Cylindrical coordinates

- Cylindrical coordinate system is defined by orthogonal curvilinear coordinate axes. The coordinate axes are everywhere mutually orthogonal.

- CCS can be defined with respect to a given Cartesian coordinate system so that the cylindrical coordinate variables \((r, \theta, z)\) are related to Cartesian coordinate variables \((x_1, x_2, x_3)\) by the following relations:
  \[
  r = \sqrt{x_1^2 + x_2^2} \quad ; \quad \theta = \arctan \frac{x_2}{x_1} \quad ; \quad z = x_3
  \]

  The inverse relations are:
  \[
  x_1 = r \cdot \cos \theta \quad ; \quad x_2 = r \cdot \sin \theta \quad ; \quad x_3 = z
  \]
Remark:

- It is important to pay attention to which quadrant a point is in when converting from $x_1$ and $x_2$ to $\Theta$.

- In cylindrical coordinate system each point cannot be defined uniquely. For example, a point with an angle $\Theta$ will be the same as the point with angle $\Theta+2\pi$ when they have the same $r$ values. Therefore, we often choose $0 \leq \Theta < 2\pi$.

- To ensure that the transformation is unique, the following procedure is proposed:

$$\Theta = \begin{cases} 
\arctan \frac{x_2}{x_1} & \text{if } x_1 > 0 \text{ and } x_2 \geq 0 \\
\arctan \frac{x_2}{x_1} + 2\pi & \text{if } x_1 > 0 \text{ and } x_2 < 0 \\
\arctan \frac{x_2}{x_1} + \pi & \text{if } x_1 < 0 \\
\frac{\pi}{2} & \text{if } x_1 = 0 \text{ and } y_2 > 0 \\
\frac{3\pi}{2} & \text{if } x_1 = 0 \text{ and } y_2 < 0 \\
0 & \text{if } x_1 = x_2 = 0
\end{cases}$$
- The basis vectors \((e_r, e_\theta, e_\varphi)\) at \(P\) are tangent to the curvilinear coordinate axes. The base vector \(e_r\) is a unit vector that points in the direction of increasing \(r\) when \(\theta\) and \(\varphi\) are held constant. Likewise for \(e_\theta\) and \(e_\varphi\).

- The base vectors are related to the base vectors aligned with the corresponding Cartesian coordinate system by:

\[
e_r = \cos \theta \, e_1 + \sin \theta \, e_2
\]

\[
e_\theta = -\sin \theta \, e_1 + \cos \theta \, e_2
\]

\[
e_\varphi = e_3
\]

- It is very important to note that, whereas \(e_1, e_2\) and \(e_3\) point in the same directions regardless of the point in the space, \(e_r\) and \(e_\theta\) are functions of the coordinate variable \(\theta\). Consequently, the expression for the gradient, divergence and curl in terms of the scalar components of a tensor will not be the same as those in Cartesian coordinates.

- The scalar components of a vector \(a\) and a second order tensor \(A\) in the orthornormal basis defined by \(e_r, e_\theta\) and \(e_\varphi\) are denoted by:

\[
[a] = \begin{bmatrix} a_r \\ a_\theta \\ a_\varphi \end{bmatrix} \quad \text{and} \quad [A] = \begin{bmatrix} A_{rr} & A_{r\theta} & A_{r\varphi} \\ A_{\theta r} & A_{\theta\theta} & A_{\theta\varphi} \\ A_{\varphi r} & A_{\varphi\theta} & A_{\varphi\varphi} \end{bmatrix}
\]
The proper orthogonal tensor \( \mathbf{Q} \), which represents the transformation of scalar components from the Cartesian orthonormal basis to the cylindrical coordinate orthonormal basis is given by

\[
\begin{bmatrix}
\mathbf{Q}
\end{bmatrix} = \begin{bmatrix}
\cos \Theta & \sin \Theta & 0 \\
-\sin \Theta & \cos \Theta & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

The transformation rules for vector \( \mathbf{a} \) and a second-order tensor \( \mathbf{A} \):

\[
\begin{bmatrix}
\mathbf{a}
\end{bmatrix} = \begin{bmatrix}
\mathbf{Q}
\end{bmatrix} \begin{bmatrix}
\mathbf{a}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\mathbf{A}
\end{bmatrix} = \begin{bmatrix}
\mathbf{Q}
\end{bmatrix} \begin{bmatrix}
\mathbf{A}
\end{bmatrix} \begin{bmatrix}
\mathbf{Q}
\end{bmatrix}^{T}
\]

Since \( \mathbf{Q} \) is proper orthogonal, it follows that

\[
\begin{bmatrix}
\mathbf{Q}^{-1}
\end{bmatrix} = \begin{bmatrix}
\mathbf{Q}^{T}
\end{bmatrix} = \begin{bmatrix}
\cos \Theta & -\sin \Theta & 0 \\
\sin \Theta & \cos \Theta & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
\begin{bmatrix}
\mathbf{a}
\end{bmatrix} = \begin{bmatrix}
\mathbf{Q}^{T}
\end{bmatrix} \begin{bmatrix}
\mathbf{a}
\end{bmatrix}
\]

\[
\begin{bmatrix}
\mathbf{A}
\end{bmatrix} = \begin{bmatrix}
\mathbf{Q}^{T}
\end{bmatrix} \begin{bmatrix}
\mathbf{A}
\end{bmatrix} \begin{bmatrix}
\mathbf{Q}
\end{bmatrix}
\]
**Del (nabla) operator**

- The del operator \( \nabla \) in cylindrical coordinate system is given by

\[
\nabla = \frac{\partial}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial}{\partial \theta} \mathbf{e}_\theta + \frac{\partial}{\partial z} \mathbf{e}_z
\]

**Proof:**
- The transformation rules for the unit base vectors:

\[
\begin{align*}
\mathbf{e}_1 &= \cos \theta \mathbf{e}_r - \sin \theta \mathbf{e}_\theta \\
\mathbf{e}_2 &= \sin \theta \mathbf{e}_r + \cos \theta \mathbf{e}_\theta \\
\mathbf{e}_3 &= \mathbf{e}_z
\end{align*}
\]
- The transformation of the partial derivatives uses the chain rule of differentiation as follows:

\[
\frac{\partial}{\partial x_i} = \frac{\partial}{\partial r} \frac{\partial r}{\partial x_i} + \frac{\partial}{\partial \theta} \frac{\partial \theta}{\partial x_i} + \frac{\partial}{\partial z} \frac{\partial z}{\partial x_i}, \quad i = 1, 2, 3
\]

where

\[
\begin{align*}
\frac{\partial r}{\partial x_1} &= \frac{x_1}{r} = \cos \theta \\
\frac{\partial r}{\partial x_2} &= \frac{x_2}{r} = \sin \theta \\
\frac{\partial \theta}{\partial x_1} &= -\frac{x_2}{r^2} = -\frac{\sin \theta}{r} \\
\frac{\partial \theta}{\partial x_2} &= \frac{x_1}{r^2} = \frac{\cos \theta}{r} \\
\frac{\partial z}{\partial x_3} &= 1
\end{align*}
\]

and other derivatives are zero.
Thus
\[
\frac{\partial}{\partial x_1} = \cos \Theta \frac{\partial}{\partial r} - \frac{\sin \Theta}{r} \frac{\partial}{\partial \Theta}
\]
\[
\frac{\partial}{\partial x_2} = \sin \Theta \frac{\partial}{\partial r} + \frac{\cos \Theta}{r} \frac{\partial}{\partial \Theta}
\]
\[
\frac{\partial}{\partial x_3} = \frac{\partial}{\partial z}
\]
- Substituting in the definition of \( \nabla \) in Cartesian coordinate system:
\[
\nabla = \frac{\partial}{\partial x_1} e_1 + \frac{\partial}{\partial x_2} e_2 + \frac{\partial}{\partial x_3} e_3
\]
\[
= \left( \cos \Theta \frac{\partial}{\partial r} - \frac{\sin \Theta}{r} \frac{\partial}{\partial \Theta} \right) \left( \cos \Theta e_r - \sin \Theta e_\Theta \right)
\]
\[
+ \left( \sin \Theta \frac{\partial}{\partial r} + \frac{\cos \Theta}{r} \frac{\partial}{\partial \Theta} \right) \left( \sin \Theta e_r + \cos \Theta e_\Theta \right)
\]
\[
+ \frac{\partial}{\partial z} e_z
\]
- after simplification:
\[
\nabla = \cos^2 \Theta \frac{\partial}{\partial r} e_r + \frac{\sin^2 \Theta}{r} \frac{\partial}{\partial \Theta} e_\Theta - \cos \Theta \sin \Theta \frac{\partial}{\partial r} e_\Theta
\]
\[
- \frac{\cos \Theta \sin \Theta}{r} \frac{\partial}{\partial \Theta} e_r + \frac{\sin^2 \Theta}{r} \frac{\partial}{\partial r} e_r + \frac{\cos^2 \Theta}{r} \frac{\partial}{\partial \Theta} e_\Theta
\]
\[
+ \cos \Theta \sin \Theta \frac{\partial}{\partial r} e_\Theta + \frac{\sin \Theta \cos \Theta}{r} \frac{\partial}{\partial \Theta} e_r + \frac{\partial}{\partial z} e_z
\]
\[
= \frac{\partial}{\partial r} e_r + \frac{1}{r} \frac{\partial}{\partial \Theta} e_\Theta + \frac{\partial}{\partial z} e_z
\]
Derivatives of the basis vectors

- the base vectors $e_r$ and $e_{\theta}$ are functions of the cylindrical coordinate variable $\theta$.
- Their derivatives with respect to $\theta$ can be easily obtained as

$$\frac{\partial e_r}{\partial \theta} = \frac{\partial}{\partial \theta} \left( \cos \theta e_1 + \sin \theta e_2 \right) = -\sin \theta e_1 + \cos \theta e_2$$

$$\frac{\partial e_{\theta}}{\partial \theta} = \frac{\partial}{\partial \theta} \left( -\sin \theta e_1 + \cos \theta e_2 \right) = -\cos \theta e_1 - \sin \theta e_2$$

- Thus

$$\frac{\partial e_r}{\partial \theta} = e_{\theta}$$

$$\frac{\partial e_{\theta}}{\partial \theta} = -e_r$$

- Other derivatives are $\theta$. 


**Gradient**

- **Scalar Field**
  - The gradient of the scalar field $f(r, \theta, z)$ is
    
    \[ \nabla f = \nabla f = \frac{\partial f}{\partial r} e_r + \frac{1}{r} \frac{\partial f}{\partial \theta} e_{\theta} + \frac{\partial f}{\partial z} e_z \]

- **Vector Field**
  - Let a smooth vector field $\mathbf{a} = a_r e_r + a_\theta e_\theta + a_z e_z$ be given. Then
    
    \[ \nabla \mathbf{a} = \mathbf{a} \otimes \nabla \]

    \[ = (a_r e_r + a_\theta e_\theta + a_z e_z) \otimes \left( \frac{\partial}{\partial r} e_r + \frac{\partial}{\partial \theta} e_\theta + \frac{\partial}{\partial z} e_z \right) \]

    \[ = \frac{\partial a_r}{\partial r} e_r \otimes e_r + \frac{\partial a_\theta}{\partial r} e_\theta \otimes e_r + \frac{\partial a_z}{\partial r} e_z \otimes e_r \]

    \[ + \frac{1}{r} \frac{\partial}{\partial \theta} (a_r e_r) \otimes e_\theta + \frac{1}{r} \frac{\partial}{\partial \theta} (a_\theta e_\theta) \otimes e_\theta + \frac{1}{r} \frac{\partial a_z}{\partial \theta} e_z \otimes e_\theta \]

    \[ + \frac{\partial a_r}{\partial z} e_r \otimes e_z + \frac{\partial a_\theta}{\partial z} e_\theta \otimes e_z + \frac{\partial a_z}{\partial z} e_z \otimes e_z \]

  where

  \[ \frac{\partial}{\partial \theta} (a_r e_r) = \frac{\partial a_r}{\partial \theta} e_r + a_r e_\theta \]

  \[ \frac{\partial}{\partial \theta} (a_\theta e_\theta) = \frac{\partial a_\theta}{\partial \theta} e_\theta - a_\theta e_r \]
Thus, in matrix form:

\[
\begin{bmatrix}
\frac{\partial a_r}{\partial r} & \frac{1}{r} \left( \frac{\partial a_r}{\partial \theta} - a_\theta \right) & \frac{\partial a_r}{\partial z} \\
\frac{\partial a_\theta}{\partial r} & \frac{1}{r} \left( \frac{\partial a_\theta}{\partial \theta} + a_r \right) & \frac{\partial a_\theta}{\partial z} \\
\frac{\partial a_z}{\partial r} & \frac{1}{r} \frac{\partial a_z}{\partial \theta} & \frac{\partial a_z}{\partial z}
\end{bmatrix}
\]

\textbf{DIVERGENCE}

- vector field

- the divergence of vector field \( \mathbf{a} \) is given by

\[
\text{div} \mathbf{a} = \mathbf{a} \cdot \nabla = (a_r \mathbf{e}_r + a_\theta \mathbf{e}_\theta + a_z \mathbf{e}_z) \cdot \\
\left( \frac{\partial}{\partial r} \mathbf{e}_r + \frac{1}{r} \frac{\partial}{\partial \theta} \mathbf{e}_\theta + \frac{\partial}{\partial z} \mathbf{e}_z \right)
\]

\[
= \frac{\partial a_r}{\partial r} + \frac{1}{r} \frac{\partial}{\partial \theta} (a_r \mathbf{e}_r) \cdot \mathbf{e}_\theta + \frac{1}{r} \frac{\partial}{\partial \theta} (a_\theta \mathbf{e}_\theta) \cdot \mathbf{e}_\theta + \frac{\partial a_z}{\partial z}
\]
where
\[
\frac{\partial}{\partial \theta} (ar \mathbf{e}_r) \cdot \mathbf{e}_\theta = ar
\]
\[
\frac{\partial}{\partial \theta} (a \mathbf{e}_\theta \mathbf{e}_\theta) \cdot \mathbf{e}_\theta = \frac{\partial a\theta}{\partial \theta}
\]
Therefore
\[
\text{div } a = \frac{\partial a_r}{\partial r} + \frac{1}{r} \left( \frac{\partial a\theta}{\partial \theta} + ar \right) + \frac{\partial a_z}{\partial z}
\]

- 2nd-order tensor field

- the divergence of the second-order tensor field \( S \) is given by the vector

\[
\text{div } S = \begin{bmatrix}
\frac{\partial S_{rr}}{\partial r} + \frac{1}{r} \frac{\partial S_{re}}{\partial \theta} + \frac{\partial S_{rz}}{\partial z} + \frac{1}{r} \left( S_{rr} - S_{\theta\theta} \right) \\
\frac{\partial S_{er}}{\partial r} + \frac{1}{r} \frac{\partial S_{ee}}{\partial \theta} + \frac{\partial S_{ez}}{\partial z} + \frac{1}{r} \left( S_{er} + S_{\theta e} \right) \\
\frac{\partial S_{zr}}{\partial r} + \frac{1}{r} \frac{\partial S_{ze}}{\partial \theta} + \frac{\partial S_{zz}}{\partial z} + \frac{1}{r} S_{zr}
\end{bmatrix}
\]
• **vector field**
  
  - the curl of vector field \( \mathbf{a} \) is

\[
\text{curl}\, \mathbf{a} = \nabla \times \mathbf{a}
\]

\[
\begin{bmatrix}
\nabla \times \mathbf{a} \\
\end{bmatrix} = 
\begin{bmatrix}
\frac{1}{r} \frac{\partial a_z}{\partial \theta} - \frac{\partial a_\theta}{\partial z} \\
\frac{\partial a_\varphi}{\partial z} - \frac{\partial a_z}{\partial r} \\
\frac{\partial a_\theta}{\partial r} + \frac{a_\varphi}{r} - \frac{1}{r} \frac{\partial a_r}{\partial \theta}
\end{bmatrix}
\]
- The basic assumption of the linear theory of elasticity is that the displacements of the body under the applied loads are very "small", and hence, the difference between the deformed and undeformed configurations is very small. This simplification allows us to impose equilibrium conditions to the undeformed configuration of the body, because it is nearly identical to its deformed configuration. Note that the deformed configuration is unknown.

- The primary goals of the theory of linear elasticity are to predict the deformations and stresses in the elastic body under external loads.

- An elastic material reverts to its original state on the removal of the loads.
Displacement-strain relations

- If the displacement gradient is infinitesimal at every material point in the body, then the body is said to experience infinitesimal deformation.

- The infinitesimal strain tensor is defined as

\[ \varepsilon = \frac{1}{2} \left( \nabla \otimes \mathbf{u} + \mathbf{u} \otimes \nabla \right) \]

where \( \mathbf{u} \) represents the displacement field.

- In a Cartesian coordinate system:

\[ \mathbf{u} = u(x,y,z) \mathbf{e}_1 + v(x,y,z) \mathbf{e}_2 + w(x,y,z) \mathbf{e}_3 \]

\[ \nabla = \frac{\partial}{\partial x} \mathbf{e}_1 + \frac{\partial}{\partial y} \mathbf{e}_2 + \frac{\partial}{\partial z} \mathbf{e}_3 \]

Thus, \( \varepsilon \) has the form

\[
\varepsilon = \begin{bmatrix}
\frac{\partial u}{\partial x} & \frac{1}{2} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) & \frac{1}{2} \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) \\
\frac{1}{2} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) & \frac{\partial v}{\partial y} & \frac{1}{2} \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \\
\frac{1}{2} \left( \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} \right) & \frac{1}{2} \left( \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} \right) & \frac{\partial w}{\partial z}
\end{bmatrix}
\]

The components of \( \varepsilon \) are usually denoted as

\[
\varepsilon = \begin{bmatrix}
\varepsilon_x & \frac{1}{2} \gamma_{xy} & \frac{1}{2} \gamma_{xz} \\
\frac{1}{2} \gamma_{yx} & \varepsilon_y & \frac{1}{2} \gamma_{yz} \\
\frac{1}{2} \gamma_{zx} & \frac{1}{2} \gamma_{zy} & \varepsilon_z
\end{bmatrix}
\]
Thus \[ E_x = \frac{\partial u}{\partial x}; \quad E_y = \frac{\partial u}{\partial y}; \quad E_z = \frac{\partial w}{\partial z} \]
\[ \partial_x = \frac{\partial u}{\partial y} + \frac{\partial w}{\partial x}; \quad \partial_x = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}; \quad \partial_y = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial y}. \]

- In a cylindrical coordinate system:

\[ u = u_r(r, \theta, z) r + u_\theta(r, \theta, z) \theta + u_z(r, \theta, z) z \]

\[ \nabla = \frac{\partial}{\partial r} E_r + \frac{1}{r} \frac{\partial}{\partial \theta} E_\theta + \frac{\partial}{\partial z} E_z \]

Consequently, \( \varepsilon \) is given as

\[
\begin{bmatrix}
\varepsilon
\end{bmatrix} =
\begin{bmatrix}
\varepsilon_{rr} & \varepsilon_{r\theta} & \varepsilon_{rz} \\
\varepsilon_{\theta r} & \varepsilon_{\theta\theta} & \varepsilon_{\theta z} \\
\varepsilon_{rz} & \varepsilon_{z\theta} & \varepsilon_{zz}
\end{bmatrix}
\]

where

\[ \varepsilon_{rr} = \frac{\partial u_r}{\partial r} \]
\[ \varepsilon_{\theta\theta} = \frac{1}{r} \left( \frac{\partial u_\theta}{\partial \theta} + u_r \right) \]
\[ \varepsilon_{zz} = \frac{\partial u_z}{\partial z} \]
\[ \varepsilon_{r\theta} = \varepsilon_{\theta r} = \frac{1}{2} \left( \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{\partial u_r}{\partial \theta} \right) \]
\[ \varepsilon_{rz} = \varepsilon_{zr} = \frac{1}{2} \left( \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right) \]
\[ \varepsilon_{\theta z} = \varepsilon_{z\theta} = \frac{1}{2} \left( \frac{\partial u_\theta}{\partial z} + \frac{1}{r} \frac{\partial u_z}{\partial \theta} \right) \]
Strain-stress relation

- In the case of infinitesimal deformation, the Cauchy stress, first Piola-Kirchhoff stress and the second Piola-Kirchhoff stress tensors are approximately equivalent. A single (kinematically infinitesimal) stress tensor

$$\begin{bmatrix} \sigma \\ \epsilon \end{bmatrix} = \begin{bmatrix} \sigma_x & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_y & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_z \end{bmatrix}$$

is used.

