The \( \nabla \) operator in cylindrical coordinate system is given by
\[
\nabla = \frac{\partial}{\partial r} e_r + \frac{1}{r} \frac{\partial}{\partial \theta} e_\theta + \frac{\partial}{\partial z} e_z.
\] (1)

The following identities are used:
\[
\frac{\partial e_r}{\partial \theta} = e_\theta, \quad \frac{\partial e_\theta}{\partial \theta} = -e_r,
\]
whereas other derivatives are zero. Furthermore:
\[
e_r \times e_r = 0, \quad e_r \times e_\theta = e_z, \quad e_r \times e_z = -e_\theta,
\]
\[
e_\theta \times e_r = -e_z, \quad e_\theta \times e_\theta = 0, \quad e_\theta \times e_z = e_r,
\]
\[
e_z \times e_r = e_\theta, \quad e_z \times e_\theta = -e_r, \quad e_z \times e_z = 0.
\] (2)

The following rule is applied:
\[
\nabla \times (\bullet) = e_i \times \frac{\partial (\bullet)}{\partial x_i}.
\] (6)

The curl of a vector \( \mathbf{v} \)
The vector field \( \mathbf{v} \) is given by
\[
\mathbf{v} = v_r e_r + v_\theta e_\theta + v_z e_z.
\] (7)

By definition:
\[
\text{curl} \mathbf{v} = \nabla \times \mathbf{v} = \left( \frac{\partial}{\partial r} e_r + \frac{1}{r} \frac{\partial}{\partial \theta} e_\theta + \frac{\partial}{\partial z} e_z \right) \times (v_r e_r + v_\theta e_\theta + v_z e_z).
\] (8)

Thus
\[
\nabla \times \mathbf{v} = e_r \times \frac{\partial (v_r e_r)}{\partial r} + e_r \times \frac{\partial (v_\theta e_\theta)}{\partial r} + e_r \times \frac{\partial (v_z e_z)}{\partial r} + \frac{1}{r} e_\theta \times \frac{\partial (v_r e_r)}{\partial \theta} + \frac{1}{r} e_\theta \times \frac{\partial (v_\theta e_\theta)}{\partial \theta} + \frac{1}{r} e_\theta \times \frac{\partial (v_z e_z)}{\partial \theta} + e_z \times \frac{\partial (v_r e_r)}{\partial z} + e_z \times \frac{\partial (v_\theta e_\theta)}{\partial z} + e_z \times \frac{\partial (v_z e_z)}{\partial z},
\] (9)

\[
\nabla \times \mathbf{v} = e_r \times \frac{\partial (v_r e_r)}{\partial r} + e_r \times \frac{\partial (v_\theta e_\theta)}{\partial r} + e_r \times \frac{\partial (v_z e_z)}{\partial r} + \frac{1}{r} e_\theta \times \frac{\partial (v_r e_r)}{\partial \theta} + \frac{1}{r} e_\theta \times \frac{\partial (v_\theta e_\theta)}{\partial \theta} + \frac{1}{r} e_\theta \times \frac{\partial (v_z e_z)}{\partial \theta} + e_z \times \frac{\partial (v_r e_r)}{\partial z} + e_z \times \frac{\partial (v_\theta e_\theta)}{\partial z} + e_z \times \frac{\partial (v_z e_z)}{\partial z},
\] (10)
\[ \nabla \times \mathbf{v} = e_r \times \left( \frac{\partial v_r}{\partial r} e_r \right) + e_r \times \left( \frac{\partial v_\theta}{\partial \theta} e_\theta \right) + e_r \times \left( \frac{\partial v_z}{\partial \theta} e_z \right) + \frac{1}{r} e_\theta \times \left( \frac{\partial v_r}{\partial \theta} e_r + r v_r e_\theta + \frac{1}{r} e_\theta \times \left( \frac{\partial v_z}{\partial \theta} e_z \right) \right) + e_z \times \left( \frac{\partial v_r}{\partial z} e_r \right) + e_z \times \left( \frac{\partial v_\theta}{\partial z} e_\theta \right) + e_z \times \left( \frac{\partial v_z}{\partial z} e_z \right), \]

(11)

\[ \nabla \times \mathbf{v} = e_r \times \left( \frac{\partial v_\theta}{\partial r} e_r \right) + e_r \times \left( \frac{\partial v_z}{\partial r} e_z \right) + e_r \times \left( \frac{\partial v_\theta}{\partial \theta} e_\theta \right) + \frac{1}{r} e_\theta \times \left( \frac{\partial v_r}{\partial \theta} e_r + r v_r e_\theta \right) + \frac{1}{r} e_\theta \times \left( \frac{\partial v_z}{\partial \theta} e_z \right) + e_z \times \left( \frac{\partial v_r}{\partial z} e_r \right) + e_z \times \left( \frac{\partial v_\theta}{\partial z} e_\theta \right) + e_z \times \left( \frac{\partial v_z}{\partial z} e_z \right), \]

(12)

\[ \nabla \times \mathbf{v} = e_r \times \left( \frac{\partial v_\theta}{\partial r} e_\theta \right) + e_r \times \left( \frac{\partial v_z}{\partial r} e_z \right) + \frac{1}{r} e_\theta \times \left( \frac{\partial v_r}{\partial \theta} e_r \right) + \frac{1}{r} e_\theta \times \left( -v_\theta e_r \right) + \frac{1}{r} e_\theta \times \left( \frac{\partial v_z}{\partial \theta} e_z \right) + e_z \times \left( \frac{\partial v_r}{\partial z} e_r \right) + e_z \times \left( \frac{\partial v_\theta}{\partial z} e_\theta \right), \]

(13)

\[ \nabla \times \mathbf{v} = \frac{\partial v_\theta}{\partial r} (e_r \times e_\theta) + \frac{\partial v_z}{\partial r} (e_r \times e_z) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} (e_\theta \times e_r) - \frac{1}{r} v_\theta (e_\theta \times e_r) + \frac{1}{r} \frac{\partial v_z}{\partial \theta} (e_\theta \times e_z) + \frac{\partial v_r}{\partial z} (e_z \times e_r) + \frac{\partial v_\theta}{\partial z} (e_z \times e_\theta), \]

(14)

\[ \nabla \times \mathbf{v} = \frac{\partial v_\theta}{\partial r} (e_\theta) + \frac{\partial v_z}{\partial r} (-e_\theta) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} (-e_\theta) - \frac{1}{r} v_\theta (-e_\theta) + \frac{1}{r} \frac{\partial v_z}{\partial \theta} (-e_\theta) + \frac{\partial v_r}{\partial z} (e_\theta) + \frac{\partial v_\theta}{\partial z} (-e_r), \]

(15)

\[ \nabla \times \mathbf{v} = \left( \frac{1}{r} \frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z} \right) e_r + \left( \frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z} \right) e_\theta + \left( \frac{\partial v_\theta}{\partial r} - \frac{1}{r} \frac{\partial v_r}{\partial \theta} + \frac{1}{r} v_\theta \right) e_z, \]

(16)

\[ [\nabla \times \mathbf{v}] = \begin{bmatrix} \frac{1}{r} \frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z} \\ \frac{\partial v_z}{\partial r} + \frac{\partial v_r}{\partial z} \\ \frac{\partial v_\theta}{\partial r} - \frac{1}{r} \frac{\partial v_r}{\partial \theta} + \frac{1}{r} v_\theta \end{bmatrix}, \]

(17)
The second-order tensor field $S$

By definition:

$$\nabla \times \mathbf{v} = \begin{bmatrix} \frac{1}{r} v_{z,\theta} - v_{\theta,z} \\ \frac{1}{r} v_{r,z} - v_{z,r} \\ v_{\theta,r} - \frac{1}{r} v_{r,\theta} + \frac{1}{r} v_{\theta} \end{bmatrix}. \quad (18)$$

The curl of a second-order tensor $S$

The second-order tensor field $S$ is given by

$$S = S_{rr} e_r \otimes e_r + S_{r\theta} e_r \otimes e_\theta + S_{rz} e_r \otimes e_z$$
$$+ S_{\theta r} e_\theta \otimes e_r + S_{\theta \theta} e_\theta \otimes e_\theta + S_{\theta z} e_\theta \otimes e_z$$
$$+ S_{z r} e_z \otimes e_r + S_{z \theta} e_z \otimes e_\theta + S_{zz} e_z \otimes e_z. \quad (19)$$

