Proceedings of the ASME 2013 International Design Engineering Technical Conferences &
Computers and Information in Engineering Conference
IDETC/CIE 2013
August 4-7, 2013, Portland, Oregon, USA

DETC2013-12324

TIME DOMAIN ANALYSIS OF THE SMITH PREDICTOR IN CASE OF PARAMETER UNCERTAINTIES: A CASE STUDY

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ABSTRACT

Time domain representation of the original Smith Predictor is presented for systems with feedback delays. It is shown that if the parameters in the internal model of the predictor are not equal to the parameters of the real system, then the dimension of the closed loop system is double of the dimension of the open-loop system. Furthermore, the time-domain representation of the corresponding control law involves terms of integrals with respect to the past similarly to the Finite Spectrum Assignment control technique. The results are demonstrated for a second order system (pendulum) subjected to the Smith Predictor. It is demonstrated that stability diagrams can be constructed using the D-subdivision method and Stepan's formulas. The sensitivity of the stability properties with respect to the parameter uncertainties in the predictor's internal model is analyzed. It is shown that the original Smith Predictor can stabilize unstable plants for some extremely detuned internal model parameters. Thus the general concept that the Smith Predictor is not capable to stabilize unstable systems is technically not true.

1. INTRODUCTION

Since its publication in 1957 [23], the Smith Predictor is one of the best known control techniques to overcome the destabilization effect of feedback delays in control systems [21]. It is known that state prediction is a fundamental concept for systems with feedback delay [4], [15]. The main idea behind predictive controllers is that the feedback delay is eliminated from the control loop using a prediction of the actual state based on an internal model of the plant. It is a general view that the original Smith Predictor can be applied only to stable open-loop systems. Still, the technique has a large practical significance since it allows the use of larger feedback gains for stable plants without losing stability. Another issue with the prediction-based controllers is that they require knowledge about the plant and the feedback delay. The slightest mismatch between the internal model used for the prediction and the actual system may destabilize the closed-loop system. There are several types of modifications of the Smith predictor to overcome these difficulties, see, e.g., [1], [7], [9], [21]. A modification that involves integrals in the past over the delay period only is the so-called Finite Spectrum Assignment [11], [26], [8], which can deal also with unstable open-loop systems.

The issue related to the preservation of stability in case of parameter perturbation is serious limitation to predictive controllers. The conditions for practical stability (i.e. stability preservation for infinitesimal modeling mismatches) for the original Smith Predictor was given in [20] and for some special cases in [19], [27] and in [3]. The effect of delay mismatches was investigated in [12].

The original Smith Predictor is usually represented in frequency domain either by its block diagram or by its transfer function. In the current paper, the time-domain representation of the Smith Predictor is presented and the closed-loop system is analyzed through the example of a second-order system.

The structure of the article is as follow. In Section 2, the Smith Predictor is presented in frequency and in time-domain. The governing equations for a conventional/inverted pendulum subjected to the Smith Predictor are derived in Section 3. Section 4 presents the stability analysis of the system using the D-subdivision method and Stepan's formulas [24]. Then the effect of parameter uncertainties on the stability is analyzed for stable plants in Section 5 and for unstable plants in Section 6. It is shown that the original Smith Predictor can stabilize unstable plants for some extremely detuned internal model parameters. The results are concluded in Section 7.

2. THE SMITH PREDICTOR

The original Smith Predictor was developed in frequency domain [23], and time-domain representations are rarely available in the literature. In this section, the time-domain equations of the Smith Predictor are presented based on its block diagram.

2.1. Frequency domain representation

The block diagram of the original Smith Predictor is shown in Figure 1. As it was mentioned in the Introduction, the point of the Smith Predictor is that the feedback delay is eliminated from the control loop using a prediction of the actual state based on an internal model of the plant. Let us denote the transfer function of the plant by $P(s)$, the transfer function of the plant used by the internal model by $\tilde{P}(s)$, the actual feedback delay by τ and the delay used by the internal model by $\tilde{\tau}$. In practice, the internal model is not perfectly accurate, therefore $P(s) \neq \tilde{P}(s)$ and $\tau \neq \tilde{\tau}$. The transfer function from the input r to the output x can be given as

$$W_{rx}(s) = \frac{C(s)P(s)}{1 + C(s)\tilde{P}(s) - C(s)\tilde{P}(s)\mathrm{e}^{-\tilde{\tau}s} + C(s)P(s)\mathrm{e}^{-\tau s}}, \quad (1)$$

where the $C(s)$ is the transfer function of the controller. If the plant and the controller are factorized as

$$P(s) = \frac{B_1(s)}{A_1(s)}, \quad \tilde{P}(s) = \frac{\tilde{B}_1(s)}{\tilde{A}_1(s)}, \quad C(s) = \frac{B_2(s)}{A_2(s)}, \quad (2)$$

then the transfer function reads

$$W_{rx}(s) = \\ = \frac{\tilde{A}_1(s)B_1(s)B_2(s)}{A_1(s)A_2(s)\tilde{A}_1(s) + A_1(s)B_2(s)\tilde{B}_1(s)(1-\mathrm{e}^{-\tilde{\tau}s}) + \tilde{A}_1(s)B_1(s)B_2(s)\mathrm{e}^{-\tau s}}. \quad (3)$$

If the plant used by the internal model perfectly matches the real plant (i.e., if $\tilde{A}_1(s) = A_1(s)$ and $\tilde{B}_1(s) = B_1(s)$), then there is a pole-zero cancellation at the zeros of $A_1(s)$. In case of an unstable plant, this presents an unstable pole-zero cancellation. This example demonstrates that if $\tilde{A}_1(s) = A_1(s)$ and $\tilde{B}_1(s) = B_1(s)$ then the original Smith Predictor can only be applied to stable plants.