- The fundamental assumption of linear elasticity is that the material obeys Hooke's law, implying that there is a linear relation between stress and strain.

- The most widely used form of the Hooke's law is written as

$$\sigma = 2\mu \varepsilon + \lambda (\varepsilon + \mu \varepsilon) I$$

where the Lamé constants $\lambda$ and $\mu$ are related to the Young's modulus and Poisson's ratio as

$$\lambda = \frac{E \cdot \nu}{(1+\nu)(1-2\nu)}$$

$$\mu = G = \frac{E}{2(1+\nu)}$$
the constitutive relation can be expressed as

$$
\overline{\mathbb{C}} = \frac{E}{1+\nu} \left( \mathcal{E} + \frac{\nu}{1-2\nu} (\text{tr} \mathcal{E}) \mathbb{I} \right)
$$

- **Equilibrium equations**

  - The linear equation of motion for the stress tensor field $\mathbf{\sigma}$ is given by

    $$
    \text{div} \mathbf{\sigma} + \mathbf{b} = \mathbf{g} \mathbf{a}
    $$

    where $\mathbf{b}$ is the body force per unit volume, $\mathbf{g}$ is the mass density and $\mathbf{a}$ is the acceleration.

  - The angular equation of motion reads as

    $$
    \mathbf{\Omega} = \mathbf{\Omega}^T
    $$

- **Compatibility condition**

  - The strain field $\mathcal{E}$ is uniquely defined for a given displacement field $\mathbf{u}$.

  - When the strain field $\mathcal{E}$ is given and the corresponding displacement field $\mathbf{u}$ is sought, then the problem is overdetermined because we have six independent equations for only three unknowns.

  - The necessary condition for $\mathcal{E}$ to be compatible is

    $$
    \nabla \times (\nabla \times \mathcal{E}) = \nabla \times \mathcal{E} \times \nabla = 0
    $$

  - This condition must be satisfied by $\mathcal{E}$ if the displacement field exists. There is no solution for $\mathbf{u}$ if this condition is not satisfied.
Summary

- Bringing together the governing equations of linear homogeneous isotropic elasticity results in a set of equations with 15 unknowns:
  - 6 stress components,
  - 6 strain components,
  - 3 displacement components.
- We have 15 basic equations:
  - 3 from the equilibrium,
  - 6 from the constitutive equation,
  - 6 from the strain-displacement relation.

- The set is a mixture of linear partial differential equations and linear algebraic equations. They need to be supplemented by appropriate boundary conditions if a unique solution is to be obtained.
- It should be noted that when the displacement field is one of the unknowns, the compatibility equations need not be explicitly considered.
- The field equations may be rearranged into different forms, for convenience, depending on the type of problem under consideration.

The Lamé-Navier equations and the Beltrami-Michell's equations are two particular reduced forms.
Linear elastic problem formulation

- two approaches can be used: displacement based and stress based formulations.

**Displacement based formulation**

- stress and strain fields are eliminated, thus leaving 3 scalar field equations for the 3 scalar displacement fields.
- primary variable is the displacement field
- useful when the boundary conditions are specified by given displacement
- avoids the use of the compatibility equations
- mostly used in 3D problems
- basis of most of the numerical methods

- inserting the strain-displacement relation into the Hooke's law:

\[
\varepsilon = \mu (\nabla \nabla \cdot \mathbf{u} + \nabla \nabla \mathbf{u}) + \frac{1}{2} \lambda + \tau (\nabla \nabla \cdot \mathbf{u} + \nabla \nabla \mathbf{u})
\]

\[
\sigma = \mu (\nabla \nabla \cdot \mathbf{u} + \nabla \nabla \mathbf{u}) + \lambda (\text{div } \mathbf{u})
\]

- Substituting into the equilibrium equation:

\[
0 = \mu \text{div} (\nabla \nabla \cdot \mathbf{u} + \nabla \nabla \mathbf{u}) + \lambda \text{div} ((\text{div } \mathbf{u}) \nabla) + \frac{\sigma}{\mu}
\]

\[
0 = \mu \nabla^2 \mathbf{u} + \mu (\mathbf{u} \cdot \nabla) \nabla + \lambda (\mathbf{u} \cdot \nabla) \nabla + \frac{\sigma}{\mu}
\]

\[
(\lambda + \mu) (\mathbf{u} \cdot \nabla) \nabla + \mu \nabla^2 \mathbf{u} + \frac{\sigma}{\mu} = 0
\]

*Lagrange-Navier Equations*

\[
(\lambda + \mu) \mathbf{u}_{,ki} + \mu \mathbf{u}_{,ii} + \frac{\sigma}{\mu} = 0
\]
- displacements are obtained by solving the second-order partial differential equations
- Using the strain-displacement relations, strain can be calculated
- Finally, stresses are obtained using the Hooke’s law

**Stress based formulation**
- primary variable is the stress field
- eliminating the displacement and strain fields generates a system of 6 equations for the six unknown stress components
- useful when the boundary conditions are specified by stresses
- Compatibility equations come into play
- can be used only in quasi-static problems

- Inserting the inverse Hooke’s law into the compatibility equations, we arrive (after some additional derivation steps) at the compatibility equations in terms of stress as:

\[ \sigma_{rs,ij} + \frac{1}{1+V} \sigma_{ij,rs} = \frac{V}{1-V} \sigma_{ij,yy} \delta_{rs} + \sigma_{is,ir} + \sigma_{ir,rs} \]

- Using it in the equilibrium equation gives:

\[ \nabla^2 \sigma + \frac{1}{1+V} (\text{tr} \sigma) \sigma = -\frac{V}{1-V} (\nabla \cdot \sigma) T - (\sigma \cdot \nabla \sigma + \nabla \sigma \cdot \sigma) \]

**MICHELL’S EQUATIONS**

\[ \sigma_{ii,kl} + \frac{1}{1+V} \sigma_{kl,ii} = -\frac{V}{1-V} \partial_{kk} \delta_{ii} - (\sigma_{ii,k} + \sigma_{i,k}) \]
- If \( \Theta \) is homogeneous, then it reduces to

\[
\nabla^2 \Theta + \frac{1}{1+\nu} \left( (\text{tr}\, \Theta) \Theta \right) \Theta = 0
\]

BELTRAMI'S EQUATIONS

- Second-order PDE system
- Once the stress field is known, the strain field is obtained using the inverse Hooke's law.
- To obtain the displacement field requires the geometric equations be integrated with the prescribed boundary conditions. It is a considerable disadvantage with respect to the displacement formulation.
Saint-Venant's principle

"According to the principle, the strains that are produced in a body by the application, to a small part of its surface, of a system of forces s\textit{t}\textit{a}\textit{t}\textit{i}c\textit{a}\textit{l}\textit{\ y} \ e\textit{q}u\textit{i}\textit{v}e\textit{r} \ to \ z\textit{e}r\textit{o} \ f\textit{o}r\textit{c}e \ a\textit{n}d \ z\textit{e}r\textit{o} \ c\textit{o}u\textit{p}e, \ a\textit{r}e \ o\textit{f} \ n\textit{e}g\textit{l}\textit{i}g\textit{i}\textit{b}l\textit{e} \ m\textit{a}g\textit{n}\textit{i}\textit{t}u\textit{u}d\textit{e} \ a\textit{t} \ d\textit{i}s\textit{t}a\textit{n}\textit{c}e\textit{s} \ w\textit{h}i\textit{c}h \ a\textit{r}e \ l\textit{a}r\textit{g}e \ c\textit{o}m\textit{p\textit{a}r\textit{e}d \ w\textit{i\textit{t}}h \ t\textit{h}e \ l\textit{i}\textit{n}\textit{e}a\textit{r} \ d\textit{i}\textit{m}\textit{e}\textit{n}\textit{s}\textit{i}\textit{s} \ o\textit{f} \ t\textit{h}e \ p\textit{a}r\textit{t}" \\

in other words:

\"The difference between the stresses caused by s\textit{t}\textit{a}\textit{t}\textit{i}c\textit{a}\textit{l}\textit{\ y} \ e\textit{q}u\textit{i}\textit{v}e\textit{r} \ l\textit{o}a\textit{d} \ s\textit{y}\textit{s}\textit{t}\textit{e}\textit{m}\textit{s} \ is \ i\textit{n}\textit{s\textit{i}g\textit{n}\textit{i}g\textit{i}\textit{a}\textit{n}t \ a\textit{t} \ d\textit{i}s\textit{t}a\textit{n}\textit{c}e\textit{s} \ g\textit{r}e\textit{a}\textit{t}e\textit{r} \ t\textit{h\textit{a}}n \ t\textit{h}e \ l\textit{a}r\textit{g}e\textit{st} \ d\textit{i}\textit{m}\textit{e}\textit{n}\textit{s}\textit{i}\textit{s} \ o\textit{f} \ t\textit{h}e \ \textit{a}r\textit{e}\textit{a} \ o\textit{v\textit{e}r \ w\textit{h}i\textit{c}h \ t\textit{h}e \ l\textit{o}\textit{a}d\textit{s} \ a\textit{r}e \ a\textit{c}\textit{t}\textit{i}n\textit{g}.\" \\

LOAD SYSTEM 1:

LOAD SYSTEM 2:
**Boundary conditions**

- In order for a boundary value problem of elasticity to be wellposed, suitable boundary conditions must be specified for every material point on the boundary of the body.

- Boundary conditions can be:
  - displacement B.C.
  - traction B.C.
  - mixed B.C.

**Example:**

![Diagram of a beam with boundary conditions](image)

**Displacement field:**

\[
\begin{bmatrix}
U \\
V \\
W
\end{bmatrix} = \begin{bmatrix}
U \\
V \\
W
\end{bmatrix}
\]

**Boundary Conditions:**

- **BC1:** \( u = v = w = 0 \) on \( x = 0 \)
- **BC2:** \( \tau_{zx} = \tau_{zy} = \sigma_z = 0 \) on \( z = \pm a \)
- **BC3:** \( \tau_{yx} = \tau_{yy} = \tau_{yzz} = 0 \) on \( y = \pm b \)
- **BC4:** \( \tau_{xy} = \tau_{zxx} = 0 \) on \( x = L \)
- **BC5:** \( \sigma_x = P/A \) on \( x = L \)
• Mathematical techniques:

- Analytical solution techniques:
  - Power series method
  - Fourier method
  - Integral Transform method
  - Complex Variable method

- Approximate solution procedures:
  - Ritz Method

- Numerical solution procedures:
  - Finite Difference Method
  - Finite Element Method
  - Boundary Element Method
Plane strain problems - Hooke's law

- the stress and strain tensors in a Cartesian coordinate system have the matrix forms

\[
\mathbf{\varepsilon} = \begin{bmatrix}
\varepsilon_x & \frac{1}{2} \gamma_{xy} & 0 \\
\frac{1}{2} \gamma_{yx} & \varepsilon_y & 0 \\
0 & 0 & 0 \\
\end{bmatrix} ; \quad \mathbf{\sigma} = \begin{bmatrix}
\sigma_x & \tau_{xy} & 0 \\
\tau_{yx} & \sigma_y & 0 \\
0 & 0 & \sigma_z \\
\end{bmatrix}
\]

where direction z is perpendicular to the plane.

- Hooke's law can be written in a shorter form as

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix}
1-\nu & \nu & 0 \\
\nu & 1-\nu & 0 \\
0 & 0 & 1-2\nu \\
\end{bmatrix} \begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix}
\]

with \( \sigma_z = \nu (\sigma_x + \sigma_y) \)

Plane stress problems - Hooke's law

\[
\begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix} = \begin{bmatrix}
\varepsilon_x & \frac{1}{2} \gamma_{xy} & 0 \\
\frac{1}{2} \gamma_{yx} & \varepsilon_y & 0 \\
0 & 0 & \varepsilon_z
\end{bmatrix} ; \quad \begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix} = \begin{bmatrix}
\sigma_x & \tau_{xy} & 0 \\
\tau_{yx} & \sigma_y & 0 \\
0 & 0 & \sigma_z
\end{bmatrix}
\]

- Hooke's law:

\[
\begin{bmatrix}
\sigma_x \\
\sigma_y \\
\tau_{xy}
\end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix}
1 & \nu & 0 \\
\nu & 1 & 0 \\
0 & 0 & \frac{1-\nu}{2}
\end{bmatrix} \begin{bmatrix}
\varepsilon_x \\
\varepsilon_y \\
\gamma_{xy}
\end{bmatrix}
\]

with \( \varepsilon_z = -\frac{\nu}{1-\nu} (\varepsilon_x + \varepsilon_y) \)
- consider a thin disk of uniform thickness rotating with a constant angular velocity \( \omega \). No external body force is applied.

- when the thickness of the disk is small in comparison with its radius, the variation of the radial and tangential stresses over the thickness can be neglected.
- it is assumed, that the displacement field exhibits cylindrical symmetry.
- Based on these assumption, the displacement field in the embedding cylindrical coordinate system can be given as

\[
\mathbf{u}_r = \mathbf{u}_r(r)
\]
- the above assumptions lead to the following strain tensor:

\[
\begin{bmatrix}
\varepsilon_r & 0 & 0 \\
0 & \varepsilon_t & 0 \\
0 & 0 & \varepsilon_z
\end{bmatrix}
\]

where \( \varepsilon_r = \frac{du_r}{dr} \)

\[
\varepsilon_t = \frac{du_t}{r}
\]

- The material obeys the Hooke's law:

\[
\sigma = \frac{E}{1+\nu} \left( \varepsilon + \frac{\nu}{1-2\nu} \varepsilon_I \right)
\]

where \( \begin{bmatrix}
\sigma_r & 0 & 0 \\
0 & \sigma_t & 0 \\
0 & 0 & \sigma_z
\end{bmatrix} = \begin{bmatrix}
\varepsilon_r & 0 & 0 \\
0 & \varepsilon_t & 0 \\
0 & 0 & \varepsilon_z
\end{bmatrix}
\]

In the case of plane stress formulation, \( \sigma_z = 0 \), because the sides of the disk are traction free.

From \( \sigma_z = 0 \), it follows that \( \varepsilon_z = -\frac{\nu}{1-\nu} (\varepsilon_r + \varepsilon_t) \)

In this case \( \varepsilon_I = \varepsilon_r + \varepsilon_t + \varepsilon_z = \frac{1-2\nu}{1-\nu} (\varepsilon_r + \varepsilon_t) \)

Thus, the stress components will be:

\[
\sigma_r = \frac{E}{1-\nu^2} (\varepsilon_r + \nu \varepsilon_t)
\]

\[
\sigma_t = \frac{E}{1-\nu^2} (\varepsilon_t + \nu \varepsilon_r)
\]
- since \( b = 0 \), the equilibrium equation reduces to

\[
\text{div} \bar{\sigma} = \rho a
\]

where the acceleration vector of a particular material point is

\[
a = -r \omega^2 \hat{e}_r
\]

Thus the equilibrium equation reads as

\[
\begin{bmatrix}
\frac{d\bar{\sigma}_r}{dr} + \frac{1}{r} (\bar{\sigma}_r - \bar{\sigma}_t) \\
0 \\
0
\end{bmatrix} = \begin{bmatrix}
-\rho \omega^2 r \\
0 \\
0
\end{bmatrix}
\]

\[
\frac{d\bar{\sigma}_r}{dr} + \frac{1}{r} (\bar{\sigma}_r - \bar{\sigma}_t) = -\rho \omega^2 r
\]
- Substituting the Hooke's law into the equilibrium equation gives

\[
\frac{d}{dr} \left[ \frac{E}{1-\nu^2} \left( \varepsilon_r + \nu \varepsilon_t \right) \right] + \frac{1}{r} \left( \frac{E}{1+\nu} \left( \varepsilon_r - \varepsilon_t \right) \right) = -\rho w^2 r
\]

\[
\frac{d}{dr} \left[ \varepsilon_r + \nu \varepsilon_t \right] + \frac{1-\nu}{r} \left( \varepsilon_r - \varepsilon_t \right) = -\frac{1-\nu^2}{E} \rho w^2 r
\]

- Inserting the strain-displacement relations:

\[
\frac{d}{dr} \left[ \frac{d u_r + \nu u_r}{r} \right] + \frac{1-\nu}{r} \left( \frac{d u_r - u_r}{r} \right) = -\frac{1-\nu^2}{E} \rho w^2 r
\]

\[
\frac{d^2 u_r}{dr^2} + \frac{1}{r} \frac{d u_r}{dr} - \frac{u_r}{r^2} = -\frac{1-\nu^2}{E} \rho w^2 r
\]

\[
\frac{d}{dr} \left[ \frac{1}{r} \frac{d(r \cdot u_r)}{dr} \right] = C_0 \cdot r
\]

Euler's differential equation
- by direct integration, we get

\[ \frac{1}{r} \frac{d}{dr} \left( u_r \cdot r \right) = \frac{1}{2} C_0 r^2 + C_1 \]

\[ \frac{d}{dr} (u_r \cdot r) = \frac{1}{2} C_0 r^3 + C_1 r \]

\[ u_r \cdot r = \frac{1}{8} C_0 r^4 + \frac{1}{2} C_1 r^2 + C_2 \]

\[ u_r = \left( \frac{1}{8} C_0 r^3 + \frac{1}{2} C_1 r + C_2 \right) \frac{1}{r} \]

Let \[ A \]

\[ B \]

\[ C \]

\[ u_r = A r^3 + B r + C \frac{1}{r} \]

unknown constants: \[ B, C \]

- Then, the strain components are given as

\[ \varepsilon_r = 3Ar^2 + B - C \frac{1}{r^2} \]

\[ \varepsilon_t = Ar^2 + B + C \frac{1}{r^2} \]

- Finally, the stress components:

\[ \sigma_r = \left( \frac{(3 + \nu)E}{1-\nu^2} A \right) r^2 + \left( \frac{E}{1-\nu} B \right) + \left( -\frac{E}{1+\nu} C \right) \frac{1}{r^2} \]

\[ \sigma_t = \left( \frac{(1+3\nu)E}{1-\nu^2} A \right) r^2 + \left( \frac{E}{1-\nu} B \right) + \left( \frac{E}{1+\nu} C \right) \frac{1}{r^2} \]
The constants $B$ and $C$ are obtained from the boundary conditions.

**Solid disk**:
- The boundary conditions are
  
  $\text{B.C. 1: } u_r(0) = 0$
  
  $\text{B.C. 2: } \sigma_r(R) = 0$

- In order for the displacement to be bounded at $r = 0$, it follows from B.C. 1 that $C = 0$

- B.C. 2 gives
  
  $0 = \frac{(3+\nu)E}{1-\nu^2}AR^2 + \frac{E}{1-\nu}B$
  
  $B = -\frac{3+\nu}{1+\nu}AR^2 = \frac{(3+\nu)(1-\nu^2)gw^2R^2}{8(1+\nu)E}$
  
  $B = \frac{(3+\nu)(1-\nu)gw^2R^2}{8E}$

- Thus, the final solutions are

<table>
<thead>
<tr>
<th>$u_r(r)$</th>
<th>$\frac{(1-\nu)(3+\nu)gw^2r}{8E} \left( R^2 - \frac{1+\nu}{3+\nu} r^2 \right)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_r(r)$</td>
<td>$\frac{(1-\nu)(3+\nu)gw^2}{8E} \left( R^2 - \frac{3+3\nu}{3+\nu} r^2 \right)$</td>
</tr>
<tr>
<td>$\sigma_t(r)$</td>
<td>$\frac{(1-\nu)(3+\nu)gw^2}{8E} \left( R^2 - \frac{1+\nu}{3+\nu} r^2 \right)$</td>
</tr>
<tr>
<td>$\sigma_r(r)$</td>
<td>$\frac{3+\nu}{8} gw^2( R^2 - r^2 )$</td>
</tr>
<tr>
<td>$\sigma_t(r)$</td>
<td>$\frac{3+\nu}{8} gw^2( R^2 - \frac{1+3\nu}{3+\nu} r^2 )$</td>
</tr>
</tbody>
</table>
- illustration of the stress distributions:

\[ \sigma_r, \sigma_t \]

\[ \sigma_t(r) \]

\[ \sigma_r(r) \]

\[ \sigma_t(R) = \frac{1-v}{4} \rho w^2 R^2 \]

- because \( v_{\text{max}} = 0.5 \), it follows that \( \sigma_t > \sigma_r \) at every point.

- \( \sigma_r \) and \( \sigma_t \) are positive everywhere.