By definition:

$$\text{curl} S = \nabla \times S =$$

$$+ \left( \frac{\partial}{\partial r} e_r \right) \times S_{rr} e_r \otimes e_r + \left( \frac{\partial}{\partial r} e_r \right) \times S_{r\theta} e_r \otimes e_\theta + \left( \frac{\partial}{\partial r} e_r \right) \times S_{rz} e_r \otimes e_z$$
$$+ \frac{1}{r} \left( \frac{\partial}{\partial \theta} e_\theta \right) \times S_{rr} e_r \otimes e_r + \frac{1}{r} \left( \frac{\partial}{\partial \theta} e_\theta \right) \times S_{r\theta} e_r \otimes e_\theta + \frac{1}{r} \left( \frac{\partial}{\partial \theta} e_\theta \right) \times S_{rz} e_r \otimes e_z$$
$$+ \left( \frac{\partial}{\partial z} e_z \right) \times S_{rr} e_r \otimes e_r + \left( \frac{\partial}{\partial z} e_z \right) \times S_{r\theta} e_r \otimes e_\theta + \left( \frac{\partial}{\partial z} e_z \right) \times S_{rz} e_r \otimes e_z$$
$$+ \frac{1}{r} \left( \frac{\partial}{\partial \theta} e_\theta \right) \times S_{\theta r} e_\theta \otimes e_r + \frac{1}{r} \left( \frac{\partial}{\partial \theta} e_\theta \right) \times S_{\theta \theta} e_\theta \otimes e_\theta + \frac{1}{r} \left( \frac{\partial}{\partial \theta} e_\theta \right) \times S_{\theta z} e_\theta \otimes e_z$$
$$+ \left( \frac{\partial}{\partial z} e_z \right) \times S_{\theta r} e_\theta \otimes e_r + \left( \frac{\partial}{\partial z} e_z \right) \times S_{\theta \theta} e_\theta \otimes e_\theta + \left( \frac{\partial}{\partial z} e_z \right) \times S_{\theta z} e_\theta \otimes e_z$$
$$+ \frac{1}{r} \left( \frac{\partial}{\partial \theta} e_\theta \right) \times S_{zz} e_z \otimes e_r + \frac{1}{r} \left( \frac{\partial}{\partial \theta} e_\theta \right) \times S_{z \theta} e_z \otimes e_\theta + \frac{1}{r} \left( \frac{\partial}{\partial \theta} e_\theta \right) \times S_{zz} e_z \otimes e_z$$
$$+ \left( \frac{\partial}{\partial z} e_z \right) \times S_{zz} e_z \otimes e_r + \left( \frac{\partial}{\partial z} e_z \right) \times S_{z \theta} e_z \otimes e_\theta + \left( \frac{\partial}{\partial z} e_z \right) \times S_{zz} e_z \otimes e_z. \quad (20)$$
\[
\text{curl} S = \nabla \times S =
\]

\[
+ e_r \times \frac{\partial (S_r e_r \otimes e_r)}{\partial r} + e_r \times (S_r e_r \otimes e_\theta) + e_r \times \frac{\partial (S_r e_r \otimes e_z)}{\partial r} \\
+ \frac{1}{r} e_\theta \times \frac{\partial (S_r e_r \otimes e_r)}{\partial \theta} + \frac{1}{r} e_\theta \times \frac{\partial (S_r e_r \otimes e_\theta)}{\partial \theta} + \frac{1}{r} e_\theta \times \frac{\partial (S_r e_r \otimes e_z)}{\partial \theta} \\
+ e_z \times \frac{\partial (S_r e_r \otimes e_r)}{\partial z} + e_z \times \frac{\partial (S_r e_r \otimes e_\theta)}{\partial z} + e_z \times \frac{\partial (S_r e_r \otimes e_z)}{\partial z} \\
+ e_r \times \frac{\partial (S_{\theta r} e_\theta \otimes e_r)}{\partial r} + e_r \times (S_{\theta \theta} e_\theta \otimes e_\theta) + e_r \times \frac{\partial (S_{\theta r} e_\theta \otimes e_z)}{\partial r} \\
+ \frac{1}{r} e_\theta \times \frac{\partial (S_{\theta r} e_\theta \otimes e_r)}{\partial \theta} + \frac{1}{r} e_\theta \times \frac{\partial (S_{\theta \theta} e_\theta \otimes e_\theta)}{\partial \theta} + \frac{1}{r} e_\theta \times \frac{\partial (S_{\theta r} e_\theta \otimes e_z)}{\partial \theta} \\
+ e_z \times \frac{\partial (S_{\theta r} e_\theta \otimes e_r)}{\partial z} + e_z \times \frac{\partial (S_{\theta \theta} e_\theta \otimes e_\theta)}{\partial z} + e_z \times \frac{\partial (S_{\theta r} e_\theta \otimes e_z)}{\partial z},
\]

(21)

\[
\text{curl} S = \nabla \times S =
\]

\[
+ e_r \times (S_{rr} e_r \otimes e_r) + e_r \times (S_{r\theta} e_r \otimes e_\theta) + e_r \times (S_{rz} e_r \otimes e_z) \\
+ \frac{1}{r} e_\theta \times \frac{\partial (S_{rr} e_r \otimes e_r)}{\partial \theta} + \frac{1}{r} e_\theta \times \frac{\partial (S_{r\theta} e_r \otimes e_\theta)}{\partial \theta} + \frac{1}{r} e_\theta \times \frac{\partial (S_{rz} e_r \otimes e_z)}{\partial \theta} \\
+ e_z \times (S_{rr} e_r \otimes e_r) + e_z \times (S_{r\theta} e_r \otimes e_\theta) + e_z \times (S_{rz} e_r \otimes e_z) \\
+ e_r \times (S_{\theta \theta} e_\theta \otimes e_r) + e_r \times (S_{\theta r} e_\theta \otimes e_\theta) + e_r \times (S_{\theta z} e_\theta \otimes e_z) \\
+ \frac{1}{r} e_\theta \times \frac{\partial (S_{\theta \theta} e_\theta \otimes e_r)}{\partial \theta} + \frac{1}{r} e_\theta \times \frac{\partial (S_{\theta r} e_\theta \otimes e_\theta)}{\partial \theta} + \frac{1}{r} e_\theta \times \frac{\partial (S_{\theta z} e_\theta \otimes e_z)}{\partial \theta} \\
+ e_z \times (S_{\theta \theta} e_\theta \otimes e_r) + e_z \times (S_{\theta r} e_\theta \otimes e_\theta) + e_z \times (S_{\theta z} e_\theta \otimes e_z),
\]

(22)
\[
\text{curl} \; S = \nabla \times S = \\
+ e_r \times (S_{rr,e_r} e_r \otimes e_r) + e_r \times (S_{r\theta,r} e_r \otimes e_\theta) + e_r \times (S_{rz,r} e_r \otimes e_z) \\
+ \frac{1}{r} e_\theta \times (S_{rr,\theta} e_r \otimes e_r + S_{r\theta} e_\theta \otimes e_r + S_{rr} e_r \otimes e_\theta) \\
+ \frac{1}{r} e_\theta \times (S_{r\theta,\theta} e_r \otimes e_\theta + S_{r\theta} e_\theta \otimes e_\theta - S_{r\theta} e_r \otimes e_\theta) \\
+ \frac{1}{r} e_\theta \times (S_{rz,\theta} e_r \otimes e_r + S_{r\theta} e_\theta \otimes e_r + S_{r\theta} e_r \otimes e_\theta) \\
+ \frac{1}{r} e_\theta \times (S_{zz,\theta} e_r \otimes e_r + S_{zz} e_r \otimes e_\theta - S_{zz} e_\theta \otimes e_r) \\
+ \frac{1}{r} e_\theta \times (S_{zz,\theta} e_r \otimes e_r - S_{zz} e_\theta \otimes e_\theta) \\
+ \frac{1}{r} e_\theta \times (S_{zz,\theta} e_r \otimes e_r) \\
+ e_z \times (S_{rr,e_z} e_z \otimes e_r) + e_z \times (S_{r\theta,z} e_z \otimes e_\theta) + e_z \times (S_{rz,z} e_z \otimes e_z) \\
+ e_r \times (S_{rr,r} e_z \otimes e_r) + e_r \times (S_{r\theta,r} e_\theta \otimes e_r) + e_r \times (S_{r\theta} e_\theta \otimes e_\theta) \\
+ \frac{1}{r} e_\theta \times (S_{zz,r} e_z \otimes e_r) + S_{zz} e_r \otimes e_\theta \\
+ e_\theta \times (S_{zz,\theta} e_r \otimes e_r) + e_\theta \times (S_{zz,\theta} e_\theta \otimes e_r) + e_\theta \times (S_{zz,\theta} e_\theta \otimes e_\theta),
\]

(23)

\[
\text{curl} \; S = \nabla \times S = \\
+ \frac{1}{r} e_\theta \times (S_{rr,\theta} e_r \otimes e_r + S_{rr} e_\theta \otimes e_r) \\
+ \frac{1}{r} e_\theta \times (S_{r\theta,\theta} e_r \otimes e_\theta) \\
+ \frac{1}{r} e_\theta \times (S_{rz,\theta} e_r \otimes e_r) \\
+ \frac{1}{r} e_\theta \times (S_{zz,\theta} e_r \otimes e_\theta) + e_z \times (S_{zz,\theta} e_\theta \otimes e_r) + e_z \times (S_{zz,\theta} e_\theta \otimes e_\theta) \\
+ e_\theta \times (S_{zz,\theta} e_r \otimes e_r) + e_\theta \times (S_{zz,\theta} e_\theta \otimes e_r) + e_\theta \times (S_{zz,\theta} e_\theta \otimes e_\theta),
\]