In the literature, the transfer function is often written from the plant input disturbance d to the output x [21], [13] as

$$W_{dx}(s) = \frac{P(s)\left(1 + C(s)\tilde{P}(s) - C(s)\tilde{P}(s)\mathrm{e}^{-\tilde{\tau}s}\right)}{1 + C(s)\tilde{P}(s) - C(s)\tilde{P}(s)\mathrm{e}^{-\tilde{\tau}s} + C(s)P(s)\mathrm{e}^{-\tau s}}. \quad (4)$$

In this case, the same factorization gives

$$W_{dx}(s) = \\ = \frac{A_1(s)A_2(s)B_1(s) + B_1(s)\tilde{B}_1(s)B_2(s)(1-\mathrm{e}^{-\tau s})}{A_1(s)A_2(s)\tilde{A}_1(s) + A_1(s)B_2(s)\tilde{B}_1(s)(1-\mathrm{e}^{-\tilde{\tau}s}) + \tilde{A}_1(s)B_1(s)B_2(s)\mathrm{e}^{-\tau s}}. \quad (5)$$

This form of the transfer function shows clearly that the poles of the open-loop system (which are the zeros of $A_1(s)$) are the poles of the closed-loop system only in case of a perfect internal model with $\tilde{A}_1(s) = A_1(s)$ and $\tilde{B}_1(s) = B_1(s)$.

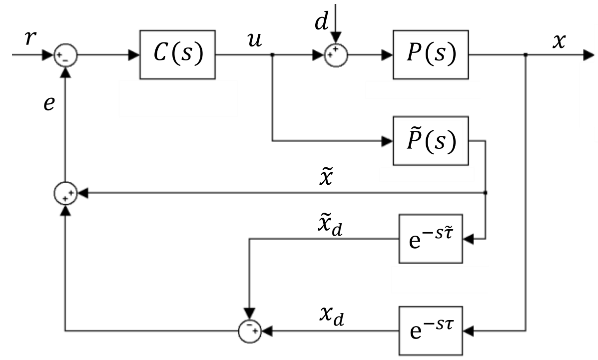

Figure 1. The block diagram of the Smith Predictor

2.2. Time domain representation

Time domain representation of the Smith Predictor can be given in the form

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \quad (6)$$

$$\dot{\tilde{\mathbf{x}}}(t) = \tilde{\mathbf{A}}\tilde{\mathbf{x}}(t) + \tilde{\mathbf{B}}\mathbf{u}(t), \quad (7)$$

$$\mathbf{u}(t) = \mathbf{D}\big(\mathbf{x}(t-\tau) - \tilde{\mathbf{x}}(t-\tilde{\tau}) + \tilde{\mathbf{x}}(t)\big), \quad (8)$$

where $\mathbf{x} \in \mathbb{R}^n$ is the vector of actual state variables, $\tilde{\mathbf{x}} \in \mathbb{R}^n$ is the vector of predicted state variables, $\mathbf{A}$ and $\tilde{\mathbf{A}}$ are the actual and the model state matrices, $\mathbf{B}$ and $\tilde{\mathbf{B}}$ are the actual and the model input matrices and matrix $\mathbf{D}$ contains the control gains. Without loss of generality it can be assumed that the reference input r is zero (if r is not zero then the variational system around the reference input r has the form of Eqs. (6)- (7)-(8)). Actually, Eq. (7) is an observer equation for the original system given by (6). Note that at the time of the development of the Smith Predictor, the state observer theory did not yet exist [8]. The corresponding control law can be given in the integral form

$$\mathbf{u}(t) = \mathbf{D}\left(\mathbf{x}(t-\tau) - \int_0^{t-\tilde{\tau}} \mathrm{e}^{\tilde{\mathbf{A}}(t-\tilde{\tau}-\theta)}\tilde{\mathbf{B}}\mathbf{u}(\theta)\mathrm{d}\theta + \int_0^{t} \mathrm{e}^{\tilde{\mathbf{A}}(t-\theta)}\tilde{\mathbf{B}}\mathbf{u}(\theta)\mathrm{d}\theta\right). \quad (9)$$

Thus, the control law involves integrals of the control input over the interval $[0, t]$. The closed-loop system can be described by a Retarded Functional Differential Equation (RFDE) of dimension $2n$ with two delays (τ and $\tilde{\tau}$) as

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{D}\big(\mathbf{x}(t-\tau) - \tilde{\mathbf{x}}(t-\tilde{\tau}) + \tilde{\mathbf{x}}(t)\big), \quad (10)$$

$$\dot{\tilde{\mathbf{x}}}(t) = \tilde{\mathbf{A}}\tilde{\mathbf{x}}(t) + \tilde{\mathbf{B}}\mathbf{D}\big(\mathbf{x}(t-\tau) - \tilde{\mathbf{x}}(t-\tilde{\tau}) + \tilde{\mathbf{x}}(t)\big). \quad (11)$$

The characteristic equation then forms as

$$\det\begin{pmatrix} s\mathbf{I} - \mathbf{A} - \mathbf{B}\mathbf{D}\mathrm{e}^{-\tau s} & -\mathbf{B}\mathbf{D}(1-\mathrm{e}^{-\tilde{\tau}s}) \\ -\tilde{\mathbf{B}}\mathbf{D}\mathrm{e}^{-\tau s} & s\mathbf{I} - \tilde{\mathbf{A}} - \tilde{\mathbf{B}}\mathbf{D}(1-\mathrm{e}^{-\tilde{\tau}s}) \end{pmatrix} = 0. \quad (12)$$

It can be seen that the Smith Predictor actually doubles the dimension of the system by employing an observer system described by (7).