- illustration of the strain distributions:

\[ \varepsilon_r, \varepsilon_t \]

\[ \varepsilon_t(r) \]

\[ \varepsilon_r(r) \]

\[ \varepsilon_t(R) = \frac{(1-v)\rho w^2 R^2}{4E} \]

\[ \varepsilon_r(R) = \frac{-v(1-v)\rho w^2 R^2}{4E} \]

- illustration of the displacement solution:

\[ u_r(r) \]

\[ u_r(r) = \frac{(1-v)\rho w^2 R^3}{4E} \]

\[ u_r(R) = \frac{(3-v)(1-v)^2 \rho w^2 R^3}{12E} \]

- if \( v > 0 \):
- The strain along the thickness is

\[ \varepsilon_z = -\frac{V}{1-V} (\varepsilon_r + \varepsilon_t) \]

\[ \varepsilon_z = -\frac{(3+V)Vg\omega^2}{4E} \left( \frac{R^2}{R^2 - \frac{2+2V}{3+V} R^2} \right) \]

---

**Annular disk**

- The boundary conditions are

  **B.C. 1**: \( 6r(R_i) = 0 \)
  **B.C. 2**: \( 6r(R) = 0 \)

- By solving the system of algebraic equations defined by B.C. 1 and B.C. 2, give the solutions

\[ B = \frac{(1-V)(3+V)g\omega^2}{8E} (R^2 + R_i^2) \]

\[ C = \frac{(1+V)(3+V)g\omega^2}{8E} R R_i^2 \]

- Thus, the final solutions are

\[ u_r(r) = \frac{(3+V)(1-V)g\omega^2}{8E} r \left( \frac{R^2 + R_i^2}{R^2} + \frac{1+V}{1-V} \left( \frac{R R_i}{r} \right)^2 - \frac{1+V}{3+V} r^2 \right) \]

\[ \varepsilon_r(r) = \frac{(3+V)(1-V)g\omega^2}{8E} \left( R^2 + R_i^2 - \frac{1+V}{1-V} \left( \frac{R R_i}{r} \right)^2 - \frac{3+3V}{3+V} r^2 \right) \]

\[ \varepsilon_t(r) = \frac{(3+V)(1-V)g\omega^2}{8E} \left( \frac{R^2 + R_i^2}{R^2} + \frac{1+V}{1-V} \left( \frac{R R_i}{r} \right)^2 - \frac{1+V}{3+V} r^2 \right) \]
\[ \sigma_r(r) = \frac{(3+\nu)\rho w^2}{8r^2}\left( R^2 - r^2 \right) \left( r^2 - R_i^2 \right) \]

\[ \sigma_t(r) = \frac{(3+\nu)\rho w^2}{8} \left( \frac{R_i^2}{r^2} \right) \left( R^2 + R_i^2 \right) - \frac{(1+3\nu)\rho w^2}{8} r^2 \]

Illustration of the stress distributions:

\[ \frac{(3+\nu)\rho w^2}{4} \left( \frac{R_i^2}{r^2} \right) \left( R^2 + \frac{1-\nu}{\nu} R_i^2 \right) \]

Illustration of the displacement solution:

\[ \frac{R \cdot \rho w^2}{4E} \left( \frac{R_i^2 (3+\nu)}{R^2} + R_i^2 (1-\nu) \right) \]
Remarks

- for both cases (solid and annular) the Mohr's equivalent stress is defined as

$$ \sigma_{eq} = \sigma_1 - \sigma_3 = \sigma_t - 0 $$

Making a small (Ri≈0) circular hole at the center of a rotating solid disk, we shall double the maximum Mohr's equivalent stress

<table>
<thead>
<tr>
<th>Solid</th>
<th>Annular with Ri=0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_t(r=0) = \frac{3+\nu}{8} \rho \omega^2 R^2$</td>
<td>$\sigma_t(r=0) = \frac{3+\nu}{4} \rho \omega^2 R^2$</td>
</tr>
</tbody>
</table>

- related problems:
  - rotating disk with variable thickness
  - rotating disk of uniform stress
  - rotating long cylinder
  - thin rotating ring or cylinder
HOLLOW CYLINDER SUBJECTED TO INNER AND OUTER Pressures

- A hollow circular cylinder is subjected to uniform pressures $p_i$ on the inner surface and $p_o$ on the outer surface.

- Find the stress, strain and displacement solutions for the following cases:

CASE A: fixed ends
CASE B: free ends
CASE C: thin disk (washer)
CASE A

- this case can be considered as a plane strain problem.
- Consequently

\[ \mathbf{u} = \begin{bmatrix} u_r \\ 0 \\ 0 \end{bmatrix}; \quad \mathbf{\varepsilon} = \begin{bmatrix} \varepsilon_r & 0 & 0 \\ 0 & \varepsilon_t & 0 \\ 0 & 0 & 0 \end{bmatrix}; \quad \mathbf{\sigma} = \begin{bmatrix} \sigma_r & 0 & 0 \\ 0 & \sigma_t & 0 \\ 0 & 0 & \sigma_2 \end{bmatrix} \]

where \( \sigma_2 = Y(\sigma_x + \sigma_y) \)
- because there is no body force acting on the body, the equilibrium equation reduces to

\[ \text{div} \mathbf{\sigma} = 0 \quad \rightarrow \quad \frac{d\sigma_r}{dr} + \frac{1}{r} (\sigma_r - \sigma_t) = 0 \]
- according to the Hooke's law, the radial and tangential stresses in plane strain case are given by:

\[ \sigma_r = \frac{E}{(1+\nu)(1-2\nu)} \left( (1-\nu)\varepsilon_r + \nu \varepsilon_t \right) \]

\[ \sigma_t = \frac{E}{(1+\nu)(1-2\nu)} \left( (1-\nu)\varepsilon_t + \nu \varepsilon_r \right) \]

where the strain components are defined as

\[ \varepsilon_r = \frac{d u_r}{dr} \]

\[ \varepsilon_t = \frac{u_r}{r} \]
- Substituting the stress formulae into the equilibrium equation gives

\[
\frac{d}{dr} \left[ (1-v)\varepsilon_r + \gamma \cdot \varepsilon_t \right] + \frac{1}{r} \left( 1-2v \right) (\varepsilon_r - \varepsilon_t) = 0
\]

\[
\frac{d}{dr} \left[ (1-v) \frac{du_r}{dr} + \gamma \frac{u_r}{r} \right] + \frac{1-2v}{r} \left( \frac{du_r}{dr} - \frac{u_r}{r} \right) = 0
\]

\[
\frac{d^2 u_r}{dr^2} + \frac{1}{r} \frac{du_r}{dr} - \frac{u_r}{r^2} = 0
\]

\[
\frac{d}{dr} \left[ \frac{1}{r} \frac{d(u_r \cdot r)}{dr} \right] = 0
\]

Euler's differential equation

- By direct integration we get

\[
u_r = \frac{1}{2} C_0 \cdot r + C_1 \cdot \frac{1}{r}
\]

- Based on this result it follows that

\[
\varepsilon_r = \frac{1}{2} C_0 - C_1 \cdot \frac{1}{r^2}
\]

\[
\varepsilon_t = \frac{1}{2} C_0 + C_1 \cdot \frac{1}{r^2}
\]

\[
\bar{\sigma}_r = \frac{E}{(1+v)(1-2v)} \left[ \frac{1}{2} C_0 - (1-2v) C_1 \cdot \frac{1}{r^2} \right]
\]

\[
\bar{\sigma}_t = \frac{E}{(1+v)(1-2v)} \left[ \frac{1}{2} C_0 + (1+2v) C_1 \cdot \frac{1}{r^2} \right]
\]
- constants $C_0$ and $C_1$ can be determined from the boundary conditions:

\[ \sigma_r(R_i) = -P_i \]
\[ \sigma_r(R_o) = -P_o \]

- Solving this system of algebraic equations we get

\[ C_0 = \frac{2(1+\nu)(1-2\nu)}{E} A \]
\[ C_1 = \frac{(1+\nu)R_i^2}{E} B \]

where

- $B = \frac{\Delta P}{1-\psi_o}$
- $A = B \cdot \psi_o - P_o$
- $\psi = \left( \frac{R_i}{r} \right)^2$
- $\psi_o = \left( \frac{R_i}{R_o} \right)^2$
- $\Delta P = P_i - P_o$

are introduced for convenience.
The strains, stresses and displacement solutions in terms of parameter $\varphi$, then become

\[ \sigma_r = A - B \cdot \varphi \]
\[ \sigma_t = A + B \cdot \varphi \]
\[ \sigma_\theta = 2\nu \cdot A \]

\[ \varepsilon_r = \frac{1 + \nu}{E} \left( (1 - 2\nu)A - B \cdot \varphi \right) \]
\[ \varepsilon_t = \frac{1 + \nu}{E} \left( (1 - 2\nu)A + B \cdot \varphi \right) \]

\[ u_r = \frac{(1 + \nu)R_i}{E} \left( (1 - 2\nu)A \cdot \frac{1}{1+\varphi} + B \cdot \varphi \right) \]
- For the case, when the fixed ends are replaced to end caps, then the equilibrium equation along the z-axis reads as

\[ \bar{g}_z \cdot (R_i^2 - R_o^2) + p_o \cdot (R_o^2 - R_i^2) = p_i \cdot (R_i^2 - R_o^2) \]

Thus, it follows that

\[ \bar{g}_z = \frac{p_i \cdot R_i^2 - p_o \cdot R_o^2}{R_o^2 - R_i^2} = \frac{p_i \cdot \varphi_i - p_o}{1 - \varphi_i} = A \]

- Stresses \( \bar{g}_r \) and \( \bar{g}_t \) are unaltered.

- The axial strain \( \varepsilon_z \) is obtained as

\[ \varepsilon_z = \frac{1 - 2\nu}{E} \cdot A \]

\[ \varepsilon_z = \frac{1}{E} \left( \bar{g}_z - \nu (\bar{g}_r + \bar{g}_t) \right) \]

whereas the radial and tangential strains are obtained by adding an extra term \(-\nu \cdot \varepsilon_z\) to the solutions derived for case A. Thus

\[ \varepsilon_r = \frac{1 + \nu}{E} \left( \frac{1 - 2\nu}{1 + \nu} A - B \cdot \varphi \right) \]

\[ \varepsilon_t = \frac{1 + \nu}{E} \left( \frac{1 - 2\nu}{1 + \nu} A + B \cdot \varphi \right) \]

- The radial displacement \( u_r \) is obtained by adding the term \(-\nu \cdot \varepsilon_z \cdot r\) to the solution derived for Case A. Thus

\[ u_r = \frac{R_i(1 + \nu)}{E} \left( \frac{1 - 2\nu}{1 + \nu} \frac{A}{\varphi^1} + B \cdot \varphi \right) \]
- The axial displacement is

\[ u_z = z \cdot \varepsilon_{zz} = \frac{1-2V}{E} A \cdot z \]

**Case C**

- For thin disks, the plane stress assumption serves a good approximation to the problem.
- Based on the fact that there is no functional difference between the plane strain and plane stress formulations in linear elasticity, the solutions obtained for plane stress case can be converted to plane strain case (and vice versa) by appropriate changing of the elastic constants as follows:

<table>
<thead>
<tr>
<th></th>
<th>( E )</th>
<th>( V )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Plane stress ( \rightarrow ) Plane strain :</td>
<td>( \frac{E}{1-V^2} )</td>
<td>( \frac{V}{1-V} )</td>
</tr>
<tr>
<td>Plane strain ( \rightarrow ) Plane stress :</td>
<td>( \frac{1+2V}{(1+V)^2 E} )</td>
<td>( \frac{V}{1+V} )</td>
</tr>
</tbody>
</table>

- Applying this conversation to the solutions obtained for Case A, we get the following expressions for the tangential and radial strains:

\[ \varepsilon_r = \frac{1+V}{E} \left( \frac{1-V}{1+V} A - B \cdot \varphi \right) \]

\[ \varepsilon_t = \frac{1+V}{E} \left( \frac{1-V}{1+V} A + B \cdot \varphi \right) \]
- for plane stress problems the strain $\varepsilon_z$ is given by

$$\varepsilon_z = -\frac{v}{E} (\sigma_r + \sigma_t) = -\frac{v}{1-v} (\varepsilon_r + \varepsilon_t)$$

$$\varepsilon_z = -\frac{2v}{E} A$$

- the radial displacement becomes

$$u_r = \frac{(1+v)R}{E} \left( \frac{1-v}{1+v} \frac{A}{\sqrt{r}} + B \sqrt{r} \right)$$

- the stresses are independent of material properties.
Airy Stress Function

- Lame'–Navier and Beltrami–Michell's equations are the field equations representing linear elasticity problems using displacement and stress based formulations.
- It is a nontrivial task to find either the displacement or stress solutions, which satisfy the specified boundary conditions.
- We can use either approximate or exact solution techniques in solving a particular problem. However, only a limited class of problems can be solved analytically.
- The use of scalar potential functions is one of the methods. There exist stress functions (potentials) and displacement functions (potentials), as well. They have no obvious physical meaning other than their use in defining stress or displacement components in terms of derivatives.
- The use of potential function can simplify the task of finding solutions.
- Potentials related to displacements:
  - Galerkin vector potential;
  - Neuber–Papkovich potential.
- Potentials related to stress components:
  - Maxwell stress function;
  - Morera stress function;
  - Airy stress function.
- The use of Airy stress function has great importance in solving 2D linear elasticity problems.
Formulation in Cartesian coordinate system:

- For static plane stress/strain problems, the equilibrium equations reduce to:

\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + b_x = 0 \\
\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + b_y = 0
\]

- Conservative body forces can be expressed in terms of a scalar potential function \( \phi(x,y) \) as

\[
\mathbf{b} = -\nabla \phi. \quad \text{Thus}
\]

\[
\frac{\partial}{\partial x} \left( \sigma_x - \phi \right) + \frac{\partial \tau_{xy}}{\partial y} = 0
\]

\[
\frac{\partial}{\partial y} \left( \sigma_y - \phi \right) + \frac{\partial \tau_{yx}}{\partial x} = 0
\]

- Define the Airy stress function \( \phi(x,y) \) such that:

\[
\sigma_x = \frac{\partial^2 \phi}{\partial y^2} + \phi \\
\sigma_y = \frac{\partial^2 \phi}{\partial x^2} + \phi \\
\tau_{xy} = \tau_{yx} = -\frac{\partial^2 \phi}{\partial x \partial y}
\]

- This representation ensures that the stress field satisfies the 2D equilibrium equations.
- because the constitutive equation is not involved in the derivation steps, therefore, Airy stress functions can be used for 2D static problems involving inelastic constitutive laws.

- every equilibrated stress field has an Airy stress function representation.

- compatibility equations for plane stress/strain problems reduce to

\[ \nabla \times \mathbf{\varepsilon} \times \nabla = 0 \quad \Rightarrow \quad \varepsilon_{x,yy} + \varepsilon_{y,xx} - 2\varepsilon_{xy,xy} = 0 \]

Using the Hooke's law it can be written as

\[ \nabla^2 (\delta_x + \delta_y) = -\beta (\delta_x,xx + \delta_y,yy) \]

where

\[ \beta = \begin{cases} \frac{1}{1-\nu} & \text{for plane strain} \\ \frac{1}{1+\nu} & \text{for plane stress} \end{cases} \]

The compatibility equation can be rewritten in terms of the Airy stress function \( \phi(x,y) \) and the body force potential \( \mathcal{J}(x,y) \) as

\[ \nabla^4 \phi + (2-\beta) \nabla^2 \mathcal{J} = 0 \]

where \( \nabla^4 \phi = \Delta^2 \phi = \phi_{xxxx} + \phi_{yyyy} + 2\phi_{xxyy} \)

- Thus, the problem is reduced to seeking solution of the equation above such that the stress components satisfy the boundary conditions.
• **No body force**

- When the body force is neglected, then the equilibrium equations (in a Cartesian coordinate system) reduce to

\[
\begin{align*}
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} &= 0 \\
\frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \sigma_y}{\partial y} &= 0
\end{align*}
\]

- The definitions for the stress components are

\[
\begin{align*}
\sigma_x &= \frac{\partial^2 \phi}{\partial y^2} \\
\sigma_y &= \frac{\partial^2 \phi}{\partial x^2} \\
\tau_{xy} &= \tau_{yx} = -\frac{\partial^2 \phi}{\partial x \partial y}
\end{align*}
\]

- The compatibility equations then become

\[
\nabla^4 \phi = 0 \quad \text{or} \quad \Delta^2 \phi = 0
\]

- Thus, the Airy stress function \( \phi \) must be bi-harmonic.
Cylindrical coordinate system

- The equilibrium equations are given by

\[
\begin{align*}
\frac{\partial \sigma_r}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{1}{r} (\sigma_r - \sigma_\theta) + b_r &= 0 \\
\frac{\partial \tau_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_\theta}{\partial \theta} + \frac{2}{r} \tau_{r\theta} + b_\theta &= 0
\end{align*}
\]

A conservative body force can be expressed as \( b_r = -q \cdot \sin \theta \). Thus

\[
b_r = -\frac{2\omega}{r} \quad ; \quad b_\theta = -\frac{1}{2} \frac{2\omega}{r \partial^2 \theta}.
\]

- The Airy stress function is defined such that

\[
\begin{align*}
\sigma_r &= \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \omega \\
\sigma_\theta &= \frac{\partial^2 \phi}{\partial r^2} + \omega \\
\tau_{r\theta} &= \frac{\partial \phi}{\partial r} \left( \frac{1}{r} \frac{\partial \phi}{\partial \theta} \right) - \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2}
\end{align*}
\]

- The tensorial form of the compatibility equation is invariant, so that

\[
\nabla^4 \phi + (2-\beta) \nabla^2 \omega = 0
\]

Since there is no dependence on \( z \), it follows that

\[
\begin{align*}
\nabla^4 (\phi) &= \nabla^2 \left[ \nabla^2 (\phi) \right] \\
\nabla^2 (\phi) &= \frac{\partial^2 (\phi)}{\partial r^2} + \frac{1}{r} \frac{\partial (\phi)}{\partial r} + \frac{1}{r^2} \frac{\partial^2 (\phi)}{\partial \theta^2}
\end{align*}
\]
Displacement solution

Cartesian coordinate system
- when the body forces are neglected, then the displacements can be expressed as

\[
\begin{align*}
U_x &= \frac{1}{2\mu} \left( \frac{1}{\beta} \frac{\partial \psi}{\partial y} - \frac{\partial \phi}{\partial x} \right) \\
U_y &= \frac{1}{2\mu} \left( \frac{1}{\beta} \frac{\partial \psi}{\partial x} - \frac{\partial \phi}{\partial y} \right)
\end{align*}
\]

where an additional potential function \( \psi(x, y) \) is introduced, which satisfies the conditions

\[
\nabla^2 \psi = 0, \quad \frac{\partial^2 \psi}{\partial x \partial y} = \nabla^2 \phi
\]

Cylindrical coordinate system

\[
\begin{align*}
U_r &= \frac{1}{2\mu} \left( \frac{r}{\beta} \frac{\partial \psi}{\partial \theta} - \frac{\partial \phi}{\partial r} \right) \\
U_\theta &= \frac{1}{2\mu} \left( \frac{r^2}{\beta} \frac{\partial \psi}{\partial r} - \frac{1}{r} \frac{\partial \phi}{\partial \theta} \right)
\end{align*}
\]

with

\[
\nabla^2 \psi = 0, \quad \frac{\partial}{\partial r} \left( r \frac{\partial \psi}{\partial \theta} \right) = \nabla^2 \phi
\]
Forms of Airy's stress function

- Depending on the type of the boundary conditions one can propose suitable stress functions providing the solution of plane elasticity problem.

- Two widely accepted classes of Airy stress functions are the polynominal and the Fournier series forms.

- The polynominal approach has severe theoretical limitations, because discontinuous boundary conditions are not representable by polynomials. Fournier series allow the study of very general loading cases such as loadings with discontinuities.

Polynominal Solutions

- A general form of the polynominal can be defined (in a Cartesian coordinate system) as

\[
\phi_N(x,y) = \sum_{i=0}^{N} A_i \cdot x^{N-i} \cdot y^i
\]

where we have \((N+1)\) independent coefficients, \(A_i \ (i=0,N)\).

- Any term containing the combination of \(x\) and \(y\) up to the third power, will automatically satisfy the biharmonic equation \(\nabla^4 \phi = 0\).

  The coefficients of higher order terms, are constrained in order \(\phi\) to be bi-harmonic.

- However, the forms of polynomial can be defined for which the combined degree of terms are not equal.
Example: Straight cantilever beam loaded by concentrated force at its end.

- at \( x=0 \) the beam is built in, whereas the free end \((x=L)\) is subjected to concentrated force \( P \).
- Find the stress distributions in the beam.
- It has been already solved in course Strength of Materials. Obtaining the solution using Airy stress function method is a much more sophisticated technique.