(24)
\[ \text{curl} \mathbf{S} = \bigtriangledown \times \mathbf{S} = \]
\[ + \frac{1}{r} \left( -S_{rr,\theta} e_z \otimes e_r - S_{rr} e_z \otimes e_\theta \right) \]
\[ + \frac{1}{r} \left( -S_{r\theta,r} e_z \otimes e_\theta + S_{r\theta} e_z \otimes e_r \right) \]
\[ + \frac{1}{r} \left( -S_{rz,\theta} e_z \otimes e_z \right) \]
\[ + (S_{rr,z} e_\theta \otimes e_r) + (S_{r\theta,z} e_\theta \otimes e_\theta) + (S_{rz,z} e_\theta \otimes e_z) \]
\[ + (S_{rr} e_z \otimes e_r) + (S_{r\theta} e_z \otimes e_\theta) + (S_{rz} e_z \otimes e_z) \]
\[ + \frac{1}{r} \left( S_{r\theta} e_z \otimes e_r \right) + \frac{1}{r} \left( S_{\theta\theta} e_z \otimes e_\theta \right) + \frac{1}{r} \left( S_{\theta z} e_z \otimes e_z \right) \]
\[ + (-S_{rr,z} e_r \otimes e_r) + (-S_{r\theta,z} e_r \otimes e_\theta) + (-S_{rz,z} e_r \otimes e_z) \]
\[ + (-S_{rr} e_\theta \otimes e_r) + (-S_{r\theta} e_\theta \otimes e_\theta) + (-S_{rz} e_\theta \otimes e_z) \]
\[ + \frac{1}{r} \left( S_{zr,\theta} e_r \otimes e_r + S_{zr} e_r \otimes e_\theta \right) \]
\[ + \frac{1}{r} \left( S_{z\theta,\theta} e_\theta \otimes e_\theta - S_{z\theta} e_\theta \otimes e_r \right) \]
\[ + \frac{1}{r} \left( S_{zz,\theta} e_\theta \otimes e_\theta \right). \]

Thus

\[
[\bigtriangledown \times \mathbf{S}] = \begin{bmatrix}
S_{z\theta,\theta} - S_{z\theta,\theta} & S_{z\theta,\theta} + S_{z\theta} & S_{z\theta,\theta} - S_{\theta z,\theta} \\
S_{rr,z} - S_{z\theta,r} & S_{r\theta,r} - S_{z\theta} & S_{r\theta,\theta} - S_{\theta z,\theta} \\
S_{r\theta} - S_{rr} & S_{\theta\theta} - S_{\theta z} & S_{\theta\theta} - S_{\theta z,\theta}
\end{bmatrix}. \tag{26}
\]
EXERCISE 1

Consider the 2D scalar field
\[ \phi = \left( x_1^2 + x_2^2 \right) \arctan \left[ \frac{x_2}{x_1} \right]. \] (1)

Determine the gradient of the scalar field \( \phi \) at point \( P \) with coordinates \( x_1 = 3 \) and \( x_2 = 4 \). Determine the gradient at \( P \) using cylindrical coordinate system.

SOLUTION

Using Cartesian coordinate system:

The scalar field \( \phi \) is illustrated in Figure 1.

By definition, the gradient of the scalar field \( \phi \) in Cartesian coordinate system is defined as
\[ \nabla \phi = \frac{\partial \phi}{\partial x_1} e_1 + \frac{\partial \phi}{\partial x_2} e_2, \] (2)

where \( e_1 \) and \( e_2 \) are the unit basis vectors of the Cartesian coordinate system.

The partial derivatives of \( \phi \) are
\[ \frac{\partial \phi}{\partial x_1} = 2x_1 \arctan \left[ \frac{x_2}{x_1} \right] - \left( x_1^2 + x_2^2 \right) \frac{x_2}{1 + x_1^2} x_1^2 = 2x_1 \arctan \left[ \frac{x_2}{x_1} \right] - x_2, \] (3)
\[ \frac{\partial \phi}{\partial x_2} = 2x_2 \arctan \left[ \frac{x_2}{x_1} \right] + \left( x_1^2 + x_2^2 \right) \frac{x_2}{1 + x_1^2} x_1 = 2x_2 \arctan \left[ \frac{x_2}{x_1} \right] + x_1. \] (4)
Therefore, the gradient (field) is given by

\[
\begin{bmatrix}
2x_1\arctan\left(\frac{x_2}{x_1}\right) - x_2 \\
2x_2\arctan\left(\frac{x_2}{x_1}\right) + x_1
\end{bmatrix}.
\]

(5)

At point \( P \) it has the numerical values

\[
\begin{bmatrix}
6\arctan\left(\frac{4}{3}\right) - 4 \\
8\arctan\left(\frac{4}{3}\right) + 3
\end{bmatrix} \approx \begin{bmatrix} 1.56377 \\ 10.4184 \end{bmatrix}.
\]

(6)

The length of this vector equals to

\[
\sqrt{1.56377^2 + 10.4184^2} \approx 10.535.
\]

(7)

The orientation of the gradient vector at \( P \) is illustrated in Figure 2.

---

Using cylindrical coordinate system:

The transformation rules between the coordinates of the Cartesian and cylindrical coordinate systems are

\[
r = \sqrt{x_1^2 + x_2^2}, \quad \theta = \arctan\left(\frac{x_2}{x_1}\right).
\]

(8)

Note that function \( \tan\theta \) has period \( \pi \), and the principal value for inverse tangent function is

\[-\frac{\pi}{2} < \arctan\left(\frac{x_2}{x_1}\right) < \frac{\pi}{2}.
\]

(9)
Thus, the angle \( \theta \) should be determined by the procedure

\[
\theta = \begin{cases} 
\arctan \left[ \frac{x_2}{x_1} \right] & \text{if } x_1 > 0 \text{ and } x_2 \geq 0 \\
\arctan \left[ \frac{x_2}{x_1} \right] + 2\pi & \text{if } x_1 > 0 \text{ and } x_2 < 0 \\
\arctan \left[ \frac{x_2}{x_1} \right] + \pi & \text{if } x_1 < 0 \\
\frac{\pi}{2} & \text{if } x_1 = 0 \text{ and } x_2 > 0 \\
\frac{3\pi}{2} & \text{if } x_1 = 0 \text{ and } x_2 < 0 \\
0 & \text{if } x_1 = x_2 = 0
\end{cases}
\]  

(10)

The cylindrical coordinates of point \( P \) are

\[
r = \sqrt{3^2 + 4^2} = 5, \\
\theta = \arctan \left[ \frac{4}{3} \right] \approx 0.927295 \approx 53.1301°.
\]  

(11)

(12)

The gradient (field) of a 2D scalar field using cylindrical coordinate system is given by the formula

\[
\nabla \phi \phi = \frac{\partial \phi}{\partial r} e_r + \frac{1}{r} \frac{\partial \phi}{\partial \theta} e_\theta.
\]  

(13)

Thus, the gradient in the first quadrant (where \( P \) is located) is given by

\[
[\nabla \phi] = \left[ \begin{array}{c} 2r\theta \\
r \end{array} \right]
\]  

(14)

Consequently, the gradient vector at \( P \) is

\[
[\nabla \phi](P) = \left[ \begin{array}{c} 2 \cdot 5 \cdot 0.927295 \\
5 \end{array} \right] = \left[ \begin{array}{c} 9.27295 \\
5 \end{array} \right].
\]  

(15)

The norm of this vector is calculated as

\[
\|[\nabla \phi(P)]\| = \sqrt{9.27295^2 + 5^2} \approx 10.535.
\]  

(16)
**EXERCISE 2**
Consider the smooth vector field \( v = v_r e_r + v_\theta e_\theta + v_z e_z \) given in a cylindrical coordinate system. Prove that the curl of \( v \) equals to
\[
\nabla \times v = \left( \frac{v_\theta}{r} - v_{\theta, z} \right) e_r + \left( v_{r, z} - v_{z, r} \right) e_\theta + \left( \frac{v_\theta}{r} + v_{\theta, r} - \frac{v_r}{r} \right) e_z. \tag{17}
\]