3. DESCRIPTION OF THE SYSTEM UNDER ANALYSIS

Stability properties of the Smith predictor for stable and unstable plants are demonstrated for the second-order system

$$\ddot{\varphi}(t) + a\varphi(t) = -q(t-\tau), \tag{13}$$

where $\varphi(t)$ is the state variable, a is the system parameter, q is the control input (normalized control force) and τ is the feedback delay. This system describes a pendulum-cart system subjected to a delayed feedback control shown in Figure 1, where the angular displacement of the pendulum is denoted by φ, the position of the pivot point is denoted by x, and the mass of the cart is negligible compared to the mass of the pendulum. For the conventional pendulum, the system parameter can be given as $a = 6g/l$, where l is the length of the pendulum and g is the gravitational acceleration. In this case the open-loop system is stable. For the inverted pendulum, $a = -6g/l$, which gives an unstable open-loop system. If the control force acting at the pivot point of the pendulum is denoted by Q, then the control input in (13) can be given as $q(t) = 6Q(t)/(ml)$, where m is the mass of the pendulum.

The state space model of the system reads

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t-\tau), \tag{14}$$

where

$$\mathbf{x}(t) = \begin{pmatrix} \varphi(t) \\ \dot{\varphi}(t) \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 0 & 1 \\ -a & 0 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \mathbf{u}(t) = -q(t). \tag{15}$$

In case of a proportional-derivative (PD) controller, the normalized control force reads

$$q(t) = p\varphi(t) + d\dot{\varphi}(t), \tag{16}$$

where p and d are the normalized proportional and derivative control gains. For the pendulum-cart model, $p = 6K_{\mathrm{p}}/(ml)$ and $d = 6K_{\mathrm{d}}/(ml)$, where K_{p} and K_{d} are the actual control gains. The control input in this case can be given as

$$\mathbf{u}(t) = \mathbf{D}\mathbf{x}(t), \tag{17}$$

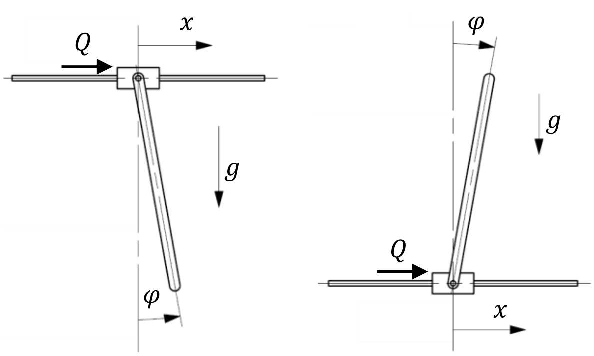

Figure 2. Mechanical model for the conventional pendulum (left) and for the inverted pendulum (right).

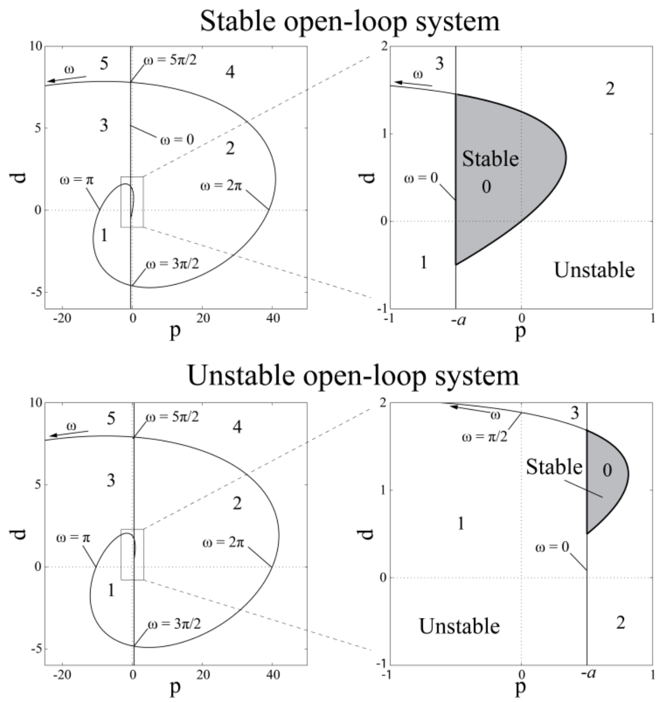

Figure 3. Stability diagram with the number of the unstable characteristic exponents for stable- (top) and unstable open-loop system (bottom) subjected to a PD controller.

with

$$\mathbf{D} = \begin{pmatrix} -p & -d \end{pmatrix}. \tag{18}$$

The stability properties for this system can be given by the D-subdivision method [24]. The stability boundaries can be derived in the form

$$\text{if } \omega = 0: \qquad p = -a, \ \ d \in \mathbb{R}, \tag{19}$$

$$\text{if } \omega \neq 0: \quad p = (\omega^2 - a)\cos(\omega\tau), \ \ d = \frac{\omega^2 - a}{\omega}\sin(\omega\tau). \tag{20}$$

The stability diagrams are shown in Figure 3 for $\tau = 1$. In the case of the unstable open-loop system, it is known that for a given system parameter a, the system cannot be stabilized if the feedback delay is larger than the critical value

$$\tau = \sqrt{-2/a} = \sqrt{l/(3g)}. \tag{21}$$

4. STABILITY ANALYSIS FOR THE STABLE PLANT

The characteristic equation can be derived either from the state space representation using Eq. (12), or from the closed-loop transfer function (4). Here, we follow the latter case. The transfer functions in (4) for the stable plant read

$$P(s) = \frac{1}{s^2 + a}, \quad \tilde{P}(s) = \frac{1}{s^2 + \tilde{a}}, \quad C(s) = p + ds. \tag{22}$$

Here, we assume that the model parameter $\tilde{a}$ is not perfectly accurate, i.e., $\tilde{a} \neq a$. Furthermore, we assume that the difference between $\tilde{a}$ and a can be extremely large. In order to see the tendencies of parameter mismatches, we will also analyze some unrealistic particular cases such as $\tilde{a} = 50a$ or $\tilde{a} = -50a$.