- Boundary conditions:

  B.C. 1: \( \gamma_{xy}(x, \pm c) = 0 \)
  B.C. 2: \( \sigma_y(x, \pm c) = 0 \)
  B.C. 3: \( \sigma_x(0, y) = 0 \)
  B.C. 4: \( \theta_c \int_{-c}^{c} \gamma_{xy}(0, y) \, dy = P \)

- The most important (and most difficult) step is to find a suitable Airy function. Even though the seeking is restricted to polynomial functions, finding the right terms is not a trivial task. Various attempts may lead us to the correct form of \( \phi(x,y) \).
- Let the first try to be
  \[ \phi = A \cdot xy^3 \]
  where \( A \) is a constant. This function satisfies the biharmonic equation \( \nabla^4 \phi = 0 \).

- The stress components are
  \[
  \sigma_x = \frac{\partial^2 \phi}{\partial y^2} = 6Ax y \\
  \sigma_y = \frac{\partial^2 \phi}{\partial x^2} = 0 \\
  \tau_{xy} = -\frac{\partial^2 \phi}{\partial x \partial y} = -3Ay^2
  \]

- It can be clearly concluded that B.C. 2 and B.C. 3 are identically satisfied, whereas B.C. 1 would be satisfied only if \( A = 0 \) which corresponds to a completely stress-free case, which is obviously not correct.

- The next attempt to formulate \( \phi \) is
  \[ \phi = A \cdot xy^3 + B \cdot xy \]

- In this case, the stress components become
  \[
  \sigma_x = 6A xy \\
  \sigma_y = 0 \\
  \tau_{xy} = -3Ay^2 - B
  \]

- B.C. 2 and B.C. 3 are satisfied again, whereas B.C. 1 gives the relation
  \[ B = -3A \cdot c^2 \]
Substituting into B.C. 4 gives

\[ \theta \int_{-c}^{c} (-3Ay^2 + 3Ac^2) dy = P \]

\[ 4A \theta c^3 = P \quad \rightarrow \quad A = \frac{P}{4 \theta c^3} \]

Therefore the final form of the Airy stress function is

\[ \phi = \frac{Pxy}{4 \theta c^3}(y^2 - 3c^2) \]

The stress solutions:

\[ \sigma_x = \frac{3Pxy}{2 \theta c^3} \]

\[ \sigma_y = 0 \]

\[ \tau_{xy} = \frac{3P}{4 \theta c^3}(c^2 - y^2) \]

Introducing the moment of inertia of the cross section w.r.t. the \( \theta \)-axis:

\[ I \theta = \frac{b \cdot (2c)^3}{12} = \frac{2}{3} bc^3 \]

The bending moment is given by

\[ M(x) = P \cdot x \]

Therefore, \( \sigma_x \) can be written as

\[ \sigma_x = \frac{M(x)}{I \theta} \cdot y \]

which is identical to the solution obtained in Strength of Materials.
- End cross sections are subjected to torque $M_t$
- Lateral surface is free from any loads
- Noncircular cross section
- Saint-Venant's assumptions:
  - Each cross section's projections on the xy-plane behave as a rigid body rotating around axis z.
  - The rotation of a cross section is proportional to the distance measured from the end. Thus, $\Theta = \varphi \cdot z$, where $\varphi$ is the twist per unit length (twist factor)
  - Out-of-plane deformations of each cross section are the same and it is proportional to $\varphi$

**Displacement field**

$$r = \sqrt{x^2 + y^2}$$
- according to the first two assumptions, the in-plane displacement components are computed as

\[ u_x = r \cdot \cos(\Theta + \kappa) - r \cdot \cos \kappa \]
\[ = x (\cos \Theta - 1) - y \cdot \sin \Theta \]
\[ u_y = r \cdot \sin(\Theta + \kappa) - r \cdot \sin \kappa \]
\[ = y (\cos \Theta - 1) + x \cdot \sin \Theta \]

- the third assumption leads to

\[ u_z = U \cdot \varphi(x, y) \]

\text{warping function} (describes the out-of-plane distortion)

- in small strain deformation case, \( \Theta \ll 1 \), thus \( \cos \Theta \approx 1 \) and \( \sin \Theta \approx \Theta \).

Thus, the linearized displacements are

\[ u_x = -y \Theta \]
\[ u_y = x \Theta \]
\[ u_z = U \varphi \]

Using \( \Theta = U z \) gives

\[
\begin{bmatrix}
  u(x, y, z) \\
  u_x(y, z) \\
  u_y(x, y) \\
  u_z(x, y)
\end{bmatrix} =
\begin{bmatrix}
  u_x(y, z) \\
  u_y(x, y) \\
  u_z(x, y)
\end{bmatrix}
\]

- \( u_x = -U \cdot y z \)
- \( u_y = U \cdot x z \)
- \( u_z = U \cdot \varphi(x, y) \)
Strain fields
- The infinitesimal strain tensor has the matrix form
\[
[\varepsilon] = \begin{bmatrix}
0 & 0 & \frac{1}{2} \gamma_{xz} \\
0 & 0 & \frac{1}{2} \gamma_{yz} \\
\frac{1}{2} \gamma_{zx} & \frac{1}{2} \gamma_{zy} & 0
\end{bmatrix}
\]
where
\[
\gamma_{xz} = \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}, \quad \gamma_{yz} = \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}
\]
\[
\gamma_{zx} = \nu \left( \frac{\partial \varphi}{\partial x} - y \right), \quad \gamma_{zy} = \nu \left( \frac{\partial \varphi}{\partial y} + x \right)
\]

Stress field
- According to the Hooke's law, it follows that
\[
[\sigma] = \begin{bmatrix}
0 & 0 & \tau_{xz} \\
0 & 0 & \tau_{yz} \\
\tau_{zx} & \tau_{zy} & 0
\end{bmatrix}
\]
Thus
\[
\tau_{xz} = G \nu \left( \frac{\partial \varphi}{\partial x} - y \right), \quad \tau_{yz} = G \nu \left( \frac{\partial \varphi}{\partial y} + x \right)
\]
- The projected shear traction:
\[
\tau = \sqrt{\tau_{xz}^2 + \tau_{yz}^2}
\]
• Equilibrium

- in case of no body force, the equilibrium equations reduce to the following three equations

\[
\frac{\partial \gamma_x}{\partial z} = 0; \quad \frac{\partial \gamma_y}{\partial z} = 0; \quad \frac{\partial \gamma_z}{\partial x} + \frac{\partial \gamma_y}{\partial y} = 0
\]

The first two are satisfied, whereas the third one gives

\[
G U \left( \frac{\partial^2 \gamma}{\partial x^2} + \frac{\partial^2 \gamma}{\partial y^2} \right) = 0 \quad G U \neq 0
\]

\[
\frac{\partial^2 \gamma}{\partial x^2} + \frac{\partial^2 \gamma}{\partial y^2} = 0
\]

\[
\nabla^2 \gamma = 0
\]

- Thus, \( \gamma(x, y) \) must be harmonic in the region of the cross section

• Boundary conditions

- the boundary curve can be defined parametrically as

\[
P = x(s) \mathbf{e}_x + y(s) \mathbf{e}_y
\]

- tangent unit vector:

\[
m = \frac{d}{ds} \mathbf{e}_x + \frac{dy}{ds} \mathbf{e}_z
\]

- the outward unit normal vector by definition:

\[
n = m \times \mathbf{e}_z \quad ; \quad \begin{bmatrix} n_x \\ n_y \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{dy}{ds} \\ -\frac{dx}{ds} \\ 0 \end{bmatrix}
\]
Since the boundary is traction-free, it follows that
\[ \overline{\mathbf{t}} \cdot \mathbf{n} = 0 \]
\[ \rightarrow \]
\[ \gamma_{xz} \cdot n_x + \gamma_{yz} \cdot n_y = 0 \]
\[ G U \left[ \left( \frac{\partial \varphi}{\partial x} - y \right)n_x + \left( \frac{\partial \varphi}{\partial y} + x \right)n_y \right] = 0 \]
\[ \left( \frac{\partial \varphi}{\partial x} - y \right) \frac{du}{ds} - \left( \frac{\partial \varphi}{\partial y} + x \right) \frac{dx}{ds} = 0 \]

Neumann-type boundary condition

- an alternative form may be expressed as

\[ \frac{\partial \varphi}{\partial x} n_x + \frac{\partial \varphi}{\partial y} n_y = y \cdot n_x - x \cdot n_y \]

\[ \frac{\partial \varphi}{\partial s} = y \frac{du}{ds} + x \frac{dx}{ds} \]

\[ \frac{\partial \varphi}{\partial s} = \frac{1}{2} \frac{d}{ds} \left( x^2 + y^2 \right) \]
Moment in a cross section
- the stress vector at \( P \) is given by
\[
\begin{align*}
\tau &= \sigma \cdot \varepsilon_z \\
\tau &= \gamma_{xz} \varepsilon_x + \gamma_{yz} \varepsilon_y
\end{align*}
\]
- the moment of the stress vector with regard to the centre of the cross section is
\[
\mathbf{r}_P \times \tau = (x \varepsilon_x + y \varepsilon_y) \times (\gamma_{xz} \varepsilon_x + \gamma_{yz} \varepsilon_y) = (x \gamma_{yz} - y \gamma_{xz}) \varepsilon_z
\]
- Thus, the resultant moment around \( z \) is
\[
M_t = \int_A (x \gamma_{yz} - y \gamma_{xz}) \, dA
\]
\[
M_t = G J_t \int_A \left[ \left( \frac{\partial \psi}{\partial y} + x \right)x - \left( \frac{\partial \psi}{\partial x} - y \right)y \right] \, dA
\]
\[
J_t : \text{torsional constant or torsional rigidity}
\]
\[
M_t = G J_t
\]
with
\[
J_t = \int_A \left( x^2 + y^2 + x \frac{\partial \psi}{\partial y} - y \frac{\partial \psi}{\partial x} \right) \, dA
\]
\( J_t \) depends only on the geometry.
• Summary

- Find $\phi(x,y)$ which is harmonic and, in addition, satisfies the traction-free boundary condition.
- Once $\phi(x,y)$ is known, stress components, strain components, and displacements can be computed.
- Since this method is a displacement-based formulation, compatibility equations are not need to be considered here.
Example 1  Circular cylinder

- for circular cross section, one may assume that
  \[ \varphi(x,y) = 0 \]
  - it is trivially harmonic, i.e. \( \Delta \varphi = 0 \) is satisfied
  - boundary condition becomes
  \[ \frac{\partial \varphi}{\partial s} = 0 = \frac{1}{2} \frac{d}{ds} (x^2 + y^2) \]
  \[ 0 = \frac{d}{ds} (x^2 + y^2) \quad \text{along the boundary curve} \]
  Integrating it gives
  \[ x^2 + y^2 = c^2 \quad \text{where } c \text{ is a constant} \]
  Selecting \( c = R \), the boundary condition is satisfied.

- For circular cross section, the torsional constant equals to the polar moment of inertia:
  \[ J_t = I_p = \int (x^2 + y^2) \, dA = \int r^2 \, dA = \frac{1}{2} \pi R^4 = \frac{D^4 \pi}{64} \]

- twist factor:
  \[ \psi = \frac{M_t}{G J_t} = \frac{2M_t}{G \pi R^4} \]

- stress components:
  \[ \gamma_{xt} = -\frac{2M_t}{J R^4} y \]
  \[ \gamma_{yt} = \frac{2M_t}{J R^4} x \]

- projected shear traction:
  \[ \gamma = \frac{2M_t}{J R^4} r = \frac{M_t}{I_p} r \]
Example 2  Elliptical cylinder

- boundary curve: 
  \[ \frac{(x/a)^2 + (y/b)^2}{a^4 + b^4} = 1 \]

- Let the warping function to be \[ \varphi(x,y) = k \cdot xy \]
- it is harmonic, i.e. \( \Delta \varphi = 0 \)
- the boundary condition becomes
  \[ (k y - y) \frac{dy}{ds} - (k x + x) \frac{dx}{ds} = 0 \]
  \[ (k-1) y \frac{dy}{ds} - (k+1) x \frac{dx}{ds} = 0 \]
  \[ \frac{1}{2} \frac{d}{ds} \left[ (k-1)y^2 - (k+1)x^2 \right] = 0 \]

Integrating it gives

\[ \frac{x^2}{a^2} - \frac{b-1}{k+1} \frac{y^2}{b^2} = C^2 \]
where \( C \) is constant

\[ \frac{x^2}{a^2} + \frac{1-b}{(1+k)c^2} \frac{y^2}{b^2} = 1 \]

Let \( c = a \), then

\[ \frac{x^2}{a^2} + \frac{1-k}{(1+k)a^2} \frac{y^2}{b^2} = 1 \]
\[ = \frac{1}{b^2} \]
\[ \frac{1}{a^2} = \frac{1 - k}{(1+k) a^2} \quad \Rightarrow \quad \frac{b^2 - a^2}{b^2 + a^2} = k \]

Thus, the warping function which is harmonic and satisfies the boundary condition:

\[ \psi(x,y) = \frac{b^2 - a^2}{b^2 + a^2} \cdot xy \]

- the torsion constant:

\[
\overline{J}_t = \int_A (x^2 + y^2 + kx^2 - ky^2) \, dA \\
= (1+k) \int_A x^2 \, dA + (1-k) \int_A y^2 \, dA \\
\quad \begin{array}{c}
\underline{I_y} \quad \underline{I_x} \\
\text{moments of inertia}
\end{array}
\]

\[
I_x = \frac{ab^3 \overline{J}_t}{4} \\
I_y = \frac{ba^3 \overline{J}_t}{4}
\]

Therefore

\[
\overline{J}_t = (1+k)I_y + (1-k)I_x
\]

\[
\overline{J}_t = \frac{a^3 b^3 \overline{J}_t}{a^2 + b^2}
\]

- the twist factor becomes

\[ \psi = \frac{M_t (a^2 + b^2)}{Ga^3 b^3 \overline{J}_t} \]
- stress components:

\[ \gamma_{x_2} = GV(\frac{b}{a} - 1)y = -GV \frac{2a^2}{a^2 + b^2}y \]

\[ \gamma_{y_2} = GV(\frac{b}{a} + 1)x = GV \frac{2b^2}{a^2 + b^2}x \]

or

\[
\begin{array}{|c|c|}
\hline
\gamma_{x_2} &=& -\frac{2Mt}{ab\sqrt{I}} \cdot \frac{y}{b^2} \\
\hline
\gamma_{y_2} &=& \frac{2Mt}{ab\sqrt{I}} \cdot \frac{x}{a^2} \\
\hline
\end{array}
\]

- the projected shear traction:

\[ \gamma = \frac{2GV}{a^2 + b^2} \sqrt{b^4x^2 + a^4y^2} \]

\[ \gamma = \frac{2Mt}{a^3b^3\sqrt{I}} \sqrt{b^4x^2 + a^4y^2} \]

- if \( a > b \), then the maximum value occurs at \( x = 0, y = \pm b \)

\[ \gamma_{\text{max}} = \frac{2GV^2a^2b}{a^2 + b^2} \]

\[ \gamma_{\text{max}} = \frac{2Mt}{a b^2\sqrt{I}} \]
Formulation using Prandtl's stress function

- If the cross section is not a circle or ellipse, then difficult to satisfy the traction-free boundary condition.
- In this case, it is better to use Prandtl's stress function.

Prandtl's stress function is defined as

\[ \begin{array}{c|c}
\tau_{x2} & \frac{\partial \Psi}{\partial y} \\
\tau_{y2} & -\frac{\partial \Psi}{\partial x} \\
\end{array} \]

\( \Psi(x,y) \): Prandtl's stress function

- This definition guarantees that the equilibrium equation is satisfied.
- The traction-free boundary condition is expressed as

\[ \tau_{x2} \cdot n_x + \tau_{y2} \cdot n_y = 0 \]
\[ \frac{\partial \Psi}{\partial y} \frac{dy}{ds} + \frac{\partial \Psi}{\partial x} \frac{dx}{ds} = 0 \]

\[ \frac{d\Psi}{ds} = 0 \] along the boundary.

It follows that \( \Psi \) must be constant on the boundary.
For solid cylinders, without loss of generality, the boundary condition can be written as

\[ \Psi = 0 \] on the boundary.
by comparing the definitions for the stress components, one finds that
\[
\frac{\partial \psi}{\partial y} = G \nu \left( \frac{\partial \psi}{\partial x} - y \right)
\]
\[
\frac{\partial \psi}{\partial x} = -G \nu \left( \frac{\partial \psi}{\partial y} + x \right)
\]
Eliminating \( \psi \) from this system gives
\[
\nabla^2 \psi = -2G \nu
\]
in the cross section

This guarantees that the compatibility equations are satisfied.

The resultant moment is given by
\[
M_t = - \int_A \left( \frac{\partial \psi}{\partial x} x + \frac{\partial \psi}{\partial y} y \right) dA
\]

Using the Green-Riemann theorem and the fact that \( \psi = 0 \) on the boundary, we obtain (for solid cylinders) that
\[
M_t = 2 \int_A \psi dA
\]

Thus, the torsional rigidity becomes
\[
J_t = \frac{2}{G \nu} \int_A \psi dA
\]
Example 3  Equilateral Triangular Cylinder

- Find the maximum shear traction!

- One can easily check, that the Prandtl's stress function which satisfies the boundary condition and the compatibility condition has the form:

\[
\psi(x, y) = -\frac{G\nu}{6a} \left( x^2 + 3ax^2 + 3ay^2 - 3xy^2 - 4a^2 \right)
\]

- The stress components are

\[
\gamma_{xz} = \frac{\partial \psi}{\partial y} = \frac{G\nu}{a} (x-a)y
\]

\[
\gamma_{yz} = -\frac{\partial \psi}{\partial x} = \frac{G\nu}{2a} (x^2 + 2ax - y^2)
\]

- The resultant moment:

\[
M_t = 2 \int \psi dA = \frac{9}{5} \sqrt{3} \cdot G\nu a^4
\]

- Torsional constant:

\[
J_t = \frac{M_t}{G\nu} = \frac{9}{5} \sqrt{3} a^4
\]
- the maximum shear traction always occurs on the boundary. On side \( x = a \), the projected shear traction becomes

\[
\gamma = \gamma_y = \frac{Gy}{2a} (3a^2 - y^2)
\]

Thus, the maximum value equals to

\[
\gamma_{\text{max}} = \frac{3}{2} Gu a = \frac{5}{6\sqrt{3}} \frac{M_4}{a^3}
\]

The maximum value occurs at the midpoint of the sides, where the points are nearest to the centroid.

- The warping function can be found as

\[
\psi(x, y) = \frac{y}{6a} (3x^2 - y^2)
\]

- the maximum out-of-plane displacement on side \( x = a \) occurs at

\[
\frac{\partial \psi(a, y)}{\partial y} = 0 \quad \Rightarrow \quad y = \pm a
\]
One-dimensional elastic-plastic model with linear hardening:

- Idealized stress-strain curve for uniaxial extension:

![Diagram of stress-strain curve]

\[ \varepsilon = \varepsilon^e + \varepsilon^p \]

- Additive split of the total strain:

- Elastic relationship is given by Hooke's law:

\[ \sigma = E \varepsilon^e = E (\varepsilon - \varepsilon^p) \]

- The yield function and the yield condition is defined as:

\[ F = \sigma_{eq} - \sigma_y \leq 0 \]

- \( \sigma_{eq} \): equivalent stress
- \( \sigma_y > 0 \): current yield stress

\( F < 0 \): elastic state
\( F = 0 \): plastic state
\( F > 0 \): non-admissible
- Plastic flow rule:

\[ \dot{\varepsilon}^p = \dot{\lambda} \cdot \frac{\partial F}{\partial \varepsilon} \]

where \( d\lambda \geq 0 \) represents the absolute value of the plastic flow increment. The sign function is defined as

\[
\text{sign}(x) = \begin{cases} 
+1 & \text{if } x > 0 \\
-1 & \text{if } x < 0 
\end{cases}
\]

- Loading/unloading conditions (Kuhn-Tucker conditions):

\[ d\lambda \geq 0 \quad F \leq 0 \quad d\lambda \cdot F = 0 \]

- Consistency condition:

\[ d\lambda \cdot dF = 0 \]

\[ F < 0 \Rightarrow d\lambda = 0 \]

\[ d\lambda > 0 \Rightarrow F = 0 \]

- However, the model is rate-independent, it is convenient to formulate the expression in rate-form.