**SOLUTION**
The \( \nabla \) operator in cylindrical coordinate system is given by
\[
\nabla = \frac{\partial}{\partial r} e_r + \frac{1}{r} \frac{\partial}{\partial \theta} e_\theta + \frac{\partial}{\partial z} e_z. \tag{18}
\]
Furthermore, we know that
\[
\frac{\partial e_r}{\partial \theta} = e_\theta \quad \text{and} \quad \frac{\partial e_\theta}{\partial \theta} = -e_r. \tag{19}
\]
In addition:
\[
\begin{align*}
e_r \times e_r &= 0 & e_r \times e_\theta &= e_z & e_r \times e_z &= -e_\theta \\
e_\theta \times e_r &= -e_z & e_\theta \times e_\theta &= 0 & e_\theta \times e_z &= e_r \\
e_z \times e_r &= e_\theta & e_z \times e_\theta &= -e_r & e_z \times e_z &= 0
\end{align*} \tag{20}
\]
The curl is defined as
\[
\text{curl} v = \nabla \times v = \left( \frac{\partial}{\partial r} e_r + \frac{1}{r} \frac{\partial}{\partial \theta} e_\theta + \frac{\partial}{\partial z} e_z \right) \times (v_r e_r + v_\theta e_\theta + v_z e_z) \tag{21}
\]
\[
\nabla \times v = \frac{\partial v_r}{\partial r} (e_r \times e_r) + \frac{\partial v_\theta}{\partial r} (e_r \times e_\theta) + \frac{\partial v_z}{\partial r} (e_r \times e_z) + \frac{1}{r} \left( \frac{\partial v_r}{\partial \theta} e_\theta \times e_r + v_r e_\theta \times \frac{\partial e_r}{\partial \theta} \right) + \frac{1}{r} \left( \frac{\partial v_\theta}{\partial \theta} e_\theta \times e_\theta + v_\theta e_\theta \times \frac{\partial e_\theta}{\partial \theta} \right) + \frac{1}{r} \left( \frac{\partial v_z}{\partial \theta} e_\theta \times e_z + v_z e_\theta \times \frac{\partial e_z}{\partial \theta} \right) + \frac{\partial v_r}{\partial z} (e_z \times e_r) + \frac{\partial v_\theta}{\partial z} (e_z \times e_\theta) + \frac{\partial v_z}{\partial z} (e_z \times e_z) \tag{22}
\]
\[
\nabla \times v = \frac{\partial v_r}{\partial r} (e_r \times e_r) + \frac{\partial v_\theta}{\partial r} (e_r \times e_\theta) + \frac{\partial v_z}{\partial r} (e_r \times e_z) + \frac{1}{r} \left( \frac{\partial v_r}{\partial \theta} e_\theta \times e_r + v_r e_\theta \times \frac{\partial e_r}{\partial \theta} \right) + \frac{1}{r} \left( \frac{\partial v_\theta}{\partial \theta} e_\theta \times e_\theta + v_\theta e_\theta \times \frac{\partial e_\theta}{\partial \theta} \right) + \frac{1}{r} \left( \frac{\partial v_z}{\partial \theta} e_\theta \times e_z + v_z e_\theta \times \frac{\partial e_z}{\partial \theta} \right) + \frac{\partial v_r}{\partial z} (e_z \times e_r) + \frac{\partial v_\theta}{\partial z} (e_z \times e_\theta) + \frac{\partial v_z}{\partial z} (e_z \times e_z) \tag{23}
\]
\[ \nabla \times \mathbf{v} = 0 + \frac{\partial v_\theta}{\partial r} e_z - \frac{\partial v_z}{\partial r} e_\theta \\
+ \frac{1}{r} \left( - \frac{\partial v_r}{\partial \theta} e_z + v_r e_\theta \times e_\theta \right) + \frac{1}{r} (0 - v_\theta e_\theta \times e_r) + \frac{1}{r} \left( \frac{\partial v_z}{\partial \theta} e_r + 0 \right) \\
+ \frac{\partial v_r}{\partial z} e_\theta - \frac{\partial v_\theta}{\partial z} e_r + 0 \quad (24) \]

\[ \nabla \times \mathbf{v} = 0 + \frac{\partial v_\theta}{\partial r} e_z - \frac{\partial v_z}{\partial r} e_\theta \\
+ \frac{1}{r} \left( - \frac{\partial v_r}{\partial \theta} e_z + 0 \right) + \frac{1}{r} (0 + v_\theta e_z) + \frac{1}{r} \left( \frac{\partial v_z}{\partial \theta} e_r + 0 \right) \\
+ \frac{\partial v_r}{\partial z} e_\theta - \frac{\partial v_\theta}{\partial z} e_r + 0 \quad (25) \]

\[ \nabla \times \mathbf{v} = \left( \frac{1}{r} \frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z} \right) e_r + \left( \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} \right) e_\theta + \left( \frac{v_\theta}{r} + \frac{\partial v_\theta}{\partial r} - \frac{1}{r} \frac{\partial v_r}{\partial \theta} \right) e_z \quad (26) \]

\[ [\nabla \times \mathbf{v}] = \begin{bmatrix} \frac{1}{r} \frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z} \\
\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} \\
v_\theta + \frac{\partial v_\theta}{\partial r} - \frac{1}{r} \frac{\partial v_r}{\partial \theta} \end{bmatrix} \quad (27) \]

\[ [\nabla \times \mathbf{v}] = \begin{bmatrix} \frac{v_{z,\theta}}{r} - v_{\theta,z} \\
v_{r,z} - v_{z,r} \\
v_\theta + \frac{v_{\theta,r}}{r} - \frac{v_{r,\theta}}{r} \end{bmatrix} \quad (28) \]
EXERCISE 3

The displacement field in a linear homogeneous isotropic material in a Cartesian coordinate system is given by

\[ u = u_x (x, y, z) \mathbf{e}_1 + u_y (x, y, z) \mathbf{e}_2 + u_z (x, y, z) \mathbf{e}_3, \]  
(1)

\[ u = ye_1 + (x - z) e_2 + xze_3, \]  
(2)

where \( \mathbf{e}_1, \mathbf{e}_2, \) and \( \mathbf{e}_3 \) represent the Cartesian unit basis vectors, whereas \( x, y \) and \( z \) are the coordinates along the orthogonal axes.

Determine the components of the infinitesimal strain tensor using cylindrical coordinate system.

SOLUTION

The components of the displacement vector field using cylindrical coordinate system is obtained by the formula

\[ \begin{bmatrix} u \end{bmatrix}_{(r,\theta,z)} = \begin{bmatrix} Q \end{bmatrix} \begin{bmatrix} u \end{bmatrix}_{(x,y,z)}, \]  
(3)

where the proper orthogonal tensor \( Q \), which describes the transformation between the basis vectors of the two coordinate systems, is given by

\[ Q = \begin{bmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}. \]  
(4)

The transformation rules for the coordinates:

\[ x = r\cos \theta, \quad y = r\sin \theta, \quad z = z. \]  
(5)

Thus, the component of \( u \) in the cylindrical coordinate system, after simplification, become

\[ \begin{bmatrix} u \end{bmatrix}_{(r,\theta,z)} = \begin{bmatrix} u_r \\ u_\theta \\ u_z \end{bmatrix} = \begin{bmatrix} (2r\cos \theta - z) \sin \theta \\ r\cos2\theta - z\cos \theta \\ rz\cos \theta \end{bmatrix}. \]  
(6)

The (symmetric) infinitesimal strain tensor in cylindrical coordinate system is defined as

\[ \begin{bmatrix} \varepsilon \end{bmatrix}_{(r,\theta,z)} = \begin{bmatrix} \varepsilon_{rr} & \varepsilon_{r\theta} & \varepsilon_{rz} \\ \varepsilon_{r\theta} & \varepsilon_{\theta\theta} & \varepsilon_{\theta z} \\ \varepsilon_{rz} & \varepsilon_{\theta z} & \varepsilon_{zz} \end{bmatrix}, \]  
(7)

where

\[ \varepsilon_{rr} = \frac{\partial u_r}{\partial r} = \sin 2\theta, \]  
(8)

\[ \varepsilon_{\theta\theta} = \frac{1}{r} \left( \frac{\partial u_\theta}{\partial \theta} + u_r \right) = -\sin 2\theta, \]  
(9)

\[ \varepsilon_{zz} = \frac{\partial u_z}{\partial z} = r\cos \theta. \]  
(10)
\[\varepsilon_{r\theta} = \frac{1}{2} \left( \frac{1}{r} \frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} \right) = \cos 2\theta,\]  
(11)

\[\varepsilon_{rz} = \frac{1}{2} \left( \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r} \right) = \frac{1}{2} \left( z \cos \theta - \sin \theta \right),\]  
(12)

\[\varepsilon_{\theta z} = \frac{1}{2} \left( \frac{\partial u_\theta}{\partial z} + \frac{1}{r} \frac{\partial u_z}{\partial \theta} \right) = -\frac{1}{2} \left( z \sin \theta + \cos \theta \right).\]  
(13)