The characteristic function $D(s)$ is obtained as the denominator of the transfer function (4). After the substitution of (22) into Eq. (4) the characteristic function can be given as

$$D(s) = (s^2 + \tilde{a})(s^2 + a) \\ +(p + sd)\left((s^2 + a) + (s^2 + \tilde{a})\mathrm{e}^{-s\tau} - (s^2 + a)\mathrm{e}^{-s\tilde{\tau}}\right) = 0. \tag{23}$$

This system corresponds to a 4th order system with two different time delays τ and $\tilde{\tau}$.

The stability charts are constructed using the D-subdivision method and Stepan's formulas. The D-subdivision method gives the domains, where the number of unstable characteristic exponents is invariant. Substitution of $s = \mathrm{i}\omega$, where $\omega \geq 0$, and decomposition into real and imaginary parts give the functions $R(\omega) = \mathrm{Re}\big(D(\mathrm{i}\omega)\big)$ and $S(\omega) = \mathrm{Im}\big(D(\mathrm{i}\omega)\big)$. The D-curves are given by the solutions of $R(\omega) = 0$ and $S(\omega) = 0$ for the control parameters p and d. If $\omega = 0$, then

$$p = -a, \qquad d \in \mathbb{R}, \tag{24}$$

similarly to the PD controller. If $\omega > 0$, then

$$p = \Big((a - \omega^2)(\tilde{a} - \omega^2)\big((-\tilde{a} + \omega^2)\cos(\omega\tau) \\ + (a - \omega^2)(-1 + \cos(\omega\tilde{\tau}))\big)\Big) / (2a^2 + \tilde{a}^2 - 4a\omega^2 \\ - 2\tilde{a}\omega^2 + 3\omega^4 + 2(a - \omega^2)(\tilde{a} - \omega^2)\cos(\tau\omega) \\ + 2(a - \omega^2)(-\tilde{a} + \omega^2)\cos(\omega(\tau - \tilde{\tau})) - 2a^2\cos(\omega\tilde{\tau}) \\ + 4a\omega^2\cos(\omega\tilde{\tau}) - 2\omega^4\cos(\omega\tilde{\tau})),$$

$$d = \Big((a - \omega^2)(\tilde{a} - \omega^2)\big((-\tilde{a} + \omega^2)\sin(\omega\tau) \\ + (a - \omega^2)\sin(\omega\tilde{\tau})\big)\Big) / (\omega(2a^2 + \tilde{a}^2 - 4a\omega^2 - 2\tilde{a}\omega^2 \\ + 3\omega^4 + 2(a - \omega^2)(\tilde{a} - \omega^2)\cos(\omega\tau) \\ + 2(a - \omega^2)(-\tilde{a} + \omega^2)\cos\big(\omega(\tau - \tilde{\tau})\big) - 2a^2\cos(\omega\tilde{\tau}) \\ + 4a\omega^2\cos(\omega\tilde{\tau}) - 2\omega^4\cos(\omega\tilde{\tau}))). \tag{25}$$

The number of unstable characteristic exponents in the domains separated by the D-curves can be given by Stepan's formulas [24]. In the case of systems with even degree ($n = 2m$ with m being an integer), the number of unstable characteristic exponents can be given as

$$N = m + (-1)^m \cdot \sum_{k=1}^{r} (-1)^{k+1} \cdot \mathrm{sgn}\, S\big((\rho_k)\big), \tag{26}$$

where $0 < \rho_r \leq \cdots \leq \rho_1$ are the positive real zeros of $R(\omega)$. The stable domains are the ones where this number is zero. Note that the stability diagram can also be determined by numerical techniques, such as the semi-discretization method [6].

A sample stability diagram can be seen in Figure 4 for two different cases: when the system parameter a is underestimated by the internal model ($\tilde{a} < a$, top panels) and when the system parameter a is overestimated ($\tilde{a} > a$, bottom panels). It can be seen that the stability diagram for the two cases are completely

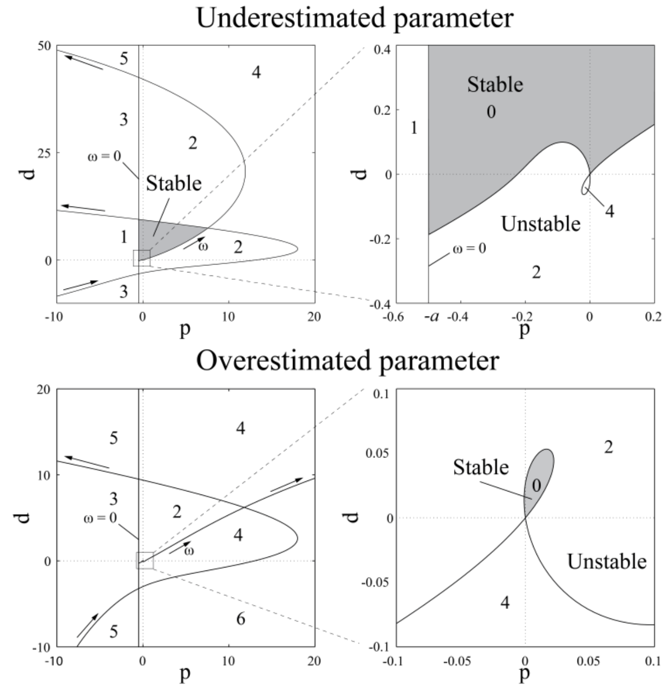

Figure 4. Stability diagram for the stable open-loop system subjected to the Smith Predictor with $a = 0.5$, $\tau = 1$, $\tilde{\tau} = 0.5\tau$ for $\tilde{a} = 0.8a$ (top) and $\tilde{a} = 1.2a$ (bottom).

different. In the case of system parameter overestimation, the stable domain is reduced to a small loop attached to the origin $(p, d) = (0,0)$.