Therefore

\[ \dot{\varepsilon}^p = \dot{\lambda} \cdot \frac{\partial F}{\partial \varepsilon} \]

\[ \dot{\lambda} \geq 0 \quad \dot{\lambda} \cdot F = 0 \quad \dot{\lambda} \cdot F = 0 \]

Hardening rules

- Linear isotropic hardening

\[ \dot{\varepsilon}_y = \dot{\varepsilon}_y^0 + H_{iso} \cdot \dot{\gamma} \]

where \( H_{iso} \): isotropic plastic hardening modulus

\( \dot{\gamma} \): internal hardening variable

\[ \dot{\gamma} = |\dot{\varepsilon}^p| \Rightarrow \gamma = \int_0^t \dot{\gamma} \, dt \]
- linear kinematic hardening

In this case, the yield function is defined as

\[ F = |\sigma - \alpha| - \bar{\sigma}_0 \]

; \quad \dot{\sigma}^p = \lambda \text{sign}(\sigma - \alpha)

\( \alpha \): "back stress" represents the centre of the elastic domain.

Ziegler's rule for the evolution of \( \alpha \):

\[ \dot{\alpha} = H_{\text{kin}} \dot{\sigma}^p \]

\( H_{\text{kin}} \): kinematic plastic hardening modulus

- linear mixed hardening

\[ H_{\text{iso}} = M \cdot H \]

\[ H_{\text{kin}} = (1 - M)H \]

\[ F = |\sigma - \alpha| - \bar{\sigma}_Y \]
Tangent elastoplastic modulus:

- Pure linear isotropic hardening:
  - From the consistency condition it follows that for plastic loading, \( F = 0 \) holds.
  - Thus:
    \[
    \frac{d}{dt} \left( |\sigma| - \sigma_Y \right) = 0
    \]
    \[
    \frac{d}{dt} |\sigma| - \sigma_Y = 0
    \]
    \[
    \frac{\partial |\sigma|}{\partial \sigma} \dot{\sigma} - \dot{\sigma}_Y = 0
    \]
    \[
    \text{sign}(\sigma) E \ddot{\varepsilon} - H \dot{\lambda} = 0
    \]
    \[
    \text{sign}(\sigma) E \dot{\varepsilon} - \text{sign}(\sigma) E \dot{\varepsilon}^p - H \dot{\lambda} = 0
    \]
    \[
    \text{sign}(\sigma) E \dot{\varepsilon} - \text{sign}(\sigma) E\varepsilon^p \dot{\lambda} - H \dot{\lambda} = 0
    \]
    \[
    \text{sign}(\sigma) E \dot{\varepsilon} - (H + E) \dot{\lambda} = 0
    \]

- Thus, the stress rate in plastic loading:
  \[
  \dot{\sigma} = E \dot{\varepsilon} - \text{sign}(\sigma) E \frac{\text{sign}(\sigma) E \dot{\varepsilon}}{E + H}
  \]
  \[
  = E \dot{\varepsilon} - \frac{E^2}{E + H} \dot{\varepsilon}
  \]
  \[
  \dot{\sigma} = \frac{EH}{E + H} \dot{\varepsilon}
  \]

\( E^{ep} \): elastoplastic tangent modulus
\[ \frac{d}{dt} \left( |\sigma - \lambda| - \sigma y_0 \right) = 0 \]
\[ \frac{d}{dt} |\sigma - \lambda| = 0 \]
\[ 2 |\sigma - \lambda| \frac{\dot{\sigma}}{(\sigma - \lambda)} = 0 \]
\[ \text{sign}(\sigma - \lambda) \dot{\sigma} = \text{sign}(\sigma - \lambda) \dot{\lambda} = 0 \]
\[ \text{sign}(\sigma - \lambda) E \dot{\varepsilon} = \text{sign}(\sigma - \lambda) E \dot{\lambda} \text{sign}(\sigma - \lambda) = \text{sign}(\sigma - \lambda) H \dot{\varepsilon} = 0 \]
\[ \text{sign}(\sigma - \lambda) E \dot{\varepsilon} - E \dot{\lambda} - H \dot{\lambda} = 0 \]

Thus, the stress rate:
\[ \dot{\sigma} = E \dot{\varepsilon} - \text{sign}(\sigma - \lambda) \frac{\text{sign}(\sigma - \lambda) E \dot{\varepsilon}}{E + H} \]

\[ \dot{\sigma} = \frac{E H \dot{\varepsilon}}{E + H} \]

mixed linear hardening:
\[ E_{\text{ef}} = \frac{E H}{E + H} \]
\[ H = H_{\text{iso}} + H_{\text{kin}} \]
Summary:

\[
\dot{\sigma} = \begin{cases} 
E\dot{\varepsilon} & \text{if } \dot{\lambda} = 0 \\
E_{sp}\dot{\varepsilon} & \text{if } \dot{\lambda} > 0 
\end{cases}
\]

Outline of the computational procedure for discretized strain-driven case:

- All variables are known at \( t_n \)
- \( \Delta \varepsilon \) is given
- Compute the trial stress as \( \sigma_{trial} = \sigma_n + E \cdot \Delta \varepsilon \)
- Evaluate the yield function for the trial state: \( F_{trial} \)
- Depending on \( F_n \) and \( F_{trial} \), the following cases can happen:
  
  \[ F_n < 0 \]
  - \( F_{trial} \leq 0 \): elastic loading \( \text{(A)} \)
  - \( F_{trial} > 0 \): elastic to plastic transition \( \text{(B)} \)

  \[ F_n = 0 \]
  - \( F_{trial} \leq 0 \): elastic unloading \( \text{(C)} \)
  - \( F_{trial} > 0 \)
    - \( \frac{\partial F}{\partial \sigma} \cdot \text{sign}(\Delta \sigma) = -1 \)
      - elastic to plastic transition \( \text{(D)} \)
    - \( \frac{\partial F}{\partial \sigma} \cdot \text{sign}(\Delta \sigma) = 1 \)
      - plastic loading \( \text{(E)} \)

Illustration of cases \( \text{A} - \text{E} \) for the case of pure isotropic hardening:
Example 1

The loading history of an uniaxial tension/compression process is given as

\[ \sigma = 200, 0, -300 \]

\[ \sigma_{\text{Y}} = 100 \text{ MPa} \]

\[ E = 225 \text{ GPa} \]

\[ H = 25 \text{ GPa} \]

\[ E^P = \frac{225 \times 25}{225 + 25} = 22.5 \text{ GPa} \]

- Determine the strains at A, B, and C
- Illustrate the evolution of the total and plastic strain
- Consider two cases: a) isotropic hardening, b) kinematic hardening

**a)** Loading path \( O \rightarrow A \)

\[ \sigma_{\text{Y}} = \sigma_{\text{Y0}} = 100 \text{ MPa} \]

\[ F_n = |\sigma_0| - \sigma_{\text{Y}} = 0 - 100 = -100 \quad \text{elastic to plastic transition} \]

\[ F_{\text{pl}} = |\sigma_A| - \sigma_{\text{Y}} = 200 - 100 = 100 \]

Boundary of the elastic domain: \( \sigma_{\text{Y0}} = 100 \text{ MPa} \)

Path \( O \rightarrow Y_0 \):

\[ \varepsilon_{\text{Y0}} = \varepsilon_0 + \frac{\sigma_{\text{Y0}} - \sigma_0}{E} = 0.444 \times 10^{-3} \]

\[ \varepsilon_{\text{Y0}}^P = \varepsilon_0^P = 0 \]

Part \( Y_0 \rightarrow A \):

\[ \varepsilon_A = \varepsilon_{\text{Y0}} + \frac{\sigma_A - \sigma_{\text{Y0}}}{E^P} = 4.888 \times 10^{-3} \]

\[ \varepsilon_{\text{Y0}}^P = \varepsilon_0^P + \frac{\sigma_A - \sigma_{\text{Y0}}}{H} = 4 \times 10^{-3} \]
loading path \( A \rightarrow B \)

\[ \sigma_Y = 200 \text{ MPa} \]

\[
\begin{align*}
F_n &= 0 \\
F_{\text{final}} &= \left| -300 \right| - 200 = 100
\end{align*}
\]

\[ \text{sign}(\sigma_A) \cdot \text{sign}(-200) = -1 \]

elastic unloading, then plastic loading

boundary of the elastic domain: \( \sigma_{Y_1} = -200 \text{ MPa} \)

path \( A \rightarrow Y_1 \):

\[
\varepsilon_{Y_1} = \varepsilon_A + \frac{\sigma_{Y_1} - \sigma_A}{E} = 3.11 \times 10^{-3}
\]

\[ \varepsilon_{Y_1}^p = \varepsilon_A^p = 4 \times 10^{-3} \]

path \( Y_1 \rightarrow B \):

\[
\varepsilon_B = \varepsilon_{Y_1} + \frac{\sigma_B - \sigma_{Y_1}}{E^p} = -1.33 \times 10^{-3}
\]

\[ \varepsilon_B^p = \varepsilon_{Y_1}^p + \frac{\sigma_B - \sigma_{Y_1}}{H} = 0 \]

loading path \( B \rightarrow C \)

\[ \sigma_Y = 300 \text{ MPa} \]

\[
\begin{align*}
F_n &= 0 \\
F_{\text{final}} &= 0 - 300 = -300
\end{align*}
\]

\[ \text{elastic unloading} \]

\[ \varepsilon_C = \varepsilon_B + \frac{\sigma_C - \sigma_B}{E} = 0 \]

\[ \varepsilon_C^p = \varepsilon_B^p = 0 \]
loading path \( O \rightarrow A \)

same as in a)

loading path \( A \rightarrow B \)

\[ \lambda_A = 100 \text{ MPa} \]

\[ F_n = 0 \]

\[ F_{\text{final}} = |\sigma_B - \lambda_A| - \delta_Y > 0 \]

\[ \text{sign}(\sigma_B - \lambda_A) \cdot \text{sign}(\sigma_B - \delta_A) = -1 \]

elastic unloading prior to plastic loading

boundary of the elastic domain: \( \delta_{Y_1} = 0 \text{ MPa} \)

path \( A \rightarrow Y_1 \):

\[ E_{Y_1} = E_A + \frac{\delta_{Y_1} - \delta_A}{E} = 4 \times 10^{-3} \]

\[ E_{Y_1}^p = E_A^p = 4 \times 10^{-3} \]

path \( Y_1 \rightarrow B \):

\[ E_B = E_{Y_1} + \frac{\delta_B - \delta_{Y_1}}{E_{\sigma_B}} = -9.333 \times 10^{-3} \]

\[ E_B^p = E_{Y_1}^p + \frac{\delta_B - \delta_{Y_1}}{H} = -8 \times 10^{-3} \]

\[ \lambda_{Y_1} = \lambda_A = 100 \text{ MPa} \]

\[ \lambda_B = \lambda_{Y_1} + H(E_B^p - E_{Y_1}^p) = -200 \text{ MPa} \]

loading path \( B \rightarrow C \)

\[ F_n = 0 \]

\[ F_{\text{final}} = |\sigma_C - \lambda_B| - \delta_Y > 0 \]

\[ \text{sign}(\sigma_C - \lambda_B) \cdot \text{sign}(\sigma_C - \delta_B) = -1 \]

elastic unloading prior to plastic process
boundary of the elastic domain: $\sigma_{y_2} = -100 \text{ MPa}$

path $B \rightarrow Y_2$:

$$\varepsilon_{y_2} = \varepsilon_B + \frac{\sigma_{y_2} - \sigma_B}{E} = -8.44 \times 10^{-3}$$

$$\varepsilon_P = \varepsilon_B = -8 \times 10^{-3}$$

path $Y_2 \rightarrow C$:

$$\varepsilon_C = \varepsilon_{y_2} + \frac{\sigma_C - \sigma_{y_2}}{E_P} = -4 \times 10^{-3}$$

$$\varepsilon_P = \varepsilon_{y_2} + \frac{\sigma_C - \sigma_{y_2}}{H} = -4 \times 10^{-3}$$
- the elastic stress solutions for pressurized hollow cylinder are

\[
\begin{align*}
\sigma_t(\varphi) &= A + B \cdot \varphi \\
\sigma_r(\varphi) &= A - B \cdot \varphi
\end{align*}
\]

with

\[
\begin{align*}
B &= \frac{\Delta p}{1 - \varphi_0} \\
A &= B \cdot \varphi_0 - \varrho_0 \\
\Delta p &= \varrho_i - \varrho_0 \\
\varphi &= \left(\frac{R_i}{r}\right)^2 \\
\varphi_0 &= \left(\frac{R_i}{R_0}\right)^2
\end{align*}
\]

- Assuming that \( \sigma_t \geq \sigma_z \geq \sigma_r \), the Mohr's equivalent stress along \( r \) becomes

\[
\sigma_{eq} = \sigma_t - \sigma_3 = \sigma_t - \sigma_r = 2B \varphi
\]

its maximum value exists at \( r = R_i \) (\( \varphi = 1 \)):

\[
\sigma_{eq} = 2B = \frac{2 \Delta p}{1 - \varphi_0}
\]

- The elastic limit load \( \Delta p_E \) at which plastic flow initiates can be expressed by applying a particular yield criterion. If Tresca's yield criterion is used for elastic - perfectly plastic material, then

\[
\Delta p_E = \frac{6Y}{2} (1 - \varphi_0)
\]
- plastic zone initiates at the inner surface

- when \( \Delta p > \Delta p_E \) then the elastically deformed zone behaves as an elastic hollow cylinder with inner pressure \(-\sigma_r(R_c)\) exerted on the inner surface \(R_c\). Thus, the following solutions are obtained for the elastic zone:

\[
\sigma_t(\varphi) = \tilde{A} + \tilde{B} \varphi \\
\sigma_r(\varphi) = \tilde{A} - \tilde{B} \varphi
\]

with

\[
\tilde{B} = \frac{\Delta p_E}{1 - \varphi_0} = \frac{6Y(1 - \varphi_0)}{2(1 - \varphi_0)} = \frac{6Y}{2}
\]

\[
\tilde{A} = \tilde{B} \cdot \tilde{\varphi}_0 - P_0 = \frac{6Y}{2} \tilde{\varphi}_0 - P_0
\]

\[
\tilde{\varphi} = \left(\frac{r}{R_c}\right)^2
\]

\[
\tilde{\varphi}_0 = \left(\frac{r}{R_c}\right)^2
\]

It is more convenient to express them using variable \( \psi \):

<table>
<thead>
<tr>
<th>( \sigma_t(\varphi) = \bar{A} + \bar{B} \psi )</th>
<th>with ( \bar{B} = \frac{6Y}{2 \Psi_c} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma_r(\varphi) = \bar{A} - \bar{B} \psi )</td>
<td>( \bar{A} = \frac{6Y}{2 \Psi_c} \tilde{\varphi}_0 - P_0 )</td>
</tr>
</tbody>
</table>

\( \Psi_c = \left(\frac{R_i}{R_c}\right)^2 \)
- From the Tresca's yield criterion, it follows that $\sigma_t - \sigma_r = \sigma_Y$ in the plastic zone. Consequently, the equilibrium equation in this domain ($R_i \leq r \leq R_e$) becomes

$$\frac{d\sigma_r}{dr} + \frac{1}{r} (\sigma_r - \sigma_t) = 0$$

$$\frac{d\sigma_r}{dr} - \frac{1}{r} \sigma_Y = 0$$

The boundary condition is specified as $\sigma_r(R_i) = -p_i$.

Thus, after integrating, one gets the solution

$$\sigma_r(r) = \sigma_Y \cdot \ln \frac{r}{R_i} - p_i$$

$$\sigma_t(r) = \sigma_r(r) + \sigma_Y = \sigma_Y (1 + \ln \frac{r}{R_i}) - p_i$$

- The stress solutions expressed with variable $\psi$:

<table>
<thead>
<tr>
<th>$\sigma_r(\psi)$</th>
<th>$- \frac{\sigma_Y}{2} \ln \psi - p_i$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_t(\psi)$</td>
<td>$- \frac{\sigma_Y}{2} (\ln \psi - 2) - p_i$</td>
</tr>
</tbody>
</table>

- The radius $R_e$ can be obtained from the boundary condition expressing that the radial stress at $R_e$ must be the same for using the elastic zone or plastic zone solutions:

$$A - B \psi_e = - \frac{\sigma_Y}{2} \ln \psi_e - p_i$$

$$\frac{\sigma_Y}{2 \psi_e} \psi_e - p_o - \frac{\sigma_Y}{2} = - \frac{\sigma_Y}{2} \ln \psi_e - p_i$$
\[ \Delta p = \frac{5y}{2} \left( 1 - \frac{\psi_0}{\psi_e} - \ln \psi_e \right) \]

For given \( \Delta p, R_i, R_o, 5y \) this equation cannot be solved analytically for \( \psi_e \) (or \( \psi_e \)).

- The ultimate pressure difference (plastic limit load) at which the whole tube becomes plastic can be obtained from the condition \( R_e = R_o \):

\[ \Delta p_p = -\frac{5y}{2} \ln \psi_0 \]

- The strain components, the radial and axial displacements and the axial stress solutions depend on the end constraints (closed end, open end, plane strain). Interested students find details in Lubliner: Plasticity theory, for instance.

- The derived solutions are valid until the initial assumption \( 6_t \geq 6_\theta \geq 6_r \) holds.

It is satisfied when

\[ \ln \frac{R_o}{R_i} \leq \begin{cases} 
\frac{(1-\nu)}{(1-2\nu)} & \text{plane strain} \\
\frac{1-\nu}{1-2\nu} \left( 1 - \psi_0 \right) & \text{closed end} \\
\frac{(1-\nu)(1-\psi_0)}{1-2\nu - \psi_0} & \text{open end}
\end{cases} \]

(detailed derivation steps can be found in Lubliner's book, for instance)
If \( \gamma = 0.3 \) then the limits for the ratio \( R_0/R_i \) becomes:

- Plane strain: 5.75
- Closed end: 5.42
- Open end: 6.19

**Residual stresses:**

- If the pressure \( \Delta P > \Delta P_E \) is completely removed by releasing the pressure, then the unloading process involves pure elastic deformation if \( \Delta P \) is not too great.

- The residual stresses are obtained by subtracting the hypothetical elastic stresses from the elastic-plastic stresses. The hypothetical elastic stresses are calculated using \( \Delta P \) in the pure elastic stress solutions.

Thus, for the elastic zone:

\[
\sigma_t^{res}(\varphi) = \bar{A} + \bar{B}\varphi - (A + B\varphi)
\]

\[
= \frac{5\gamma}{2\gamma_c} \varphi_0 - P_0 + \frac{5\gamma}{2\gamma_c} \varphi - \frac{\Delta P}{1-\varphi_0} (\varphi_0 + P_0 - \frac{\Delta P}{1-\varphi_0} \varphi
\]

\[
= \frac{5\gamma}{2\gamma_c} (\varphi_0 + \varphi) - \frac{\Delta P}{1-\varphi_0} (\varphi_0 + \varphi)
\]

\[
= \left(\frac{5\gamma}{2\gamma_c} - \frac{\Delta P}{1-\varphi_0}\right)(\varphi_0 + \varphi)
\]

\[
\sigma_r^{res}(\varphi) = \bar{A} - \bar{B}\varphi - (A - B\varphi)
\]

\[
= \left(\frac{5\gamma}{2\gamma_c} - \frac{\Delta P}{1-\varphi_0}\right)(\varphi_0 - \varphi)
\]
whereas for the plastic zone:

\[ \sigma_{t}^{\text{res}}(\varphi) = -\frac{\bar{\sigma}_y}{2} (\ln \varphi - 2) - p_i - (A + B \varphi) \]

\[ = -\frac{\bar{\sigma}_y}{2} (\ln \varphi - 2) - p_i - \frac{\Delta P}{1-\varphi_0} \varphi_0 + p_0 - \frac{\Delta P}{1-\varphi_0} \varphi \]

\[ = -\frac{\bar{\sigma}_y}{2} (\ln \varphi - 2) - \frac{\Delta P}{1-\varphi_0} (\varphi_0 + \varphi) - \Delta P \]

\[ = -\frac{\bar{\sigma}_y}{2} (\ln \varphi - 2) - \Delta P \frac{1+\varphi}{1-\varphi_0} \]

\[ \sigma_{r}^{\text{res}}(\varphi) = -\frac{\bar{\sigma}_y}{2} \ln \varphi - p_i - (A - B \varphi) \]

\[ = -\frac{\bar{\sigma}_y}{2} \ln \varphi - p_i - \frac{\Delta P}{1-\varphi_0} \varphi_0 + p_0 + \frac{\Delta P}{1-\varphi_0} \varphi \]

\[ = -\frac{\bar{\sigma}_y}{2} \ln \varphi - \Delta P \frac{1-\varphi}{1-\varphi_0} \]

- The largest value of the Mohr's equivalent stress \(|\sigma_t^{\text{res}} - \sigma_r^{\text{res}}|\) occurs at \(\varphi = 1\):

\[ \sigma_{\text{eqmax}}^{\text{res}} = \bar{\sigma}_y \left( \frac{\Delta P}{\Delta P_E} - 1 \right) \]

Thus, renewed yielding occurs, when \(\Delta P/\Delta P_E = 2\).