Thus,

\[
\begin{bmatrix}
\varepsilon_{rr} & \varepsilon_{r\theta} & \varepsilon_{rr} \\
\varepsilon_{r\theta} & \varepsilon_{\theta\theta} & \varepsilon_{rz} \\
\varepsilon_{rr} & \varepsilon_{rz} & \varepsilon_{zz}
\end{bmatrix}
= \begin{bmatrix}
\sin 2\theta & \cos 2\theta & \frac{1}{2} (z \cos \theta - \sin \theta) \\
\cos 2\theta & -\sin 2\theta & -\frac{1}{2} (z \sin \theta + \cos \theta) \\
\frac{1}{2} (z \cos \theta - \sin \theta) & -\frac{1}{2} (z \sin \theta + \cos \theta) & r \cos \theta
\end{bmatrix}.
\]  
(14)

This result can be obtained in a different, but equivalent way as follows. First the strain tensor is obtained in the Cartesian coordinate system as

\[
\begin{bmatrix}
\varepsilon_{xx} & \frac{1}{2} \gamma_{xy} & \frac{1}{2} \gamma_{xz} \\
\frac{1}{2} \gamma_{xy} & \varepsilon_{yy} & \frac{1}{2} \gamma_{yz} \\
\frac{1}{2} \gamma_{xz} & \frac{1}{2} \gamma_{yz} & \varepsilon_{zz}
\end{bmatrix}
= \begin{bmatrix}
\frac{1}{2} (\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y}) & \frac{1}{2} (\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}) & \frac{1}{2} (\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}) \\
\frac{1}{2} (\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}) & \frac{1}{2} (\frac{\partial u_y}{\partial y} + \frac{\partial u_y}{\partial x}) & \frac{1}{2} (\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}) \\
\frac{1}{2} (\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}) & \frac{1}{2} (\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}) & \frac{1}{2} (\frac{\partial u_z}{\partial z} + \frac{\partial u_z}{\partial z})
\end{bmatrix},
\]  
(15)

\[
\begin{bmatrix}
\varepsilon_{xx} & \frac{1}{2} \gamma_{xy} & \frac{1}{2} \gamma_{xz} \\
\frac{1}{2} \gamma_{xy} & \varepsilon_{yy} & \frac{1}{2} \gamma_{yz} \\
\frac{1}{2} \gamma_{xz} & \frac{1}{2} \gamma_{yz} & \varepsilon_{zz}
\end{bmatrix}
= \begin{bmatrix}
0 & 1 & \frac{1}{2} z \\
1 & 0 & -\frac{1}{2} \\
\frac{1}{2} z & -\frac{1}{2} & x
\end{bmatrix}.
\]  
(16)

The components of the second order tensor \(\varepsilon\) in the cylindrical basis is obtained as

\[
\begin{bmatrix}
\varepsilon_{rr} & \varepsilon_{r\theta} & \varepsilon_{rr} \\
\varepsilon_{r\theta} & \varepsilon_{\theta\theta} & \varepsilon_{rz} \\
\varepsilon_{rr} & \varepsilon_{rz} & \varepsilon_{zz}
\end{bmatrix}
= \begin{bmatrix}
\sin 2\theta & \cos 2\theta & \frac{1}{2} (z \cos \theta - \sin \theta) \\
\cos 2\theta & -\sin 2\theta & -\frac{1}{2} (z \sin \theta + \cos \theta) \\
\frac{1}{2} (z \cos \theta - \sin \theta) & -\frac{1}{2} (z \sin \theta + \cos \theta) & r \cos \theta
\end{bmatrix}.
\]  
(19)
EXERCISE 4
The infinitesimal strain tensor is given in a Cartesian coordinate system with its matrix as
\[
\varepsilon = \begin{bmatrix}
A \cdot x_2^3 & B \cdot x_1 x_2 & 0 \\
B \cdot x_1 x_2 & A \cdot x_1^2 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\] (20)

Determine the relation between parameters \(A\) and \(B\) required for there to exist a continuous single-valued displacement field that corresponds to this strain field.

SOLUTION
The compatibility equations read as
\[
\nabla \times (\nabla \times \varepsilon) = 0,
\] (21)
where
\[
\nabla \times \varepsilon = \frac{\partial \varepsilon_{bc}}{\partial x_a} e_a \times (e_b \otimes e_c) = \varepsilon_{bc,a} \varepsilon_{abd} (e_d \otimes e_c).
\] (22)

Thus
\[
[\nabla \times \varepsilon] = \begin{bmatrix}
\varepsilon_{31,2} - \varepsilon_{21,3} & \varepsilon_{32,2} - \varepsilon_{22,3} & \varepsilon_{33,2} - \varepsilon_{23,3} \\
\varepsilon_{11,3} - \varepsilon_{31,1} & \varepsilon_{12,3} - \varepsilon_{32,1} & \varepsilon_{13,3} - \varepsilon_{33,1} \\
\varepsilon_{21,1} - \varepsilon_{11,2} & \varepsilon_{22,1} - \varepsilon_{12,2} & \varepsilon_{23,1} - \varepsilon_{13,2}
\end{bmatrix}.
\] (23)

Let \(\Xi = \nabla \times \varepsilon\), then
\[
[\nabla \times (\nabla \times \varepsilon)] = \begin{bmatrix}
\Xi_{31,2} - \Xi_{21,3} & \Xi_{32,2} - \Xi_{22,3} & \Xi_{33,2} - \Xi_{23,3} \\
\Xi_{11,3} - \Xi_{31,1} & \Xi_{12,3} - \Xi_{32,1} & \Xi_{13,3} - \Xi_{33,1} \\
\Xi_{21,1} - \Xi_{11,2} & \Xi_{22,1} - \Xi_{12,2} & \Xi_{23,1} - \Xi_{13,2}
\end{bmatrix}.
\] (24)

For this particular problem we have
\[
[\nabla \times \varepsilon] = \begin{bmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
B x_2 - 2A x_2 & 2A x_1 - B x_1 & 0
\end{bmatrix}
\] (25)
and
\[
[\nabla \times (\nabla \times \varepsilon)] = \begin{bmatrix}
B - 2A & 0 & 0 \\
0 & B - 2A & 0 \\
0 & 0 & 0
\end{bmatrix}.
\] (26)

Therefore, the following relation must hold:
\[
B = 2A.
\] (27)
EXERCISE 5

A elastic thin disk is shrink-fitted to a rigid shaft. The diameter of the shaft is denoted by $d$, whereas the inner and outer radius of the elastic disk are $R_i$ and $R_o$. The elastic parameters of the annular disk are $E$ and $\nu$, whereas the mass density is $\rho$. The outer pressure is neglected.

Data:

\[
\begin{align*}
  d &= 50 \text{ mm}, \\
  R_i &= 24.95 \text{ mm}, \\
  R_o &= 100 \text{ mm}, \\
  E &= 200 \text{ GPa}, \\
  \nu &= 0.3, \\
  \rho &= 7800 \text{ kg/m}^3.
\end{align*}
\]

Determine a) the pressure exerted on the interface of the two bodies after assembling b) the angular velocity at which the assembly becomes to loose c) the value of radius at which the maximum radial stress exist when the disk rotates with the determined angular velocity. Plot the stress and displacement solutions in the annular disk when it is shrink fitted to the rigid shaft. Plot the stress and displacement solutions in the annular disk when it becomes to loose due to the angular velocity.

SOLUTION

a) Since thin disk is considered, the solutions obtained for plane stress case can be used here. Therefore, we have the following formulae:

\[
\begin{align*}
  \sigma_r (\varphi) &= A - B \cdot \varphi, \\
  \sigma_t (\varphi) &= A + B \cdot \varphi, \\
  u_r (\varphi) &= \frac{(1 + \nu) R_i}{E} \left( \frac{1 - \nu}{1 + \nu} \frac{A}{\sqrt{\varphi}} + B \sqrt{\varphi} \right),
\end{align*}
\]

where

\[
\begin{align*}
  A &= B \cdot \varphi_0 - p_o, \\
  B &= \frac{p_i - p_o}{1 - \varphi_0}, \\
  \varphi &= \left( \frac{R_i}{r} \right)^2, \\
  \varphi_0 &= \left( \frac{R_o}{R_i} \right)^2.
\end{align*}
\]

The shaft is modelled as rigid body. Thus, after assembling the two bodies, the inner points of the annular disk move with the displacement

\[
\delta = \frac{d}{2} - R_i = 0.05 \text{ mm}.
\]

This radial displacement is obtained from (3) by substituting $\varphi = 1$:

\[
\begin{align*}
  u_r (\varphi = 1) &= \frac{(1 + \nu) R_i}{E} \left( \frac{1}{1 + \nu} \frac{A}{1 + B} \right).
\end{align*}
\]
By substituting \( A \) and \( B \) with \( p_o = 0 \) gives
\[
    u_r (R_i) = \frac{p_i R_i}{E} \left( \frac{2}{1 - \varphi_0} - (1 - \nu) \right).
\]
(10)

Solving \( u_r (R_i) = \delta \) for \( p_i \) gives the result
\[
    p_i = \frac{E (1 - \varphi_0) \delta}{R_i (1 + \nu + (1 - \nu) \varphi_0)} = \frac{E (R_o^2 - R_i^2) \delta}{(1 - \nu) R_i^3 + (1 + \nu) R_i R_o^2}.
\]
(11)

Thus, the pressure between the two bodies is
\[
    p_i = 279.74 \text{ MPa.}
\]
(12)

The stress and displacement solutions become:
\[
    A = 18.57, \quad B = 298.31, \quad (13) \quad (14)
\]
\[
    \sigma_r (\varphi) = 18.57 - 298.31 \cdot \varphi, \quad (15) \quad (16)
\]
\[
    \sigma_t (\varphi) = 18.57 - 298.31 \cdot \varphi,
\]
\[
    u_r (\varphi) = 0.0483787 \sqrt{\varphi} + 0.00162161 \frac{1}{\sqrt{\varphi}}, \quad (17)
\]
where stresses are in [MPa], whereas the displacement is given in [mm]. Solutions (15)-(17) are illustrated in Figure 1 and Figure 2.