5. ANALYSIS OF THE UNCERTAINTIES IN THE PARAMETERS FOR STABLE PLANTS

In order to explore the relation between stability (or stabilizability) and the parameter mismatches, a series of stability diagrams are presented in Figures 5 and 6 for different size of mismatches. These diagrams can be considered projections of the 4 dimensional stability chart in the parameter space $(p, d, \tilde{a}, \tilde{\tau})$. For the ideal case, when $\tilde{a} = a$ and $\tilde{\tau} = \tau$, the stability boundaries are given by $p > -a$ and $d > 0$, which corresponds to the stability condition for the delay free system.

As Figure 5 and 6 show, the parameter uncertainties of the internal model have a strong effect on the stability of the closed-loop system. Even slight overestimation of the system parameter can almost destabilize the plant. Slight underestimation of the system parameter does not really affect the stability picture, but stability boundaries change more radically for large underestimation. The effect of the uncertainty in the time delay is smaller than that of the system parameter. In general, mismatches in the delay cause a reduction of the stable domains.

Figure 7 shows the transition of the stability diagram for a wide range of variation of the model parameter $\tilde{a}$, while all the other parameters a, τ and $\tilde{\tau}$ are kept constant. Arrows show how the D-curves are changing with the difference between $\tilde{a}$

and a. The number of the unstable characteristic exponents is also presented in Figure 7.

If $\tilde{a} = a$ then the internal model is perfectly accurate. The stable domain is the quarter plane defined by $p > -a$ and $d > 0$, which corresponds to the delay free system. Note that the D-curve $p = -a$ does not change for any system parameter mismatch.

If $\tilde{a} > a$ then the stable domain suddenly reduces to a tiny loop, which size gets larger for increasing parameter mismatch. For the extreme case $\tilde{a} = 50a$, the stable domain is close to that of the delayed state feedback subjected to a PD controller (denoted by dashed line). The reason for this is that if $\tilde{a} \gg a$, then the value of the predicted state $\tilde{\mathbf{x}}(t)$ can be neglected compared to the actual state $\mathbf{x}(t)$. In this case the Smith

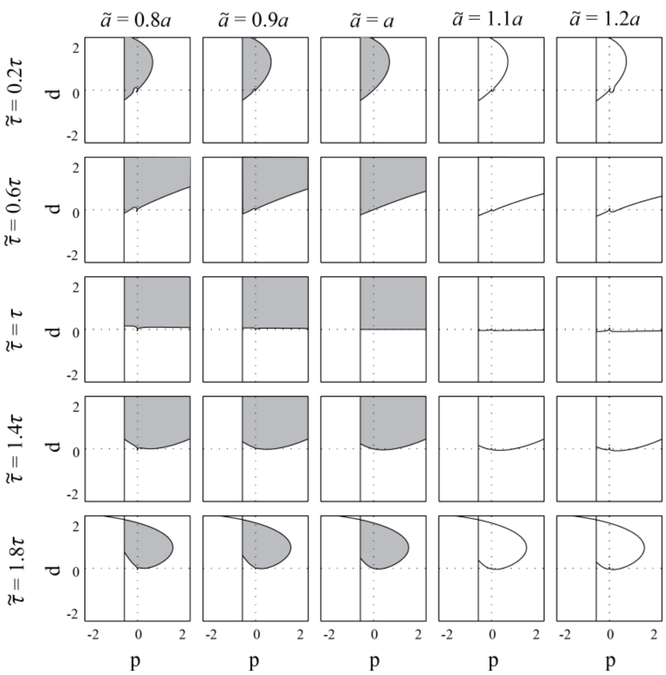

Figure 5. Stability diagrams for small parameter mismatch ($a = 0.5$, $\tau = 1$)

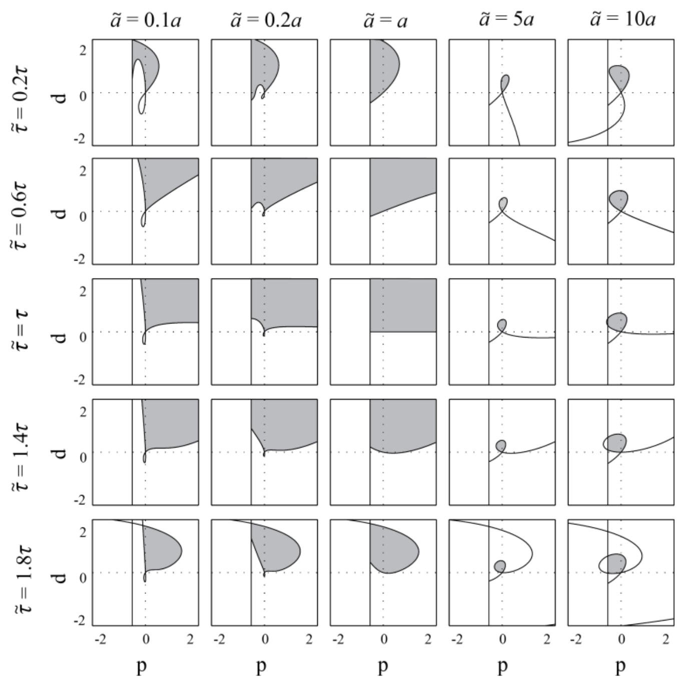

Figure 6. Stability diagrams for large parameter mismatch ($a = 0.5$, $\tau = 1$)