- Therefore, the unloading is elastic if \(\Delta P < 2\Delta P_E\).

- The largest wall ratio for which the unloading is elastic can be found from the relation

\[ \Delta P_p < 2\Delta P_E \]
\[-\frac{6\gamma}{2} \ln \varphi_0 < 6\gamma (1-\varphi_0)\]

\[\Rightarrow \]

\[\varphi_{0,\text{max}} = 0.203188\]

\[\Rightarrow \]

\[\left(\frac{R_o}{R_i}\right)_{\text{max}} = 2.218\]

\[\Rightarrow \]

\[\Delta p_{\text{max}} = 0.797 6\gamma\]

- When \(R_o/R_i > 2.218\) and \(\Delta p > 2\Delta p_e\) then unloading produces a new plastic zone.
Example

A hollow cylinder is made of linear elastic—perfectly plastic material. The inner radius is denoted with $R_i$, whereas the outer radius with $R_o$. The ends are open ends.

Data:
- $R_i = 20\text{cm}$
- $R_o = 50\text{cm}$
- $\sigma_Y = 160\text{ MPa}$
- $\gamma = 0.3$
- $p_i = 1000\text{ bar}$
- $p_o = 0$

The inner pressure is $p_i$, whereas the outer pressure is omitted.

Determine:
- a) the elastic and plastic limit loads
- b) the radius $R_e$ at which the elastic and plastic zones are connected
- c) stress distributions
- d) residual stresses

Solution:

a) $\Psi_o = \left(\frac{R_i}{R_o}\right)^2 = \left(\frac{20}{50}\right)^2 = 0.16$

$\Delta p_E = \frac{\sigma_Y}{2} (1-\Psi_o) = 67.2\text{ MPa}$

$\Delta p_p = -\frac{\sigma_Y}{2} \ln \Psi_o = 146.607\text{ MPa}$

Since $\Delta p = 100\text{ MPa}$ is greater than $\Delta p_E$, but smaller than $\Delta p_p$, it follows that elastic-plastic transition occurs.
b) The equation, which implicitly defines \( \psi_c \):

\[
\Delta p = \frac{6\gamma}{2} \left( 1 - \frac{\psi_o}{\psi_c} - \ln \psi_c \right)
\]

The solution is sought in the range \( \psi_o < \psi_c < 1 \)

\[
100 = 80 \left( 1 - \frac{0.16}{\psi_c} - \ln \psi_c \right)
\]

\[
\ln \psi_c = -0.25 - \frac{0.16}{\psi_c}
\]

\[
\psi_c = 0.595233
\]

\[
R_c = \frac{R_i}{\psi_c} = 25.923 \text{cm}
\]

C) Stress solutions in the elastic zone:

\[
\bar{B} = \frac{6\gamma}{2\psi_c} = 134.401 \text{ MPa}
\]

\[
\bar{A} = \bar{B} \cdot \psi_o - p_o = 21.5042 \text{ MPa}
\]

\[
\sigma_r(\psi) = \bar{A} - \bar{B} \psi = 21.5042 - 134.401 \psi
\]

\[
\sigma_t(\psi) = \bar{A} + \bar{B} \psi = 21.5042 + 134.401 \psi
\]

These are valid in the range \( \psi_o < \psi < \psi_c \)
Stress solutions in the plastic zone:

\[ \sigma_r(\psi) = -\frac{5r}{2} \ln \psi - p_i = -80 \ln \psi - 100 \]

\[ \sigma_t(\psi) = -\frac{5r}{2} (\ln \psi - 2) - p_i = -80 \ln \psi + 60 \]

These solutions are valid in the range \( \psi_c < \psi < 1 \).

The stress solutions can be used if \( \sigma_r \leq \sigma_z \leq \sigma_t \) holds. For open ends tube it is satisfied when

\[ \ln \frac{R_o}{R_i} \leq \frac{(1-\nu)(1-\psi_o)}{1-2\nu-\psi_o} \]

\[ 0.916 \leq 2.45 \Rightarrow \text{satisfied} \]

The stress values at particular values of \( \psi \):

<table>
<thead>
<tr>
<th>( \psi )</th>
<th>( \sigma_r )</th>
<th>( \sigma_t )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-100</td>
<td>60</td>
</tr>
<tr>
<td>( \psi_c )</td>
<td>-58.496</td>
<td>101.504</td>
</tr>
<tr>
<td>( \psi_o )</td>
<td>0</td>
<td>43.008</td>
</tr>
</tbody>
</table>
d) Since $\Delta P < 2 \Delta P_E$, the unloading process involves only elastic deformation.

- The elastic stress components are:

\[ B = \frac{\Delta P}{1 - \psi_0} = 119.048 \text{ MPa} \]
\[ A = B \cdot \psi_0 - P_0 = 19.0476 \text{ MPa} \]

\[ \sigma_r^{\text{el}} (\psi) = 19.0476 - 119.048 \psi \]
\[ \sigma_t^{\text{el}} (\psi) = 19.0476 + 119.048 \psi \]
Thus, the residual stresses in the elastic zone:

\[
\begin{align*}
\sigma_r^{res}(\psi) &= \sigma_r(\psi) - \sigma_r^{el}(\psi) = 2.4566 - 15.3535\psi \\
\sigma_t^{res}(\psi) &= \sigma_t(\psi) - \sigma_t^{el}(\psi) = 2.4566 + 15.3535\psi
\end{align*}
\]

In the plastic zone:

\[
\begin{align*}
\sigma_r^{res}(\psi) &= \sigma_r(\psi) - \sigma_r^{el}(\psi) = -80\ln\psi + 119.048\psi - 119.048 \\
\sigma_t^{res}(\psi) &= \sigma_t(\psi) - \sigma_t^{el}(\psi) = -80\ln\psi - 119.048\psi + 40.952
\end{align*}
\]

Particular values:

<table>
<thead>
<tr>
<th>(\psi)</th>
<th>(\sigma_r)</th>
<th>(\sigma_t)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>-78.095</td>
</tr>
<tr>
<td>(\psi_c)</td>
<td>-6.682</td>
<td>11.596</td>
</tr>
<tr>
<td>(\psi_o)</td>
<td>0</td>
<td>4.913</td>
</tr>
</tbody>
</table>
The radius at which the maximum residual radial stress exists is derived as

\[ \frac{d}{d\varphi} \sigma_r^{\text{res}}(\varphi) = 0 \implies \varphi^* = \frac{6\gamma}{2\Delta \rho} (1-\varphi_o) \]

\[ \varphi^* = 0.672 \]

\[ R^* = \frac{R_i}{\varphi^*} = 24.3975 \text{ cm} \]

Thus, the maximum residual radial stress:

\[ \sigma_r^{\text{res}}(\varphi^*) = -7.2479 \text{ MPa} \]
3

Theory of small strain elastoplasticity

3.1 Analysis of stress and strain

3.1.1 Stress invariants

Consider the Cauchy stress tensor $\sigma$. The characteristic equation of $\sigma$ is

$$\sigma^3 - I_1 \sigma + I_2 \sigma - I_3 \delta = 0,$$

where the scalar stress invariants $I_1$, $I_2$ and $I_3$ are computed according to the formulas (de Souza Neto et al., 2008):

$$I_1 = \text{tr} \sigma, \quad I_2 = \frac{1}{2} ( (\text{tr} \sigma)^2 - \text{tr} (\sigma^2) ), \quad I_3 = \det \sigma.$$  \hspace{1cm} (3.2)

These stress invariants can be written in a simpler form using Cauchy principal stresses:

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3, \quad I_2 = \sigma_1 \sigma_2 + \sigma_2 \sigma_3 + \sigma_3 \sigma_1, \quad I_3 = \sigma_1 \sigma_2 \sigma_3.$$  \hspace{1cm} (3.3)

The deviatoric stress tensor is obtained by subtracting the hydrostatic (or spherical) stress tensor $p$ from the Cauchy stress tensor:

$$s = \sigma - p = \sigma - p\delta, \quad \text{where} \quad p = \frac{1}{3} \text{tr} \sigma.$$  \hspace{1cm} (3.4)

Its scalar invariants are

$$J_1 = \text{tr} s = 0, \quad J_2 = \frac{1}{2} \text{tr} (s^2), \quad J_3 = \det s = \frac{1}{3} \text{tr} (s^3).$$  \hspace{1cm} (3.5)
The characteristic equation of the deviatoric stress $s$ is
\[ s^3 - J_2 s - J_3 \delta = 0, \] (3.6)
where the stress invariants $J_2$ and $J_3$ can be expressed using the principal stress of $s$:
\[ J_1 = s_1 + s_2 + s_3 = 0, \quad J_2 = \frac{1}{2} (s_1^2 + s_2^2 + s_3^2), \quad J_3 = s_1 s_2 s_3 = \frac{1}{3} (s_1^3 + s_2^3 + s_3^3). \] (3.7)
The relations between the invariants $J_1$, $J_2$, $J_3$ and $I_1$, $I_2$, $I_3$ are
\[ J_2 = \frac{1}{3} (I_1^2 - 3I_2), \quad J_3 = \frac{1}{27} (2I_1^3 - 9I_1I_2 + 27I_3). \] (3.8)

### 3.1.2 Haigh–Westergaard stress space

In the study of elastoplasticity theory, it is usually convenient if we can somehow illustrate the meaning of expressions using geometrical representation. The basis of such illustrations is the introduction of the so-called **Haigh–Westergaard stress space** (Chen and Han, 2007; Haigh, 1920; Westergaard, 1920), where principal stresses are taken as coordinate axes. In this principal stress space, it is possible to illustrate a certain stress state\(^1\) as a geometrical point with coordinates $\sigma_1$, $\sigma_2$ and $\sigma_3$, as shown in Figure 3.1.

\[ \text{Figure 3.1: Haigh–Westergaard stress space.} \]

The straight line for which $\sigma_1 = \sigma_2 = \sigma_3$ defines the **hydrostatic axis**, while the planes perpendicular to this axis are the **deviatoric planes**. The particular deviatoric plane containing the origin $O$ is called as **$\pi$-plane**. The distance of a deviatoric plane from the origin is measured with the parameter $\zeta$ as
\[ \zeta = \| \zeta \| = \sqrt{3} p. \] (3.9)

\(^1\)It should be noted that two stress matrices with the same eigenvalues but with different eigenvector orientations are mapped to the same geometrical point in the principal stress space. Consequently, this type of illustration does not provide information about the stress orientation with respect to the material body.
3.1. ANALYSIS OF STRESS AND STRAIN

The deviatoric part $\rho$ is defined as

$$\rho = OP - ON,$$  \hspace{1cm} (3.10)

which can be represented by the vector components

$$[\rho] = \begin{bmatrix} \sigma_1 \\
\sigma_2 \\
\sigma_3 
\end{bmatrix} - \begin{bmatrix} p \\
p \\
p 
\end{bmatrix} = \begin{bmatrix} s_1 \\
s_2 \\
\sigma_3 
\end{bmatrix}$$  \hspace{1cm} (3.11)

with length

$$\rho = \| \rho \| = \sqrt{2J_2} = \| s \|.$$  \hspace{1cm} (3.12a)

The Lode angle measures the angle between the deviatoric projection of the $\sigma_1$ axis and the radius vector of the current stress point (Jirásek and Bážant, 2002). It is defined by the relation

$$\cos 3\theta = \frac{3\sqrt{3}}{2} \frac{J_3}{\sqrt{J_2^3}}.$$  \hspace{1cm} (3.13)

Consequently, the principal stresses can be expressed as

$$\sigma_1 = \frac{\zeta}{\sqrt{3}} + \sqrt{\frac{2}{3}} \rho \cos \theta,$$  \hspace{1cm} (3.14)

$$\sigma_2 = \frac{\zeta}{\sqrt{3}} + \sqrt{\frac{2}{3}} \rho \cos \left( \theta - \frac{2\pi}{3} \right),$$  \hspace{1cm} (3.15)

$$\sigma_3 = \frac{\zeta}{\sqrt{3}} + \sqrt{\frac{2}{3}} \rho \cos \left( \theta + \frac{2\pi}{3} \right).$$  \hspace{1cm} (3.16)

Here, $\sigma_1 \geq \sigma_2 \geq \sigma_3$.

3.1.3 Linear elastic stress-strain relation

The general form of the linear elastic stress-strain relation for isotropic material can be written as

$$\sigma = D^e : \varepsilon,$$  \hspace{1cm} (3.17)

where $\varepsilon$ is the small strain tensor, whereas $D^e$ denotes the fourth-order elasticity tensor, which can be formulated in general form as (Doghri, 2000)

$$D^e = 2G T + K \delta \otimes \delta.$$  \hspace{1cm} (3.18)

Expression (3.17) represents the Hooke’s law. In (3.18), $G$ stands for the shear modulus, while $K$ denotes the bulk modulus. Their connections to the Young’s modulus $E$ and to the Poisson’s ratio $\nu$ are (Chen and Saleeb, 1982; Sadd, 2009)

$$G = \frac{E}{2(1 + \nu)}, \quad K = \frac{E}{3(1 - 2\nu)}.$$  \hspace{1cm} (3.19)
The inverse relation of (3.17) has the form
\[ \varepsilon = \mathbf{C}^e : \sigma, \]  
where \( \mathbf{C}^e \) denotes the fourth-order elastic compliance tensor, the inverse of \( \mathbf{D}^e \) (Doghri, 2000):
\[ \mathbf{C}^e = \frac{1}{2G} \mathbf{I} - \frac{\nu}{E} \delta \otimes \delta = \frac{1}{2G} \mathbf{T} + \frac{1}{9K} \delta \otimes \delta. \]  

### 3.1.4 Decomposition of the strain

The additive decomposition of the total strain into elastic and plastic parts is a fundamental assumption in the small strain elastoplasticity theory. It means the relation
\[ \varepsilon = \varepsilon^e + \varepsilon^p, \]  
where \( \varepsilon \) denotes the total strain, whereas \( \varepsilon^e \) and \( \varepsilon^p \) stand for the elastic and the plastic parts. The additive decomposition is also adopted for the strain rates. For one-dimensional case, Figure 3.2 illustrates the strain decomposition.

![Figure 3.2: Strain decomposition in uniaxial case.](image)

Furthermore, the strain tensor can be decomposed additively as
\[ \varepsilon = \mathbf{e} + \varepsilon, \]  
where \( \mathbf{e} \) denotes the deviatoric strain tensor, whereas the volumetric strain tensor, \( \varepsilon \), is given by
\[ \varepsilon = \varepsilon\delta, \quad \varepsilon = \frac{1}{3} \text{tr}\varepsilon. \]  

The decomposition into elastic and plastic parts is valid for the deviatoric and the volumetric strain, and for the strain rate quantities, as well.
3.2 Yield criteria

The law defining the elastic limit under an arbitrary combination of stresses is called yield criterion. In general three-dimensional case, where the stress state is described by six independent stress components, the yield criterion can be imagined as a yield surface in the six-dimensional stress space. This yield surface divides the whole stress space into elastic and plastic domains. Therefore, the yield criterion can be represented as a yield surface. In the Haigh–Westergaard stress space, the yield surface constitutes a three-dimensional surface with the definition

\[ F(\sigma, \sigma_Y) = 0, \]  \hspace{1cm} (3.25)

where \( F(\sigma, \sigma_Y) \) denotes the yield function, whereas \( \sigma_Y \) represents the yield stress. \( F = 0 \) means yielding or plastic deformation, while for elastic deformation we have \( F < 0 \). Thus, the yield criterion is expressible in the form

\[ F(\sigma, \sigma_Y) \leq 0. \]  \hspace{1cm} (3.26)

A particular yield function depends on the definition of the equivalent stress and the characteristic of the yield stress. For isotropic materials the yield criterion can be written in terms of the scalar invariants of the total stress (Chen and Han, 2007):

\[ F(I_1, I_2, I_3, \sigma_Y) \leq 0. \]  \hspace{1cm} (3.27)

3.2.1 The von Mises yield criterion

The von Mises yield criterion states that plastic yielding occurs, when the octahedral shearing stress reaches a critical value \( k = \sigma_Y / \sqrt{3} \) (von Mises, 1913). This behavior can be written using the yield function

\[ F(\sigma, \sigma_Y) = \sqrt{\frac{3}{2}}s : s - \sigma_Y \]  \hspace{1cm} (3.28)

or in an alternative way:

\[ F(\sigma, \sigma_Y) = \sqrt{J_2} - k \equiv \frac{1}{\sqrt{2}}\|s\| - k. \]  \hspace{1cm} (3.29)

The yield function (3.28) can be reformulated in a simpler, but equivalent form as

\[ F(s, R) = \|s\| - R, \]  \hspace{1cm} (3.30)

where \( R = \sqrt{\frac{2}{3}}\sigma_Y \) (Simo and Hughes, 1998). The yield surface corresponding to this yield criterion is a cylinder parallel to the hydrostatic axis (see Figure 3.3). Consequently, its locus on a particular deviatoric plane (including the \( \pi \)-plane) is a circle with radius \( R \).
3.2.2 The Drucker–Prager yield criterion

The Drucker–Prager yield criterion is a simple modification of the von Mises criterion, in which the hydrostatic stress component is also included to introduce pressure-sensitivity (Drucker and Prager, 1952). The yield function for this case can be written as (Chen, 2007; de Souza Neto et al., 2008; Jirásek and Bažant, 2002)

\[
F(\sigma, \sigma_Y, \alpha) = \frac{1}{\sqrt{2}} \|s\| + 3\alpha p - k, \tag{3.31}
\]

where \(\alpha\) is an additional material parameter. The yield surface in the principal stress space is represented by a circular cone around the hydrostatic axis (see Figure 3.4).

The angle \(\kappa\) in the meridian plane is defined as

\[
\tan \kappa = \sqrt{6\alpha}. \tag{3.32}
\]
3.3 Plastic flow rules

The material starts to deform plastically, when the yield surface is reached. Upon further loading, the deformation produces plastic flow. The direction of the plastic strain rate is defined according to the plastic flow rule

\[ \dot{\varepsilon}^p = \dot{\lambda} \frac{\partial g}{\partial \sigma}, \]  

where the scalar function \( \dot{\lambda} \) denotes the plastic multiplier (or consistency parameter), whereas \( g \) is the plastic potential function, which itself is a function of the stresses. The plastic flow rule is called associative if the plastic potential function in (3.33) equals to the yield function. Otherwise, the flow rule is termed non-associative. For the associative case, the direction of the strain rate is the outward normal of the yield surface, whereas for non-associative flow rule it is the gradient of the plastic potential surface.

3.4 Hardening laws

In uniaxial experiment, it is observed that the yield stress associated to a material can vary upon plastic loading. Furthermore, for some class of materials the yield stress in the reverse load direction (compression) is different than for tension. These phenomena can be modelled using various hardening laws. The simplest case is the perfectly plastic material, for which, the yield stress remains unchanged under loading. In this case the yield function becomes

\[ F(\sigma, \sigma_{Y0}) = \bar{\sigma}(\sigma) - \sigma_{Y0}, \]  

where \( \sigma_{Y0} \) indicates the initial yield stress, whereas \( \bar{\sigma}(\sigma) \) stands for the effective (or equivalent) stress.

3.4.1 Isotropic hardening

The hardening behavior is termed isotropic if the shape of the yield surface remains fixed, whereas the size of the yield surface changes under plastic deformation. In other words, the yield surface expands without translation under plastic loading.