![Figure 1: Illustration of the stress solutions vs. parameter \( \varphi \)](image)

The solutions above can be expressed in terms of the radius \( r \) as
\[
    \sigma_r (r) = 18.57 - 185698.58 \frac{1}{r^2}, \quad (18)
\]
\[
    \sigma_t (r) = 18.57 + 185698.58 \frac{1}{r^2}, \quad (19)
\]
\[
    u_r (r) = 0.0000649945 \cdot r + 1.20704 \frac{1}{r}. \quad (20)
\]
These solutions are plotted in Figure 3 and Figure 4.

Figure 3: Illustration of the stress solutions vs. radius $r$
b) The displacement and stress solutions in a rotating thin annular disk are given by

\[
\sigma_r (r) = \frac{(3 + \nu) \rho \omega^2}{8r^2} \left( R_o^2 - r^2 \right) \left( r^2 - R_i^2 \right),
\]  
(21)

\[
\sigma_t (r) = \frac{(3 + \nu) \rho \omega^2}{8} \left( \frac{R_o^2 R_i^2}{r^2} + R_o^2 + R_i^2 \right) - \frac{(1 + 3\nu) \rho \omega^2}{8} r^2,
\]  
(22)

\[
u_r (r) = \frac{(3 + \nu) (1 - \nu) \rho \omega^2 r}{8E} \left( \frac{1 + \nu R_o^2 R_i^2}{1 - \nu} \right) - \frac{R_i^2}{r^2} \left( R_o^2 + R_i^2 - \frac{1 + \nu}{3 + \nu} r^2 \right).
\]  
(23)

When the assembly (where the inner shaft is a rigid body) becomes to loose, then the displacement of the inner points move with \( \delta \). Thus, the angular velocity can be obtained from the equation

\[
u_r (R_i) = \delta.
\]  
(24)

The displacement value at the inner radius is obtained from (23) as

\[
u_r (R_i) = \frac{R_i \rho \omega^2}{4E} \left( R_i^2 (1 - \nu) + (3 + \nu) R_o^2 \right).
\]  
(25)

The angular velocity can be expressed as

\[
\omega = 2 \sqrt{\frac{E \delta}{R_i \rho (R_i^2 (1 - \nu) + (3 + \nu) R_o^2)}},
\]  
(26)

\[
\omega = 2479.97 \cdot 1.1 \text{ s} = 2479.97 \cdot \frac{60}{2\pi} \text{ rpm} = 23676.3 \text{ rpm}.
\]  
(27)

\[
\omega = 2479.97 \cdot 1 \text{ s} = 2479.97 \cdot \frac{60}{2\pi} \text{ rpm} = 23676.3 \text{ rpm}.
\]  
(28)

Note that, the use of dimensions [N], [mm] and [N/mm²]in (26) requires to convert the dimension of \( \rho \) to

\[
\frac{\text{kg}}{\text{m}^3} = \frac{\text{Ns}^2}{\text{m}^4} = 10^{-12} \frac{\text{Ns}^2}{\text{mm}^4}.
\]  
(29)
Thus, the stress and displacement solutions at angular velocity $\omega$ are

\[
\sigma_r (r) = 210.203 - 0.0197884 \cdot r^2 - \frac{123183.49}{r^2},
\]

\[
\sigma_t (r) = 210.203 - 0.0113933 \cdot r^2 + \frac{123183.49}{r^2},
\]

\[
u_r (r) = -2.72841 \times 10^{-8} \cdot r^3 + 0.000735709 \cdot r + \frac{0.800693}{r}.
\]

These solutions are plotted in Figure 5 and Figure 6.

---

**Figure 5:** Illustration of the stress solutions vs. radius $r$

---

**Figure 6:** Illustration of the radial displacement solution vs. radius $r$
Stress and displacement solutions can be expressed in terms of parameter $\varphi$ as

\[
\sigma_r (\varphi) = 210.203 - 197.887 \cdot \varphi - \frac{12.3183}{\varphi},
\]

\[
\sigma_t (\varphi) = 210.203 + 197.887 \cdot \varphi - \frac{7.09238}{\varphi},
\]

\[
u_r (\varphi) = -\frac{0.000423761}{\varphi^{3/2}} + \frac{0.0183559}{\sqrt[3]{\varphi}} + 0.0320919 \sqrt[3]{\varphi}.
\]

These solutions are plotted in Figure 7 and Figure 8.

Figure 7: Illustration of the stress solutions vs. radius $r$

Figure 8: Illustration of the radial displacement solution vs. radius $r$

c) The value of the radius at which the maximum radial stress exist is given by

\[
R_* = \sqrt{R_t R_o} = \sqrt{24.95 \cdot 100} = 49.94997 \text{ mm}.
\]
EXERCISE 6
An elastic solid thin disk with material parameters

\[ E = 1 \text{ GPa}, \quad \rho = 1000 \frac{\text{kg}}{\text{m}^3}, \quad \nu = 0.4 \]

is given. Determine the angular velocity \( \omega_1 \) at which the diameter of the disk becomes 102% of its original value \( D = 1000 \text{ mm} \).

SOLUTION
The displacement of the material points at the outer radius is

\[ \delta = u_r (R_o) = \frac{1.02D}{2} - \frac{D}{2} = 10 \text{ mm}. \quad (37) \]

The displacement solution in the disk is given by

\[ u_r (r) = \frac{(1 - \nu)(3 + \nu) \rho \omega^2 r}{8E} \left( R_o^2 - \frac{1 + \nu}{3 + \nu} r^2 \right). \quad (38) \]

Its value at \( r = R_o \) is

\[ u_r (R_o) = \frac{(1 - \nu) \rho \omega^2 R_o^3}{4E}. \quad (39) \]

Thus the angular velocity can be expressed as

\[ \omega_1 = \sqrt{\frac{4E\delta}{\rho R_o^3 (1 - \nu)}} = \sqrt{\frac{4 \cdot (1 \times 10^9) \cdot 0.01}{1000 \cdot 0.5^3 (1 - 0.4)}} = 730.3 \frac{1}{\text{s}}. \quad (40) \]

The result is measured in radian/sec. It can be converted into rpm (round per minute) as

\[ \omega_1 = (730.3) \cdot \frac{60}{2\pi} \text{ rpm} \quad \omega_1 = 6973.85 \text{ rpm}. \quad (41) \]

Remark. Take care with the dimension of the density \( \rho \). If \( E \) is given in [MPa] and geometry is given in [mm], then \( \rho \) has to be given according to the rule

\[ \frac{\text{kg}}{\text{m}^3} = \frac{\text{Ns}^2}{\text{m}^4} = 10^{-12} \frac{\text{Ns}^2}{\text{mm}^4}. \quad (42) \]

For example:

\[ 7800 \frac{\text{kg}}{\text{m}^3} = 7800 \times 10^{-12} \frac{\text{Ns}^2}{\text{mm}^4}. \quad (43) \]
Example 1

- An uniaxial tension/compression loading process consists of three loading segments:

|       |  \(|\sigma|\) | \(\varepsilon\) |
|-------|-------------|-----------------|
| A     | 150         | ?               |
| B     | ?           | 0               |
| C     | 200         | ?               |

- The initial state is a stress-free configuration with zero plastic strain.
- The material is linear elastic with linear isotropic hardening plastic property.

The material constants are

\[ E = 200 \text{ GPa} \]
\[ \sigma_y = 100 \text{ MPa} \]
\[ H = 20 \text{ GPa} \]

- Determine the unknown values of strain and stress at A, B, and C.