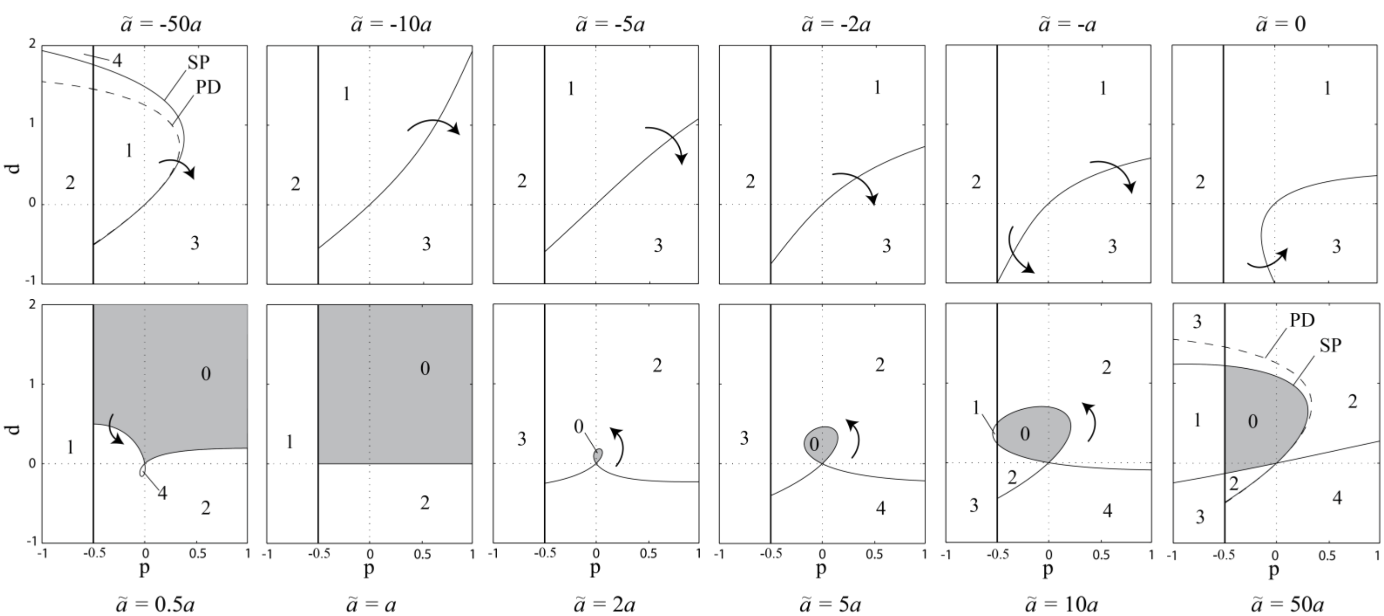

Figure 7. Stability diagrams for the stable open-loop system with parameter mismatch $a = 0.5$, $\tau = \tilde{\tau} = 1$ (PD = original PD with time delay τ; SP = Smith Predictor)

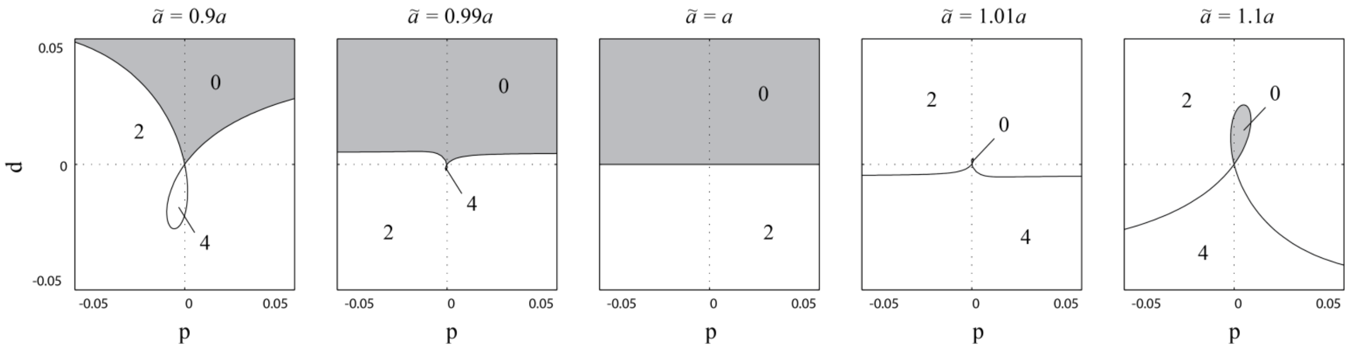

Figure 8. Stable domain in case of small mismatch in the system and model parameter ($a = 0.5,\ \tau = \tilde{\tau} = 1$)

predictor is practically equivalent to a delayed PD controller (note that if $\tilde{\mathbf{x}}(t) = 0$ then Eq. (8) gives $\mathbf{u}(t) = \mathbf{D}\mathbf{x}(t-\tau)$).