3.4.1.1 Linear isotropic hardening

If the material behavior, in the plastic region of the uniaxial stress-strain curve, is modelled with linear schematization, then we arrive at the linear isotropic hardening rule:

\[ \sigma_Y(\bar{\varepsilon}^p) = \sigma_{Y0} + H\bar{\varepsilon}^p, \]  

where \( \bar{\varepsilon}^p \) stands for the effective plastic strain.
where the slope of the curve is given by the constant plastic hardening modulus \( H \), whereas \( \bar{\varepsilon}^p \) denotes the accumulated (or cumulative) plastic strain, which defined by (Chen and Han, 2007)

\[
\bar{\varepsilon}^p = \sqrt{\frac{2}{3}} \int_0^t \| \dot{\varepsilon}^p \| \, dt. 
\]

(3.36)

An alternative, but equivalent, way to define the linear isotropic hardening is (Simo and Hughes, 1998)

\[
R(\gamma) = R_0 + h\gamma, 
\]

(3.37)

where

\[
R_0 = \sqrt{\frac{2}{3}} \sigma_{Y0}, \quad h = \frac{2}{3} H, \quad \gamma = \int_0^t \| \dot{\varepsilon}^p \| \, d\tau. 
\]

(3.38)

### 3.4.1.2 Nonlinear isotropic hardening

Nonlinear empirical idealization of the plastic hardening, in most cases, provides more accurate prediction of the material behavior. The most commonly used forms for the nonlinear isotropic hardening rule are the power law and the exponential law hardening (Doghri, 2000):

\[
\sigma_Y(\bar{\varepsilon}^p) = \sigma_{Y0} + H_1 (\bar{\varepsilon}^p)^m, \quad \text{and} \quad \sigma_Y(\bar{\varepsilon}^p) = \sigma_{Y0} + \sigma_{Y\infty} \left(1 - e^{-m\bar{\varepsilon}^p}\right), 
\]

(3.39)

where \( H_1, m \) and \( \sigma_{Y\infty} \) are material parameters. There exist some other nonlinear schematizations, which can be found in the textbook of Skrzypek (1993), for instance.

### 3.4.2 Kinematic hardening

The kinematic hardening rule assumes that during plastic flow, the yield surface translates in the stress space and its shape and size remains unchanged. This hardening model based on the Bauschinger effect observed in uniaxial tension-compression test for some material (Bauschinger, 1881; Lemaitre and Chaboche, 1990). The use of kinematic hardening rules involves the modification (shifting) the stress tensor \( \sigma \) with the so-called back-stress (or translation) tensor \( \alpha \), in the yield function. Thus, the yield function becomes \( F(\sigma - \alpha, \sigma_Y) \). Depending of the evolution of the back-stress tensor, a few kinematic hardening models exist. Two widely used rules are presented in the following.

#### 3.4.2.1 Linear kinematic hardening

The simplest evolutionary equation for the back-stress tensor \( \alpha \) is the Prager’s linear hardening rule (Chen and Han, 2007; de Souza Neto et al., 2008; Prager, 1955, 1956):

\[
\dot{\alpha} = \frac{2}{3} H \dot{\varepsilon}^p = h \dot{\varepsilon}^p. 
\]

(3.40)
3.5. ELASTIC-PLASTIC CONSTITUTIVE MODELS

3.4.2.2 Nonlinear kinematic hardening

Among different type of nonlinear kinematic hardening rules, the Armstrong–Frederick’s type is the most widely used and adopted one (Armstrong and Frederick, 1966; Frederick and Armstrong, 2007; Jirásek and Bažant, 2002). This rule introduces a fading memory effect of the strain path as

$$\dot{\alpha} = \frac{2}{3} H \dot{\varepsilon}^p - B \ddot{\varepsilon}^p \alpha,$$

(3.41)

where $B$ is a material constant.

3.4.3 Combined linear hardening

By combining the isotropic and kinematic hardening rules we arrive at the combined hardening (or mixed hardening) rule, by which the characteristics of real materials can be predicted more accurately. The combined linear hardening rules involves both the linear isotropic hardening rule (3.35) and the linear evolutionary equation (3.40) for the back-stress.

The plastic hardening modulus corresponding to the isotropic and to the kinematic hardening can be defined as

$$H_{iso} = MH, \quad H_{kin} = (1 - M) H, \quad h_{iso} = \frac{2}{3} H_{iso}, \quad h_{kin} = \frac{2}{3} H_{kin},$$

(3.42)

where the combined hardening parameter $M \in [0, 1]$ defines the share of the isotropic part in the total amount of hardening (Axelsson and Samuelsson, 1979; Chen and Han, 2007; Simo and Hughes, 1998). In this case, $H_{iso}$ has to be used in (3.35), whereas $H_{kin}$ replaces $H$ in (3.40). Consequently, $M = 1$ means purely isotropic hardening, while $M = 0$ denotes purely kinematic hardening.

3.5 Elastic-plastic constitutive models

3.5.1 Introduction

This section presents the constitutive equations of the two elastic-plastic constitutive models under consideration in this dissertation. Besides the formulation of the corresponding elastoplastic tangent tensors, the inverse forms of the constitutive equations are also presented.

3.5.2 Associative von Mises elastoplasticity model with combined linear hardening

Based on (3.30), the yield function of the von Mises elastoplasticity model with combined linear hardening is given by

$$F = \|\xi\| - R,$$

(3.43)
where $\xi = s - \alpha$ denotes the deviatoric relative (or reduced) stress. The plastic flow direction, according to (3.33), is defined by the associative flow rule

$$\dot{\varepsilon}^p = \lambda N, \quad N = \frac{\partial F}{\partial \sigma} = \frac{\xi}{\|\xi\|}, \quad \|\dot{\varepsilon}^p\| = \dot{\lambda} = \dot{\gamma},$$

(3.44)

where $N$ represents the outward normal of the yield surface. The linear isotropic hardening rule (3.37) takes the form

$$R(\gamma) = R_0 + h_{iso} \gamma.$$  

(3.45)

The evolutionary law for the back-stress according to the Prager’s linear hardening rule (3.40) is defined as

$$\dot{\alpha} = h_{kin} \dot{\varepsilon}^p = \lambda h_{kin} N = \lambda h_{kin} \frac{\xi}{\|\xi\|}.$$  

(3.46)

The loading/unloading conditions can be expressed in the Kuhn-Tucker form as (de Souza Neto et al., 2008; Luenberger and Ye, 2008; Simo and Hughes, 1998)

$$\dot{\lambda} \geq 0, \quad F \leq 0, \quad \dot{\lambda} F = 0.$$  

(3.47)

The plastic multiplier can be derived from the consistency condition $\dot{F} = 0$ using with combination of (3.17) and (3.22):

$$\dot{F} = \frac{\partial F}{\partial \sigma} : \dot{\sigma} + \frac{\partial F}{\partial \alpha} : \dot{\alpha} - \dot{R} = N : D^e : \dot{\varepsilon} - (2G + h) \dot{\lambda},$$

(3.48)

$$\dot{\lambda} = \frac{N : D^e : \dot{\varepsilon}}{2G + h} = \frac{2G\xi : \dot{\varepsilon}}{(2G + h)\|\xi\|}.$$  

(3.49)

The fourth-order elastoplastic tangent tensor $D^{ep}$, which relates the strain rate $\dot{\varepsilon}$ to the stress rate $\dot{\sigma}$ is computed from

$$\dot{\sigma} = D^e : \dot{\varepsilon}^e = D^e : \dot{\varepsilon} - D^e : \dot{\varepsilon}^p = D^e : \dot{\varepsilon} - \frac{N : D^e : \dot{\varepsilon}}{2G + h} D^e : N$$

$$= \left( D^e - \frac{D^e : N \otimes N : D^e}{2G + h} \right) : \dot{\varepsilon}.$$  

(3.50)

(3.51)

Therefore the rate-form elastic-plastic constitutive equation has the following form:

$$\dot{\sigma} = D^{ep} : \dot{\varepsilon}.$$  

(3.52)

where

$$D^{ep} = D^e - \frac{D^e : N \otimes N : D^e}{2G + h} = D^e - \frac{4G^2}{(2G + h)\|\xi\|^2} \xi \otimes \xi.$$  

(3.53)
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The constitutive equation (3.52) can be separated into deviatoric and hydrostatic (spherical) parts as follows

\[
\dot{s} = 2G\dot{\varepsilon} - \frac{4G^2}{(2G + h) \|\xi\|^2} (\xi : \dot{\varepsilon}) \xi
\]  
(3.54)

and

\[
\dot{p} = 3K\dot{\varepsilon}.
\]  
(3.55)

It can be clearly concluded, that in this model, the hydrostatic part of the total stress is governed by pure elastic law. Therefore, the plastic deformation affects only the deviatoric stress components.

The evolutionary equation for the back-stress can be expressed by combining (3.46) and (3.49):

\[
\dot{\alpha} = \frac{2Gh (1 - M)}{(2G + h) \|\xi\|^2} (\xi : \dot{e}) \xi.
\]  
(3.56)

The evolution law of the parameter \( R \) related to the yield stress is obtained by taking the time derivative of (3.45):

\[
\dot{R} = \frac{2GMh}{(2G + h) \|\xi\|^2} (\xi : \dot{e}).
\]  
(3.57)

In this description, \( R \) represents the radius of the yield surface (cylinder). Finally, the definition for the rate of the deviatoric relative stress is given by

\[
\dot{\xi} = \dot{s} - \dot{\alpha} = 2G\dot{\varepsilon} - \frac{2G}{\|\xi\|^2} \left(1 - \frac{Mh}{2G + h}\right) (\xi : \dot{\varepsilon}) \xi.
\]  
(3.58)

Inverse elastoplastic constitutive equation

The inverse elastic-plastic constitutive equation, which relates the stress rate to the strain rate, is given by the relation

\[
\dot{\varepsilon} = \mathbf{C}^{\text{ep}} : \dot{\sigma},
\]  
(3.59)

where the fourth-order elastoplastic compliance tangent tensor \( \mathbf{C}^{\text{ep}} \) can be derived by inverting the elastoplastic tangent tensor \( \mathbf{D}^{\text{ep}} \) using the Sherman–Morrison formula (Sherman and Morrison, 1949; Szabó, 1985):

\[
\mathbf{C}^{\text{ep}} = \left(\mathbf{D}^{\text{ep}}\right)^{-1} = \mathbf{C}^{\varepsilon} + \frac{1}{h} \mathbf{N} \otimes \mathbf{N} = \mathbf{C}^{\varepsilon} + \frac{1}{h \|\xi\|^2} \xi \otimes \xi.
\]  
(3.60)

The constitutive equation (3.59) can be separated into deviatoric and hydrostatic parts as follows

\[
\dot{\varepsilon} = \frac{1}{2G} \dot{s} + \frac{1}{h \|\xi\|^2} (\xi : \dot{s}) \xi.
\]  
(3.61)
and

\[ \dot{\epsilon} = \frac{1}{3K} \dot{p} \tag{3.62} \]

Combining (3.61) with (3.57), (3.58) and (3.56) we arrive at

\[ \dot{R} = \frac{M}{\|\xi\|} (\xi : \dot{s}) \tag{3.63} \]

\[ \dot{\alpha} = \frac{1 - M}{\|\xi\|^2} (\xi : \dot{s}) \xi, \tag{3.63} \]

\[ \dot{\xi} = \dot{s} - \frac{1 - M}{\|\xi\|^2} (\xi : \dot{s}) \xi \]

### 3.5.3 Non-associative Drucker–Prager elastoplasticity model with linear isotropic hardening

The yield function (3.31) for the Drucker–Prager model with linear isotropic hardening can be formulated as

\[ F = \frac{1}{\sqrt{2}} \|s\| + 3\alpha p - k. \tag{3.64} \]

Since non-associative case is considered, the plastic flow potential function has to be defined. A commonly adopted form is given by (Chen and Han, 2007)

\[ g = \frac{1}{\sqrt{2}} \|s\| + 3\beta p, \tag{3.65} \]

where \( \beta \) is a material parameter. The gradients of the yield function and the plastic potential function, with respect to \( \sigma \) are the following:

\[ N = \frac{\partial F}{\partial \sigma} = \frac{s}{\sqrt{2} \|s\|} + \alpha \delta, \tag{3.66} \]

\[ Q = \frac{\partial g}{\partial \sigma} = \frac{s}{\sqrt{2} \|s\|} + \beta \delta. \tag{3.67} \]

The non-associative flow rule for the plastic strain rate is defined using (3.33) as

\[ \dot{\epsilon}^p = \lambda Q = \dot{\lambda} \left( \frac{s}{\sqrt{2} \|s\|} + \beta \delta \right). \tag{3.68} \]

The norm of plastic strain rate and the rate of the accumulated plastic strain (3.36) are the following:

\[ \|\dot{\epsilon}^p\| = \dot{\lambda} \sqrt{\frac{1}{2} + 3\beta^2}, \quad \dot{\bar{\epsilon}}^p = \dot{\lambda} \sqrt{\frac{1}{3} + 2\beta^2}. \tag{3.69} \]

The linear isotropic hardening rule (3.35) for this model becomes (Chen and Han, 2007)

\[ k (\dot{\epsilon}^p) = \left( \alpha + \frac{1}{\sqrt{3}} \right) \sigma_Y (\dot{\epsilon}^p). \tag{3.70} \]
The loading/unloading conditions can be expressed in the Kuhn–Tucker form (3.47). The plastic multiplier can be obtained from the consistency condition \( \dot{F} = 0 \), using with combination of (3.17) and (3.22):

\[
\dot{F} = \frac{\partial F}{\partial \sigma} : \dot{\sigma} - \dot{k} = N : D^e : \dot{\varepsilon} - \dot{\lambda} \left( N : D^e : Q + H \left( \alpha + \frac{1}{\sqrt{3}} \right) \sqrt{\frac{1}{3} + 2\beta^2} \right),
\]

(3.71)

\[
\dot{\lambda} = \frac{N : D^e : \dot{\varepsilon}}{N : D^e : Q + H \left( \alpha + \frac{1}{\sqrt{3}} \right) \sqrt{\frac{1}{3} + 2\beta^2}} = \frac{1}{\tilde{h}} \left( \frac{2G}{\sqrt{2} \|s\|} s : \dot{\varepsilon} + 3K\alpha \dot{\varepsilon} \right),
\]

(3.72)

where the scalar parameter \( \tilde{h} \) is defined as

\[
\tilde{h} = G + 9K\alpha\beta + H \left( \alpha + \frac{1}{\sqrt{3}} \right) \sqrt{\frac{1}{3} + 2\beta^2}.
\]

(3.73)

The elastoplastic tangent tensor is derived from

\[
\dot{\sigma} = D_{ep} : \dot{\varepsilon},
\]

(3.76)

where

\[
D_{ep} = D^e - \frac{D^e : Q \otimes N : D^e}{\tilde{h}} = D^e - \frac{1}{\tilde{h}} \left( \frac{2G^2}{\|s\|^2} s \otimes s + \frac{6KG\alpha}{\sqrt{2} \|s\|} s \otimes \delta + \frac{6KG\beta}{\sqrt{2} \|s\|} \delta \otimes s + 9K^2\alpha\beta \delta \otimes \delta \right).
\]

(3.77)

The constitutive equation (3.76) can be separated into deviatoric and hydrostatic parts as follows

\[
\dot{s} = 2G \dot{\varepsilon} - \frac{2G^2}{h \|s\|^2} \left( s : \dot{\varepsilon} + \frac{9K\alpha \|s\| \dot{\varepsilon}}{\sqrt{2G}} \right) s
\]

(3.79)

and

\[
\dot{p} = 3K \dot{\varepsilon} - \frac{3\sqrt{2}KG\beta}{h \|s\|} \left( s : \dot{\varepsilon} + \frac{9K\alpha \|s\| \dot{\varepsilon}}{\sqrt{2G}} \right).
\]

(3.80)

**Inverse elastoplastic constitutive equation**

The inverse of the constitutive law (3.76) is defined as

\[
\dot{\varepsilon} = C_{ep} : \dot{\sigma},
\]

(3.81)

where the fourth-order elastoplastic compliance tangent tensor \( C_{ep} \) is obtained by the inversion of
(3.77) using the Sherman–Morrison formula (Sherman and Morrison, 1949; Szabó, 1985):

\[
C^{ep} = (D^{ep})^{-1} = C^e + \frac{1}{j} Q \otimes N
\]

\[
= C^e + \frac{1}{j} \left( \frac{1}{2\|s\|^2} s \otimes s + \frac{\alpha}{\sqrt{2}\|s\|} s \otimes \delta + \frac{\beta}{\sqrt{2}\|s\|} \delta \otimes s + \alpha\beta \delta \otimes \delta \right),
\]

(3.82)

where the scalar parameter \( j \) is defined as

\[
j = \tilde{h} - G - 9K\alpha\beta = H \left( \alpha + \frac{1}{\sqrt{3}} \right) \sqrt{\frac{1}{3} + 2\beta^2}.
\]

(3.84)

The inverse constitutive law (3.81) can be separated into deviatoric and hydrostatic part as follows:

\[
\dot{e} = \frac{1}{2G} \dot{s} + \frac{1}{2j\|s\|^2} \left( s : \dot{s} + 3\sqrt{2}\|s\| \alpha \dot{p} \right) s
\]

(3.85)

and

\[
\dot{\varepsilon} = \left( \frac{1}{3K} + \frac{3\alpha\beta}{j} \right) \dot{p} + \frac{\beta (s : \dot{s})}{\sqrt{2}\|s\| j}.
\]

(3.86)
- **Endre Reuss** (1 July 1900 - 10 May 1968)
- 1953-68: Professor at the Department of Applied Mechanics
- 1955-57: Dean of the Faculty
- 1930 ZAMM paper: "Berücksichtigung der elastischen Formänderung in der Plastizitätstheorie".
- 3D elastoplastic constitutive equation of elastic-perfectly plastic materials

**Governing equations**

- **additive decomposition of the strain**:
  \[
  \dot{\varepsilon} = \dot{\varepsilon}^e + \dot{\varepsilon}^p
  \]

- **linear elastic material law for the elastic strain**:
  \[
  \varepsilon^e = D^e : \dot{\varepsilon}^e = D^e : (\dot{\varepsilon} - \dot{\varepsilon}^p)
  \]
  \[
  D^e = 2G\tau + K\partial \otimes \partial
  \]
  \[
  \dot{s} = 2G \dot{\varepsilon}^e \quad \text{and} \quad \dot{s} = \dot{s} + \dot{p}
  \]
  \[
  \dot{p} = K(4\pi \dot{\varepsilon}^e)\partial
  \]
  \[
  \varepsilon = \text{dev} [\dot{s}] = T : \dot{s}
  \]
  \[
  \varepsilon = \text{dev} [\dot{\varepsilon}] = T : \dot{\varepsilon}
  \]
- Mises yield criterion:
\[ F = \frac{1}{2} \sqrt{3} \mathbf{\sigma} : \mathbf{\sigma} - 6\mathbf{\gamma} = \mathbf{0} \]

- no hardening involved:
\[ 6\mathbf{\gamma} = \text{const} \]

- associative flow rule:
\[ \dot{\mathbf{\varepsilon}}^P = \frac{2F}{\mathbf{\sigma}} \lambda \]
\[ = \mathbf{\dot{\lambda}} \cdot \frac{\partial}{\partial \mathbf{\sigma}} \left( \frac{3}{2} \mathbf{\sigma} : \mathbf{\sigma} \right) = \mathbf{\dot{\lambda}} \sqrt{\frac{3}{2}} \frac{\mathbf{\sigma}}{|| \mathbf{\sigma} ||} : \mathbf{I} \]
\[ = \mathbf{\dot{\lambda}} \sqrt{\frac{3}{2}} \frac{\mathbf{\sigma}}{|| \mathbf{\sigma} ||} = \mathbf{\dot{\lambda}} \frac{3}{2} \frac{\mathbf{\sigma}}{6\mathbf{\gamma}} \]

It is a deviatoric tensor. Thus \( \dot{\mathbf{\varepsilon}}^P = \dot{\mathbf{\varepsilon}}^P \) and \( \text{tr} \dot{\mathbf{\varepsilon}}^P = 0 \) \( \Rightarrow \) no volume change

- Determination of the plastic multiplier

- Evaluating the consistency condition:
\[ \dot{F} = 0 \]
\[ \frac{d}{dt} \left( \frac{1}{2} \sqrt{\frac{3}{2}} || \mathbf{\sigma} || - 6\mathbf{\gamma} \right) = 0 \]
\[ \frac{d}{dt} || \mathbf{\sigma} || = 0 \]
\[ \frac{\mathbf{\sigma}}{|| \mathbf{\sigma} ||} : \dot{\mathbf{\sigma}} = 0 \]
\[ \frac{\dot{\mathbf{s}}}{\sqrt{2/3} \, 6Y} = 0 \]

\[ \mathbf{s} : \dot{\varepsilon} = 0 \]

\[ \mathbf{s} : (2G \dot{\varepsilon} - 2G \dot{\varepsilon}^p) = 0 \]

\[ \mathbf{s} : \dot{\varepsilon} - \mathbf{s} : \dot{\varepsilon}^p = 0 \]

\[ \mathbf{s} : \dot{\varepsilon} - \frac{3}{2} \lambda \frac{\mathbf{s} : \mathbf{S}}{6Y} = 0 \]

\[ \lambda = \frac{\mathbf{s} : \dot{\varepsilon}}{6Y} \]

Thus, the plastic strain rate:

\[ \dot{\varepsilon}^p = \frac{3 \mathbf{s} : \dot{\varepsilon}}{2 \, 6Y^2} \mathbf{s} \]

- Determination of the elastic-plastic continuum tangent tensor

- The elastic constitutive law can be reformulated as

\[ \dot{\mathbf{s}} = \mathbf{D}^e : \dot{\varepsilon} - \mathbf{D}^e : \dot{\varepsilon}^p \]
\[ \begin{align*}
\dot{\varepsilon} &= D^e : \dot{\varepsilon} - \frac{3G(\frac{1}{2} : \ddot{\varepsilon})}{6Y^2} = \ddot{\sigma} \\
\dot{\sigma} &= D^e : \dot{\varepsilon} - \frac{3G \varepsilon \otimes \dot{\varepsilon}}{6Y^2} : \dot{\varepsilon} = \ddot{\sigma} \\
D^{ep} &= D^e - \frac{3G \varepsilon \otimes \dot{\varepsilon}}{6Y^2} \\
\dot{\varepsilon} &= C^{ep} : \dot{\varepsilon}
\end{align*} \]

- In this model, \( D^{ep} \) is non-invertible!
- \( \varepsilon = C^{ep} : \dot{\varepsilon} \) doesn't exist!