**loading path 0→A**

\[ \Delta\sigma = \sigma_A - \sigma_0 = 150 \text{ MPa} \]

\[
\begin{align*}
F_{n} &= 0 - 150 = -150 \\
F_{\text{final}} &= 0 + 150 = 150
\end{align*}
\]

\[
E_{\text{ep}} = \frac{E \cdot H}{E + H} = 18.182 \text{ GPa}
\]
path $0 \rightarrow Y_0$:

\[
\varepsilon_{Y_0} = 0 + \frac{\bar{\sigma}_{Y_0} - \sigma_0}{E} = 0.5 \times 10^{-3}
\]

\[
\varepsilon_{Y_0}^p = \varepsilon_0^p = 0
\]

path $Y_0 \rightarrow A$:

\[
\varepsilon_A = \varepsilon_{Y_0} + \frac{\bar{\sigma}_A - \bar{\sigma}_{Y_0}}{E^p} = 3.25 \times 10^{-3}
\]

\[
\varepsilon_A^p = \varepsilon_{Y_0}^p + \frac{\bar{\sigma}_A - \bar{\sigma}_{Y_0}}{H} = 2.5 \times 10^{-3}
\]

loading path $A \rightarrow B$:

\[
\bar{\sigma}_{\text{trial}} = \bar{\sigma}_A + E (\varepsilon_B - \varepsilon_A) = -500 \text{MPa}
\]

\[
\begin{align*}
F_n &= 0 \\
F_{\text{trial}} &= 500 - 150 = 350
\end{align*}
\]

\[
\text{sign}(\bar{\sigma}_A) \cdot \text{sign}(\Delta \sigma) = -1
\]

elastic to plastic transition

the new yield stress in the opposite direction:

\[
\bar{\sigma}_{Y_1} = -\bar{\sigma}_A = -150 \text{MPa}
\]

path $A \rightarrow Y_1$:

\[
\varepsilon_{Y_1} = \varepsilon_A + \frac{\bar{\sigma}_{Y_1} - \bar{\sigma}_A}{E} = 1.75 \times 10^{-3}
\]

\[
\varepsilon_{Y_1}^p = \varepsilon_A^p = 2.5 \times 10^{-3}
\]

path $Y_1 \rightarrow B$:

\[
\bar{\sigma}_B = \bar{\sigma}_{Y_1} + E^p (\varepsilon_B - \varepsilon_{Y_1}) = -181.818 \text{MPa}
\]

\[
\varepsilon_B^p = \varepsilon_{Y_1}^p + \frac{\bar{\sigma}_B - \bar{\sigma}_{Y_1}}{H} = 0.909 \times 10^{-3}
\]
loading path $B \rightarrow C$:

$$F_n = 0$$
$$F_{lab} = |\sigma_c - \sigma_B| = 18.1818$$

The new yield stress:

$$\sigma_{Y_2} = -\sigma_B = 181.818 \text{ MPa}$$

path $B \rightarrow Y_2$:

$$\varepsilon_{Y_2} = \varepsilon_B + \frac{\sigma_{Y_2} - \sigma_B}{E} = 1.818 \times 10^{-3}$$
$$\varepsilon_{Y_2}^p = \varepsilon_B = 0.909 \times 10^{-3}$$

path $Y_2 \rightarrow C$:

$$\varepsilon_c = \varepsilon_{Y_2} + \frac{\sigma_c - \sigma_{Y_2}}{E^p} = 2.818 \times 10^{-3}$$
$$\varepsilon_c^p = \varepsilon_{Y_2}^p + \frac{\sigma_c - \sigma_{Y_2}}{H} = 1.818 \times 10^{-3}$$

Results table:

<table>
<thead>
<tr>
<th></th>
<th>$\sigma$ [MPa]</th>
<th>$\varepsilon \times 10^3$</th>
<th>$\varepsilon^p \times 10^3$</th>
<th>$\varepsilon^e \times 10^3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Y_0$</td>
<td>100</td>
<td>0.5</td>
<td>0</td>
<td>0.5</td>
</tr>
<tr>
<td>A</td>
<td>150</td>
<td>3.25</td>
<td>2.5</td>
<td>0.75</td>
</tr>
<tr>
<td>$Y_1$</td>
<td>-150</td>
<td>1.75</td>
<td>2.5</td>
<td>-0.75</td>
</tr>
<tr>
<td>B</td>
<td>-181.81</td>
<td>0</td>
<td>0.909</td>
<td>-0.909</td>
</tr>
<tr>
<td>$Y_2$</td>
<td>181.81</td>
<td>1.818</td>
<td>0.909</td>
<td>0.909</td>
</tr>
<tr>
<td>C</td>
<td>200</td>
<td>2.818</td>
<td>1.818</td>
<td>1</td>
</tr>
</tbody>
</table>
Example 2

- An assembly consists of three parts as illustrated in the Figure:

- Part ① is a circular tube, whereas part ② is a circular cylinder.
- Both parts are made of linear elastic - perfectly plastic materials, but with different material properties.

- Determine a) the elastic (F_E) and plastic (F_p) limit loads
  b) the residual stresses after unloading from F_p
  c) the displacement of the rigid plate for F_E and for F_p

\[
\begin{align*}
L &= 1 \text{m} \\
L &= 40 \text{cm} \\
d_2 &= 15 \text{cm} \\
t &= 2 \text{cm} \\
\sigma_{Y_1} &= 200 \text{ MPa} \\
E_1 &= 200 \text{ GPa} \\
\sigma_{Y_2} &= 150 \text{ MPa} \\
E_2 &= 150 \text{ GPa}
\end{align*}
\]
Solution:

- the section areas:

\[ A_1 = \frac{d_1^2 - (d_1-2t)^2}{4} = 238.761 \text{ cm}^2 \]
\[ A_2 = \frac{d_2^2 J_1}{4} = 176.715 \text{ cm}^2 \]

a) - the elastic limit strains:

\[ \varepsilon_{Y_1} = \frac{6Y_1}{E_1} = 1 \times 10^{-3} \]
\[ \varepsilon_{Y_2} = \frac{6Y_2}{E_2} = 1 \times 10^{-3} \]

- the corresponding displacements:

\[ U_{Y_1} = \varepsilon_{Y_1} \cdot L_1 = \varepsilon_{Y_1} \cdot L = 1 \text{ mm} \]
\[ U_{Y_2} = \varepsilon_{Y_2} \cdot L_2 = \varepsilon_{Y_2} \cdot \frac{L}{2} = 0.5 \text{ mm} \]

- Thus, part 2 deforms plastically first.

- From the equilibrium of the rigid part, it follows that

\[ F = 6_1 \cdot A_1 + 6_2 \cdot A_2 \]
\[ F = E_1 \varepsilon_1 A_1 + E_2 \varepsilon_2 A_2 \]
\[ F = E_1 \cdot \frac{U}{L} A_1 + E_2 \frac{U}{L/2} A_2 \]

\[ F = \frac{U}{L} \left( E_1 \cdot A_1 + 2E_2 A_2 \right) \]

\[ U = \frac{F \cdot L}{E_1 A_1 + 2E_2 A_2} \]
\[ F_E = \frac{U_Y z}{L} (E_1 A_1 + 2E_2 A_2) \]

\[ F_E = 5038.335 \text{ kN} \]

- The expression for the slvesses (if \( F \leq F_E \)):

\[ \sigma_1 = \frac{F \cdot E_1}{E_1 A_1 + 2E_2 A_2} \]

\[ \sigma_2 = \frac{2F \cdot E_2}{E_1 A_1 + 2E_2 A_2} \]

- at \( F = F_E \):

\[ \sigma_{1E} = 100 \text{ MPa} ; \ E_{1E} = 0.5 \times 10^{-3} ; \ u_E = 0.5 \text{ mm} \]

\[ \sigma_{2E} = 150 \text{ MPa} ; \ E_{2E} = 1 \times 10^{-3} \]

- the plastic limit load is obtained from the equilibrium of the rigid part written at \( F = F_p \):

\[ F_p = \sigma_{1} \cdot A_1 + \sigma_{2} \cdot A_2 \]

\[ F_p = 17425.945 \text{ kN} \]

- at \( F = F_p \):

\[ \sigma_{1p} = 200 \text{ MPa} ; \ E_{1p} = 1 \times 10^{-3} ; \ u_p = 1 \text{ mm} \]

\[ \sigma_{2p} = 150 \text{ MPa} ; \ E_{2p} = 2 \times 10^{-3} \]
b) the residual stress is obtained by subtracting the elastic stress from the current stress.