If $\tilde{a} < a$ then the stable domain gets smaller. In this case a similar small loop appears at the origin $(p, d) = (0,0)$, but this is an unstable loop with 4 unstable characteristic exponents (see panel $\tilde{a} = 0.5a$). If $\tilde{a} \leq 0$ (i.e., $\tilde{a}/a \leq 0$) then there are no stable domains. Note that for $\tilde{a} = -50a$, the D-curves approximates the D-curves of the delayed PD controller similarly to the case $\tilde{a} = 50a$, but in this case the bounded area is associated with one unstable characteristic exponent.

The transition of the stability diagrams around $\tilde{a} = a$ is presented in Figure 8 for small parameter mismatches. For $\tilde{a} = 0.9a$ there is a loop attached to the origin associated with 4 unstable characteristic exponents. As $\tilde{a} \to a$, the loop gets smaller and smaller and disappears at $\tilde{a} = a$. If $\tilde{a}$ is just larger than a, then the stability diagram turns inside out, the small loop becomes stable and the domain which was stable for $\tilde{a} \leq a$ becomes unstable. This demonstrates that the effect of parameter uncertainties on the stability is not symmetric. If the system parameter is slightly underestimated then the stability charts does not change significantly. However, if the system parameter is slightly overestimated then the stability charts changes radically. Thus the Smith Predictor is very sensitive to infinitesimal parameter uncertainties even for stable plants.

6. SMITH PREDICTOR FOR UNSTABLE PLANTS

It is a general view that the original Smith Predictor is capable to compensate the feedback delay for stable plants only. This is admittedly true if the plant used by the internal model perfectly matches the real plant, since in this case the poles of the closed-loop system contains the poles of the open-loop system as it was shown by Eq. (5). However, this argument is not valid for model parameter mismatches. In this section, it is shown that the original Smith Predictor can stabilize unstable plants for some extremely large parameter mismatches.

Figure 9 shows the transition of the stability diagrams for different model parameters in case of an unstable plant with system parameter $a = -0.5$ and feedback delays $\tau = \tilde{\tau} = 1$. This figure is the counterpart of Figure 7 for an unstable plant. The wandering of the D-curves for increasing parameter mismatch can be followed by the arrows on the figure. The number of the unstable characteristic exponents is also presented. For the two extreme cases $\tilde{a} = \pm 50a$, the D-curves for the corresponding delayed PD controller are presented by dashed lines.

As it can be seen, a stable parameter region arises if the system parameter of the internal model is tuned to negative multiples of the actual system parameter (see panels $\tilde{a} = -2a$, $\tilde{a} = -5a$, $\tilde{a} = -10a$ and $\tilde{a} = -50a$ in Figure 9). The stable domain arises if the tangent of the parametric curve at $\omega \to 0$ is vertical, i.e., if

$$\lim_{\omega \to 0} \frac{\frac{\mathrm{d}}{\mathrm{d}\omega} d}{\frac{\mathrm{d}}{\mathrm{d}\omega} p} \to \infty, \tag{27}$$

where p and d are given by Eq. (25). After a long but straightforward calculation, the denominator of Eq. (27) gives

$$3\tilde{a}\left(\tilde{a}^2(2 + a\tau^2) + 2a^3\tilde{\tau}^2 + a^2\tilde{a}\tilde{\tau}(-4\tau + \tilde{\tau})\right) = 0. \tag{28}$$

From here, the critical model parameter $\tilde{a}_{crit}$ where the stable domain arises can be expressed as

$$\tilde{a}_{crit} = a\frac{\tilde{\tau}\left(a(4\tau - \tilde{\tau}) + \sqrt{-16a + a^2(8\tau^2 - 8\tau\tilde{\tau} + \tilde{\tau}^2)}\right)}{4 + 2a\tau^2}. \tag{29}$$

In this case the unstable plant ($a < 0$) is modeled by a stable system ($\tilde{a} > 0$). If $\tilde{a} \to \infty$ then the stable domain tends to that of the conventional delayed PD controller, because, in this case, the predicted state $\tilde{\mathbf{x}}(t)$ can be neglected compared to the actual state $\mathbf{x}(t)$ and the controller behaves like a pure delayed PD controller. If $\tilde{a}$ gets closer to the actual system parameter then the stable domain disappears and the system cannot be stabilized correspondently to the literature.

The left panel of Figure 10 shows the relation between parameters $\tilde{a}_{crit}$ and a for a fixed time delay. Although the system parameter a is negative (which corresponds to an unstable open-loop system) there is a positive $\tilde{a}_{crit}$ value such that for $\tilde{a} > \tilde{a}_{crit}$ the closed loop system can be stabilized. For $a = -0.5$, Eq. (25) gives $\tilde{a}_{crit} = 0.73$. The corresponding stability chart where the stable domain is just about to born is shown in the right panel of Figure 10.