- The constitutive equation can be separated into deviatoric and hydrostatic part:

\[ \begin{align*}
\dot{\varepsilon} &= 2G \dot{\varepsilon} - \frac{3G}{6Y^2} (\varepsilon : \dot{\varepsilon}) \varepsilon \\
\dot{\sigma} &= K(\text{tr} \dot{\varepsilon}) \sigma
\end{align*} \]

- It forms a system of ordinary differential equations
- For a general strain history, \( \varepsilon(t) \), the analytical solution cannot be obtained due to the complex structure of the constitutive equation.
Analytical solution for linear strain path

- all variables are known at \( t_n \)
- strain input is given as

\[
\dot{\varepsilon}(t) = \varepsilon_n + \Delta t \cdot \dot{\varepsilon}
\]

where \( \dot{\varepsilon} \) is constant and \( \Delta t = t - t_n \)
- illustration of the loading in the deviatoric plane:

\[
S_n = \psi_n
\]

Let

\[
\begin{align*}
\dot{\varepsilon} : \varepsilon &= \| \dot{\varepsilon} \| \| \varepsilon \| \cos \psi \\
&= -\frac{2}{3} \gamma \| \dot{\varepsilon} \| \cos \psi
\end{align*}
\]

Then:

\[
\begin{align*}
0 \leq \psi_n &\leq \frac{\pi}{2} : \text{plastic loading} \\
\frac{\pi}{2} \leq \psi_n &\leq \pi : \text{elastic unloading}
\end{align*}
\]
The expression for \( \hat{\mathbf{s}} \) then can be expressed as

\[
\hat{\mathbf{s}} = 2G \mathbf{e} - \frac{3G}{6\gamma^2} \sqrt{\frac{2}{3}} 6\gamma \| \mathbf{e} \| \cos \gamma \mathbf{s}
\]

\[
\hat{\mathbf{s}} : \mathbf{e} = 2G \| \mathbf{e} \|^2 - \frac{3G}{6\gamma^2} \left( \sqrt{\frac{2}{3}} 6\gamma \| \mathbf{e} \| \cos \gamma \right)^2
\]

\[
\hat{\mathbf{s}} : \mathbf{e} = 2G \| \mathbf{e} \|^2 - 2G \| \mathbf{e} \|^2 \cos^2 \gamma
\]

\[
\hat{\mathbf{s}} : \mathbf{e} = 2G \| \mathbf{e} \|^2 \sin^2 \gamma
\]

Another way to express the term \( \hat{\mathbf{s}} : \mathbf{e} \) is

\[
\hat{\mathbf{s}} : \mathbf{e} = \frac{d}{dt}(\hat{\mathbf{s}} : \mathbf{e}) = -\| \mathbf{s} \| \| \mathbf{e} \| \sin \gamma \cdot \mathbf{e}
\]

Combining the two expressions:

\[
2G \| \mathbf{e} \|^2 \sin^2 \gamma = -\| \mathbf{s} \| \| \mathbf{e} \| \sin \gamma \cdot \mathbf{e}
\]

\[
\frac{1}{\sin \gamma} \mathbf{e} = -\frac{2G \| \mathbf{e} \|^2}{\| \mathbf{s} \|}
\]

\[
\frac{1}{\sin \gamma} d\gamma = -\frac{2G \| \mathbf{e} \|^2}{\| \mathbf{s} \|} dt
\]

\[
\psi \int_0^{\Delta t} \frac{1}{\sin \gamma} d\gamma = -\frac{2G \| \mathbf{e} \|^2}{\| \mathbf{s} \|} \int_0^{\Delta t} dt
\]

\[
\left[ \ln \left( \tan \frac{\psi}{2} \right) \right]_\psi^{\psi_n} = -\frac{2G \| \mathbf{e} \| \Delta t}{\| \mathbf{s} \|}
\]
\[ \Psi = 2 \cdot \arctan \left[ \exp \left( -\frac{2G \| \dot{\epsilon} \| \Delta t}{\| \epsilon \|} \right) \right] \cdot \tan \frac{\Psi_n}{2} \]

- Substituting back into \( \dot{\epsilon} \) gives

\[ \dot{\epsilon} = -\frac{3G}{6\gamma^2} (\epsilon : \dot{\epsilon}) \epsilon + 2G \dot{\epsilon} \]

\[ \underbrace{\alpha(t)}_{\text{a(t)}} \quad \underbrace{b(t) \cdot C}_{\text{b(t) \cdot C}} \]

where \( C = \dot{\epsilon} \) is constant and \( b(t) = 2G \).

\[ \dot{\epsilon} = a(t) \cdot \epsilon + b(t) \cdot C \]

It is a non-homogeneous linear DE in tensorial form. The general solution is

\[ \epsilon(t) = A(t) \cdot \epsilon_n + B(t) \cdot C \]

\[ A(t) = \exp \left[ \int_0^{\Delta t} a(s) \, ds \right] \]

\[ B(t) = \int_0^{\Delta t} \exp \left[ \int_0^{\Delta t} a(\tau) \, d\tau \right] b(s) \, ds \]

For this model:

\[
\begin{array}{c|c}
A & \frac{\sin \Psi(t)}{\sin \Psi_n} \\
B & \frac{\| \epsilon \| \sin (\Psi_n - \Psi(t))}{\| \dot{\epsilon} \| \sin \Psi_n}
\end{array}
\]
Example

- A linear elastic - perfectly plastic material is in plastic state at $t_n$.
  The material constants are: $E = 200$ GPa, $\nu = 0.3$
- The stress tensor at $t_n$ is given by
  \[
  \begin{bmatrix}
  \sigma_{11} \\
  \sigma_{22} \\
  \sigma_{33} \\
  \sigma_{12} \\
  \sigma_{23} \\
  \sigma_{31}
  \end{bmatrix}
  =
  \begin{bmatrix}
  60 & 50 & 0 \\
  50 & 120 & 0 \\
  0 & 0 & 0
  \end{bmatrix}
  \text{MPa}
  \]
- Determine the components of the stress tensor at the end of the strain increment
  \[
  [\Delta \varepsilon] = \begin{bmatrix}
  90 & 60 & 0 \\
  60 & 0 & 0 \\
  0 & 0 & 0
  \end{bmatrix} \times 10^{-6}
  \]

Solution:

- $n$th state variables :
  \[
  \begin{align*}
  \overline{\sigma}_n &= \frac{1}{3} \text{tr} \sigma_n \cdot \sigma = 60 \sigma \quad ; \\
  [\overline{\sigma}_n] &= \begin{bmatrix}
  60 & 0 & 0 \\
  0 & 60 & 0 \\
  0 & 0 & 60
  \end{bmatrix} \text{MPa} \\
  \underline{\sigma}_n &= \sigma_n - \overline{\sigma}_n \\
  [\underline{\sigma}_n] &= \begin{bmatrix}
  0 & 50 & 0 \\
  50 & 60 & 0 \\
  0 & 0 & -60
  \end{bmatrix} \text{MPa}
  \end{align*}
  \]

  \[
  \| \underline{\sigma}_n \| = \sqrt{12200} = 110.4536 \text{ MPa}
  \]
- Thus, the yield stress:
  \[
  \begin{align*}
  F_n &= \sqrt{\frac{3}{2} \underline{\sigma}_n : \underline{\sigma}_n - 6\gamma} = 0 \\
  \Rightarrow \\
  6\gamma &= \frac{3}{2} \| \underline{\sigma}_n \| \\
  \gamma &= 135.2775 \text{ MPa}
  \end{align*}
  \]
deviatoric strain increment:

\[ \Delta e = \dot{e} \cdot \Delta t = \Delta \varepsilon - \frac{1}{3} tr(\Delta \varepsilon) \delta \]

\[
\begin{bmatrix} \Delta \varepsilon \end{bmatrix} = \begin{bmatrix} 60 & 60 & 0 \\ 60 & -30 & 0 \\ 0 & 0 & -30 \end{bmatrix} \times 10^{-6}
\]

\[ \|\Delta e\| = \|\dot{e}\| \cdot \Delta t = \sqrt{12600} \times 10^{-6} = 0.11225 \times 10^{3} \]

initial angle:

\[ \cos \psi_n = \frac{S_n \cdot \Delta \varepsilon}{\|S_n\| \|\Delta \varepsilon\|} \Rightarrow \psi_n = 1.06565 = 61.0574^\circ \]

material constants:

\[ G = \frac{E}{2(1+\nu)} = 76.9231 \text{ GPa} \]

\[ K = \frac{E}{3(1-2\nu)} = 166.67 \text{ GPa} \]

final angle:

\[ \psi_{n+1} = 2 \times \arctan \left[ \exp \left[ - \frac{2G \|\Delta \varepsilon\|}{\|S_n\|} \right] \cdot \tan \frac{\psi_n}{2} \right] \]

\[ \psi_{n+1} = 0.934266 = 53.5295^\circ \]

Thus

\[ A = \frac{\sin \psi_{n+1}}{\sin \psi_n} = 0.918934 \]

\[ B = \frac{\|S_n\| \cdot \sin (\psi_n - \psi_{n+1})}{\|\Delta \varepsilon\| \cdot \sin \psi_n} = 147.34 \text{ GPa} \]
Thus, the deviatoric stress tensor at the end of the increment:

\[ \mathbf{S}_{n+1} = A \mathbf{S}_n + B \Delta \mathbf{\varepsilon} \]

\[
\begin{bmatrix}
S_{n+1} \\
\end{bmatrix} =
\begin{bmatrix}
8.839 & 54.785 & 0 \\
54.785 & 50.717 & 0 \\
0 & 0 & -59.556
\end{bmatrix} \text{ MPa}
\]

- Checking the yield criterion:

\[ F_{n+1} = \sqrt{\frac{3}{2} \mathbf{S}_{n+1} : \mathbf{S}_{n+1} - 6\sigma} = 0 \text{ satisfied} \]

- Hydrostatic stress increment:

\[ \Delta \sigma = K \cdot tr(\Delta \mathbf{\varepsilon}) \cdot \sigma = 15 \sigma \text{ MPa} \]

\[
\begin{bmatrix}
\Delta \sigma \\
\end{bmatrix} =
\begin{bmatrix}
15 & 0 & 0 \\
0 & 15 & 0 \\
0 & 0 & 15
\end{bmatrix} \text{ MPa}
\]

\[ \mathbf{P}_{n+1} = \mathbf{P}_n + \Delta \mathbf{P} ; \quad \mathbf{P}_{n+1} =
\begin{bmatrix}
175 & 0 & 0 \\
0 & 75 & 0 \\
0 & 0 & 75
\end{bmatrix} \text{ MPa}
\]

Thus, the final stress:

\[ \mathbf{\sigma}_{n+1} = \mathbf{S}_{n+1} + \mathbf{P}_{n+1} \]

\[
\begin{bmatrix}
\mathbf{\sigma}_{n+1} \\
\end{bmatrix} =
\begin{bmatrix}
83.839 & 54.785 & 0 \\
54.785 & 125.717 & 0 \\
0 & 0 & 15.444
\end{bmatrix} \text{ MPa}
\]
- demonstration of the use of the RRM for the Prandt-Reuss equations

- the discretized solutions are given as

\[ \Phi_{n+1} = \Phi_n + K \cdot \text{tr}(\Delta \varepsilon) \delta \]

\[ \Sigma_{n+1} = \Sigma_n + 26 \Delta \varepsilon^p = \Sigma_n + 26 \Delta \varepsilon - 26 \Delta \varepsilon^p \]

\[ = \Sigma_{\text{trial}} - 26 \Delta \varepsilon^p \]

- implicit approximation of the plastic strain increment:

\[ \Delta \varepsilon^p = \Delta \varepsilon \approx \Delta \lambda \cdot \frac{\partial F_{n+1}}{\partial \Sigma_{n+1}} = \Delta \lambda \cdot \frac{3}{2} \frac{S_{n+1}}{6Y} \]

- Inserting into the expression of \( \Sigma_{n+1} \) gives

\[ \Sigma_{n+1} = \Sigma_{\text{trial}} - 36 \Delta \lambda \frac{S_{n+1}}{6Y} \]

It follows that

\[ \frac{S_{n+1}}{\| S_{n+1} \|} = \frac{\Sigma_{\text{trial}}}{\| \Sigma_{\text{trial}} \|} \]

- Taking the double contraction with \( \| S_{n+1} \| / \| \Sigma_{n+1} \| : \)

\[ \| S_{n+1} \| = \| \Sigma_{\text{trial}} \| - 36 \Delta \lambda \frac{\| S_{n+1} \|}{6Y} \]

\[ \sqrt{\frac{2}{3}} 6Y = \| \Sigma_{\text{trial}} \| - 36 \Delta \lambda \sqrt{\frac{2}{3}} \]

\[ \Delta \lambda = \frac{1}{36} \left( \sqrt{\frac{2}{3}} \| \Sigma_{\text{trial}} \| - 6Y \right) \]
Thus, the plastic strain increment becomes

\[
\Delta e^p = \Delta \lambda \frac{3}{2} \frac{\| S_{n+1} \| \cdot S_{trial}}{\| S_{trial} \| \cdot 6\gamma}
\]

\[
\Delta e^p = \Delta \lambda \frac{\sqrt{3}}{2} \frac{S_{trial}}{\| S_{trial} \|}
\]

Illustration in the deviatoric plane.
Example

- A linear elastic–perfectly plastic material is in a plastic state at tn.
- The material constants are
  \[ K = 166.67 \, \text{GPa} \]
  \[ G = 76.9231 \, \text{GPa} \]
- The stress tensor at tn:
  \[
  \begin{bmatrix}
  \sigma_{nx} \\
  \sigma_{ny} \\
  \sigma_{nz}
  \end{bmatrix}
  =
  \begin{bmatrix}
  60 & 50 & 0 \\
  50 & 120 & 0 \\
  0 & 0 & 0
  \end{bmatrix} \, \text{MPa}
  \]
- The strain increment is
  \[
  \begin{bmatrix}
  \Delta \varepsilon_{nx} \\
  \Delta \varepsilon_{ny} \\
  \Delta \varepsilon_{nz}
  \end{bmatrix}
  =
  \begin{bmatrix}
  90 & 60 & 0 \\
  60 & 0 & 0 \\
  0 & 0 & 0
  \end{bmatrix} \times 10^{-6}
  \]
- Determine \( \varepsilon_{n+1} \) using the RRM.

Solution:

\[
\varepsilon_{n+1} = \varepsilon_{n} - \frac{1}{3} \left( \text{tr} \, \varepsilon_{n} \right) \Delta \varepsilon
\]

\[
\begin{bmatrix}
\sigma_{n+1} \\
\sigma_{n+1} \\
\sigma_{n+1}
\end{bmatrix}
= \begin{bmatrix}
0 & 50 & 0 \\
50 & 60 & 0 \\
0 & 0 & -60
\end{bmatrix} \, \text{MPa}
\]

\[
\varepsilon_{n+1} = \frac{1}{3} \left( \text{tr} \, \varepsilon_{n} \right) \Delta \varepsilon = 60 \Delta \varepsilon
\]

\[
\Delta \varepsilon = \Delta \varepsilon - \frac{1}{3} \left( \text{tr} \, \Delta \varepsilon \right) \Delta \varepsilon
\]

\[
\begin{bmatrix}
\Delta \varepsilon_{nx} \\
\Delta \varepsilon_{ny} \\
\Delta \varepsilon_{nz}
\end{bmatrix}
= \begin{bmatrix}
20 & 60 & 0 \\
60 & -30 & 0 \\
0 & 0 & -30
\end{bmatrix} \times 10^{-6}
\]
- trial deviatoric stress:

$$\mathbf{\bar{S}}_{\text{trial}} = \mathbf{S}_n + 2G \Delta \varepsilon$$

$$\begin{bmatrix} \mathbf{\bar{S}}_{\text{trial}} \end{bmatrix} = \begin{bmatrix} 9.2308 & 59.2308 & 0 \\ 59.2308 & 55.3846 & 0 \\ 0 & 0 & -64.6154 \end{bmatrix}$$

$$\|\mathbf{\bar{S}}_{\text{trial}}\| = 119.768 \text{ MPa}$$

- yield stress:

$$6_Y = \sqrt{\frac{3}{2} \mathbf{S}_n : \mathbf{S}_n} = 135.2775 \text{ MPa}$$

- plastic multiplier increment:

$$\Delta \lambda = \frac{1}{3G} \left( \sqrt{\frac{3}{2}} \| \mathbf{\bar{S}}_{\text{trial}} \| - 6_Y \right) = 49.4337 \times 10^{-6}$$

- plastic strain increment:

$$\Delta \varepsilon^p = \Delta \varepsilon^p = \Delta \lambda \sqrt{\frac{3}{2}} \frac{\mathbf{\bar{S}}_{\text{trial}}}{\|\mathbf{\bar{S}}_{\text{trial}}\|}$$

$$\begin{bmatrix} \Delta \varepsilon^p \end{bmatrix} = \begin{bmatrix} 4.6662 & 29.9416 & 0 \\ 29.9416 & 27.9974 & 0 \\ 0 & 0 & -32.6636 \end{bmatrix} \times 10^{-6}$$

- Thus

$$\mathbf{S}_{\text{int}} = \mathbf{S}_{\text{trial}} - 2G \Delta \varepsilon^p$$

$$\begin{bmatrix} \mathbf{S}_{\text{int}} \end{bmatrix} = \begin{bmatrix} 8.5129 & 54.6244 & 0 \\ 54.6244 & 51.0773 & 0 \\ 0 & 0 & -59.5902 \end{bmatrix} \text{ MPa}$$
- Hydrostatic stress solution:
  \[ \sigma_{hh} = \sigma_n + K(\text{tr}\Delta \varepsilon) \sigma = 175 \sigma \text{ MPa} \]

- Thus, the total stress becomes
  \[ \sigma_{\text{total}} = \sigma_{\text{previous}} + \sigma_{\text{hydrostatic}} \]

\[
\begin{bmatrix}
\sigma_{\text{total}}
\end{bmatrix} =
\begin{bmatrix}
83.5129 & 54.6244 & 0 \\
54.6244 & 126.0943 & 0 \\
0 & 0 & 15.5902
\end{bmatrix} \text{ Mpa}
\]

- The exact solutions are (see previous example):
  \[
  \begin{bmatrix}
  \sigma_{\text{exact}}_{\text{total}}
  \end{bmatrix} =
  \begin{bmatrix}
  83.839 & 54.785 & 0 \\
  54.785 & 125.717 & 0 \\
  0 & 0 & 15.444
  \end{bmatrix} \text{ Mpa}
  \]

  \[
  \begin{bmatrix}
  \sigma_{\text{exact}}_{\text{total}}
  \end{bmatrix} =
  \begin{bmatrix}
  8.839 & 54.785 & 0 \\
  54.785 & 50.717 & 0 \\
  0 & 0 & -59.556
  \end{bmatrix} \text{ Mpa}
  \]

- The relative error of \( \sigma_{\text{total}} \) can be defined as

\[
\text{error} = \frac{\| \sigma_{\text{total}} - \sigma_{\text{exact}}_{\text{total}} \|}{\| \sigma_{\text{exact}}_{\text{total}} \|} \times 100 = 0.4868 \%
\]