- the elastic stresses are computed according to the elastic solution:

\[
\sigma_{1p}^e = \frac{F_p \cdot E_1}{E_1 A_1 + 2E_2 A_2} = 147.389 \text{ MPa}
\]

\[
\sigma_{2p}^e = \frac{2F_p E_2}{E_1 A_1 + 2E_2 A_2} = 221.0833 \text{ MPa}
\]

- the residual stresses:

\[
\sigma_{1p}^p = \sigma_{1p} - \sigma_{1p}^e = 52.611 \text{ MPa}
\]

\[
\sigma_{2p}^p = \sigma_{2p} - \sigma_{2p}^e = -71.0833 \text{ MPa}
\]

- the elastic and plastic strains:

\[
\varepsilon_{1p}^e = \frac{\sigma_{1p}^e}{E_1} = 0.737 \times 10^{-3}
\]

\[
\varepsilon_{2p}^e = \frac{\sigma_{2p}^e}{E_2} = 1.4739 \times 10^{-3}
\]

\[
\varepsilon_{1p}^p = \varepsilon_{1p} - \varepsilon_{1p}^e = 0.263 \times 10^{-3}
\]

\[
\varepsilon_{2p}^p = \varepsilon_{2p} - \varepsilon_{2p}^e = 0.526 \times 10^{-3}
\]

- the displacement of the rigid part:

\[
\mu = \varepsilon_{1p}^p \cdot L_1 = \varepsilon_{2p}^p \cdot L_2 = 0.263 \text{ mm}
\]
A thin annular disk rotates together with the rigid part surrounding it as it is illustrated in the Figure.

- The angular velocity is \( \omega \).
- The inner and outer radii of the disk are \( a \) and \( b \), respectively. (when \( \omega = 0 \), the disk is fitted with no gap to the rigid part.)
- The mass density of the disk is \( \rho \).
- The material parameters of the disk are: \( E, \nu \)

- Derive the analytical expression for \( \sigma_r(r) \) and \( \sigma_t(r) \), respectively.
- Determine the pressure acting between the two parts for the following data:
  
  \[ a = 20 \text{ mm} \]
  \[ b = 50 \text{ mm} \]
  \[ \omega = 6000 \text{ rpm} \]
  \[ \rho = 5000 \text{ kg/m}^3 \]
  \[ E = 150 \text{ GPa} \]; \( \nu = 0.25 \)

- Plot the stresses, strains and displacement distributions vs. \( r \).
In[1] := Quit[]

In[2] := data = \{a \rightarrow 20, b \rightarrow 50, \omega \rightarrow 2 \pi \frac{6000}{60}, \text{YOUNG} \rightarrow 150000, \rho \rightarrow 5000 \times 10^{-12}, \nu \rightarrow 0.25\};

In[3] := C0 = - \frac{1 - \nu^2}{\text{YOUNG}} * \rho * \omega^2;

diffeq = u''[r] + \frac{1}{r} \frac{u'[r]}{r} - \frac{u[r]}{r^2} = C0 * r;

BC1 = u[b] = 0;

BC2 = u'[a] + \nu \frac{u[a]}{a} = 0;

In[4] := sol = DSolve[\{diffeq, BC1, BC2\}, u[r], r];

In[5] := u = FullSimplify[sol[[1, 1, 2]]]

Out[5] := \\{(b - r) (b + r) (-1 + \nu^2) (b^2 r^2 (-1 + \nu) - a^2 (b^2 + r^2) (1 + \nu) + a^4 (3 + \nu)) \rho \omega^2\} / \\{8 r \text{YOUNG} (-b^2 (-1 + \nu) + a^2 (1 + \nu))\}

In[6] := er = FullSimplify[D[u, r]]

et = FullSimplify[u / r];

ez = FullSimplify[-\frac{\nu}{1 - \nu} (er + et)];

In[7] := or = FullSimplify[\frac{\text{YOUNG}}{1 - \nu^2} (er + \nu * et)]

ot = FullSimplify[\frac{\text{YOUNG}}{1 - \nu^2} (et + \nu * er)]

Out[7] := \\{(a - r) (a + r) (-a^2 (3 + \nu) (b^2 (-1 + \nu) - r^2 (1 + \nu)) + b^2 (-1 + \nu) (b^2 (1 + \nu) - r^2 (3 + \nu))) \rho \omega^2\} / \\{8 r^2 (-b^2 (-1 + \nu) + a^2 (1 + \nu))\}

Out[8] := \\{(a^4 (3 + \nu) (b^2 (-1 + \nu) + r^2 (1 + \nu)) - b^2 r^2 (-1 + \nu) (b^2 (1 + \nu) - r^2 (1 + 3 \nu)) - a^2 (1 + \nu) (b^2 (-1 + \nu) + r^2 (1 + 3 \nu))) \rho \omega^2\} / \\{8 r^2 (-b^2 (-1 + \nu) + a^2 (1 + \nu))\}

In[9] := p = FullSimplify[or /. (r \rightarrow b)]
p /. data

Out[9] := \\{(a - b) (a + b) (-b^2 (-1 + \nu) + a^2 (3 + \nu)) \rho \omega^2\} / \\{(b^2 (-1 + \nu) + a^2 (1 + \nu))\}

Out[10] := -1.38538
In[15]:= \(er = \text{Expand}[\text{er} /. \text{data}]
\)
\(et = \text{Expand}[\text{et} /. \text{data}]
\(ez = \text{Expand}[\text{ez} /. \text{data}]
\(or = \text{Expand}[\text{or} /. \text{data}]
\(ot = \text{Expand}[\text{ot} /. \text{data}]
\(u = \text{Expand}[\text{u} /. \text{data}]
\)

Out[15]= \(3.38131 \times 10^{-6} - \frac{0.001185}{r^2} - 4.62638 \times 10^{-9} r^2
\)

Out[16]= \(3.38131 \times 10^{-6} + \frac{0.001185}{r^2} - 1.54213 \times 10^{-9} r^2
\)

Out[17]= \(-2.25421 \times 10^{-6} + 2.05617 \times 10^{-9} r^2
\)

Out[18]= \(0.676263 - \frac{142.2}{r^2} - 0.000801905 r^2
\)

Out[19]= \(0.676263 + \frac{142.2}{r^2} - 0.000431795 r^2
\)

Out[20]= \(\frac{0.001185}{r} + 3.38131 \times 10^{-6} r - 1.54213 \times 10^{-9} r^3
\)

In[21]:= \(ri = a /. \text{data};
\)
\(ro = b /. \text{data};
\)

In[23]:= \(\text{Plot}[u, \{r, ri, ro\}, \text{PlotStyle} \rightarrow \{\text{Orange, Thick}\}]
\)

Out[23]=

\(\text{Plot}[\{\text{er, et, ez}\}, \{r, ri, ro\},
\text{PlotStyle} \rightarrow \{\{\text{Red, Thick}\}, \{\text{Blue, Thick}\}, \{\text{Green, Thick, Dashed}\}\}]
\)

Out[24]=
\begin{align*}
\text{In[25]:=} & \quad \text{Plot}\{\{\sigma r, \sigma t\}, \{r, \text{ri}, \text{ro}\}, \text{PlotStyle} \rightarrow \{\text{Red, Thick}, \{\text{Blue, Thick}\}\}\}\text{\textbackslash n}\text{Out[25]=}\text{\textbackslash n}  \\
\text{In[26]:=} & \quad r = \frac{a}{\sqrt{\phi}} \text{/. data;}
\text{In[27]=} \begin{align*}
\text{er} & = \frac{1.85055 \times 10^{-6}}{\phi} - 2.9625 \times 10^{-6} \phi \\
\text{et} & = 3.38131 \times 10^{-6} - \frac{6.1685 \times 10^{-7}}{\phi} + 2.9625 \times 10^{-6} \phi \\
\text{ez} & = -2.25421 \times 10^{-6} + \frac{8.22467 \times 10^{-7}}{\phi} \\
\text{sr} & = 0.676263 - \frac{0.320762}{\phi} - 0.355501 \phi \\
\text{st} & = 0.676263 - \frac{0.172718}{\phi} + 0.355501 \phi \\
u & = \frac{-0.000012337}{\phi^{3/2}} + \frac{0.0000676263}{\sqrt{\phi}} + 0.0000592501 \sqrt{\phi} \\
\text{Out[32]=} & \quad r \mathbf{0} = \left(\frac{a}{b}\right)^{2} \text{/. data;}
\end{align*}
\end{align*}
In[34]:= 
Plot[u, \{\varphi, \varphi_0, 1\}, PlotStyle \rightarrow \{Orange, Thick\}, AxesOrigin \rightarrow \{\varphi_0, 0\}]

Out[34]=

In[35]:= 
Plot[\{er, et, ez\}, \{\varphi, \varphi_0, 1\}, PlotStyle \rightarrow 
\{(Red, Thick), (Blue, Thick), (Green, Thick, Dashed)\}, AxesOrigin \rightarrow \{\varphi_0, 0\}]

Out[35]=

In[36]:= 
Plot[\{sx, st\}, \{\varphi, \varphi_0, 1\}, 
PlotStyle \rightarrow \{(Red, Thick), (Blue, Thick)\}, AxesOrigin \rightarrow \{\varphi_0, 0\}]

Out[36]=