems!
- Determine the law of motion  $\varphi(t)$  if the initial conditions are  $\varphi(t=0) = \varphi_0 = 0.015 \text{ rad}$  and  $\dot{\varphi}(t=0) = 0 \text{ rad/s}$ !

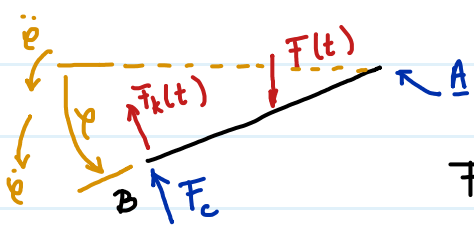
*pinos értékek arra az esetre, amikor  $\omega_0$  és útgörjesztés egyszerre van, az alábbi ábrán alapján*



1.

ΣTA'

Din. alapt.



$$\Theta_A \ddot{\varphi} = \frac{l}{2} \cos \varphi F(t) - l \cos \varphi [F_k(t) + F_c]$$

$$F_k(t) = (l \sin \varphi - r_0 \cos(\omega t)) k \approx l \varphi k - r_0 \cos(\omega t) k$$

$$\Theta_A = \frac{1}{3} m l^2$$

$$F_c \approx \frac{\dot{\varphi} l \cdot c}{\approx v_B}$$

din. mozg. egy.  $\Theta_A \ddot{\varphi} + \dot{\varphi} l^2 c + \varphi l^2 k = l \cos(\omega t) \left[ \frac{1}{2} F_0 + r_0 k \right]$

ref. alak:  $\ddot{\varphi} + 2\zeta \omega_n \dot{\varphi} + \omega_n^2 \varphi = f_0 \omega_n^2 \cos(\omega t)$

$$\omega_n = \sqrt{\frac{l^2 k}{\Theta_A}} = 20 \frac{\text{rad}}{\text{s}} \quad 2\zeta \omega_n = \frac{l^2 c}{\Theta_A} \Rightarrow \zeta = \frac{1}{2} \frac{l^2 c}{\omega_n \Theta_A} = 0,7 [-]$$

$$f_0 \omega_n^2 = \frac{1}{2} l F_0 + l r_0 k \Rightarrow f_0 = l \frac{1/2 F_0 + r_0 k}{\omega_n^2} = 0,0125 \text{ rad}$$

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} = 14,283 \text{ rad/s}$$

$$2 \quad \varphi(t) = \varphi_h(t) + \varphi_p(t)$$

$$\text{homogén: } \varphi_h(t) = A e^{\lambda t}$$

$$\text{karr. egy.: } \lambda^2 + 2\zeta \omega_n \lambda + \omega_n^2 = 0$$

$$\text{karr. gyök: } \lambda_{1,2} = -\zeta \omega_n \pm i \omega_d$$

$$\varphi_h(t) = e^{-\zeta \omega_n t} (C_1 \cos(\omega_d t) + C_2 \sin(\omega_d t))$$

$$\text{partikuláris: } \varphi_p(t) = \phi \cos(\omega t + \Delta)$$

$$\lambda = \frac{\omega}{\omega_n}$$

$$-\phi \omega^2 \cos(\omega t - \Delta) - 2\zeta \omega_n \phi \omega \sin(\omega t - \Delta) + \omega_n^2 \phi \cos(\omega t - \Delta) = f_0 \omega_n^2 \cos(\omega t) \quad | \cdot \omega_n^2$$

$$\phi \left[ \cos(\omega t - \Delta) [1 - \lambda^2] - \sin(\omega t - \Delta) 2\zeta \lambda \right] = f_0 \cos(\omega t) \quad ; \quad \tilde{\delta} = \Delta + \psi$$

$$\text{megj.: } A \cos \alpha - B \sin \alpha = C \rightarrow D \left( \frac{A}{D} \cos \alpha - \frac{B}{D} \sin \alpha \right) = C, \text{ ahol } D = \sqrt{A^2 + B^2}$$

$$\frac{A}{D} = \cos \delta; \quad \frac{B}{D} = \sin \delta, \text{ hiszen } \sqrt{\cos^2 \delta + \sin^2 \delta} = \frac{1}{D} \sqrt{A^2 + B^2} = \frac{D}{D} = 1$$

$$D (\cos \alpha \cdot \cos \delta - \sin \alpha \cdot \sin \delta) = D \cos(\alpha + \delta) = C$$

$$\phi \sqrt{(1 - \lambda^2)^2 + 4\zeta^2 \lambda^2} \cos(\omega t + \tilde{\delta}) = f_0 \cos(\omega t) \quad ; \quad N = \frac{1}{\sqrt{(1 - \lambda^2)^2 + 4\zeta^2 \lambda^2}}$$

$$\phi \cos(\omega t + \tilde{\delta}) = N f_0 \cos(\omega t) \quad ; \quad \phi = N f_0 \quad ; \quad \tilde{\delta} = 0$$

$$\text{tehát } N = 0,4092 [-]$$

$$\phi = 0,00511 \text{ rad} = 0,293^\circ$$

$$\hookrightarrow \Delta = -\psi$$

$$\text{megj.: } \frac{\sin \delta}{\cos \delta} = \tan \delta = \frac{B}{A}$$

$$\psi = \arctg\left(\frac{2\zeta \lambda}{1 - \lambda^2}\right) = -1,034 + j\pi \text{ rad}$$

$$\text{tg } \pi \text{ periodikus: } \psi \in [0, \pi]$$

$$\psi = 2,1077 \text{ rad} = 120,763^\circ$$