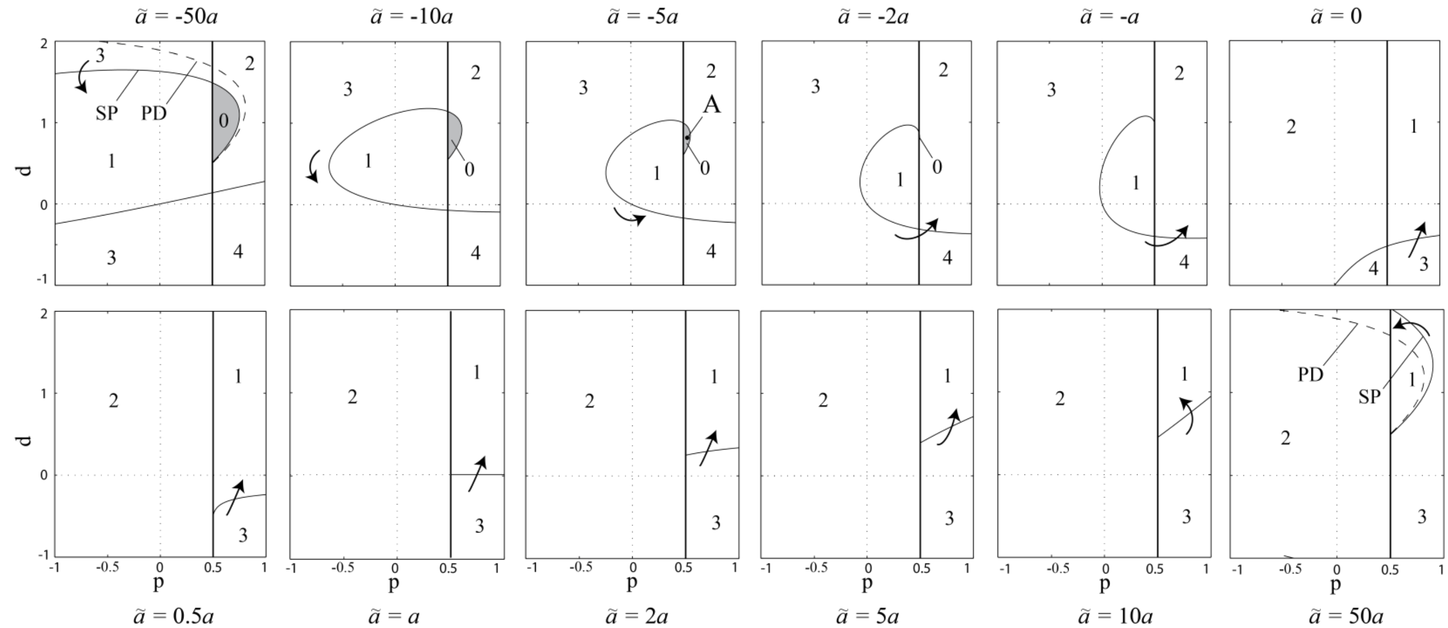

Figure 9. Stability diagram for the unstable open-loop system with parameter mismatch $a = -0.5,\ \tau = \tilde{\tau} = 1$ (PD = original PD with τ time delay, SP = Smith Predictor)

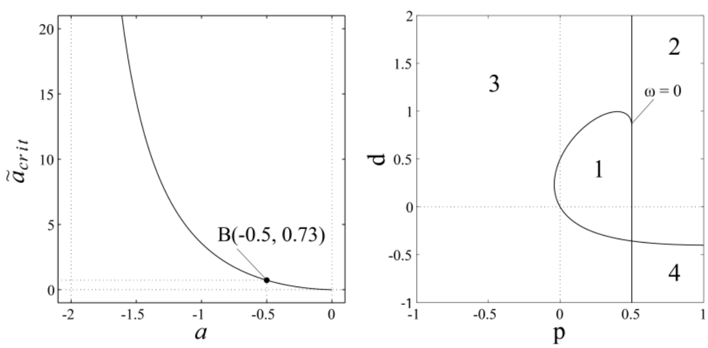

Figure 10. Critical model parameter (left) and stability chart according to point B (right) ($\tau = \tilde{\tau} = 1$)

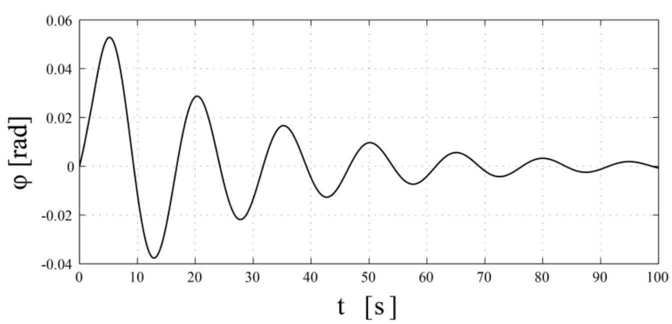

Figure 11. Time domain simulation for the parameterpoint A ($a = -0.5, \tau = \tilde{\tau} = 1, \tilde{a} = -5a, p = 0.55, d = 0.8$)

It can also be seen, that if the denominator of Eq. (29), tends to zero, then the critical model parameter tends to infinity. This limit case gives

$$4 + 2a\tau^2 = 0 \quad \rightarrow \quad a_{crit} = -2/\tau^2, \tag{30}$$

which is equivalent to the critical system parameter for the delayed PD controller: if $a < a_{crit} = -2/\tau^2$, then the system cannot be stabilized by a PD controller (see Eq. (21)). Consequently, if $a < a_{crit} = -2/\tau^2$, then the system cannot be stabilized by the Smith predictor neither. However, if $a > a_{crit}$, then there exists a model parameter $\tilde{a} > \tilde{a}_{crit}$, where $\tilde{a}_{crit}$ is given by Eq. (29), for which the system can be stabilized by the Smith predictor.

Although it can be seen that an extreme tuning of the original Smith Predictor may stabilize an unstable plant, the real mechanism behind this stabilization is in fact a delayed PD controller. Therefore the practical relevance of this stabilization is limited. Still, this example points out that the general concept that the Smith Predictor is not capable to stabilize unstable systems is technically not true.

Figure 11 shows a time domain simulation for the system's response in case of an unstable plant. The parameters are set according to point A in panel $\tilde{a} = -5a$ of Figure 9.

7. CONCLUSIONS

The original Smith Predictor was analyzed in time domain through the example of a stable- and unstable open-loop system (a conventional and an inverse pendulum). It was shown that the closed-loop system can be described by the system of RFDEs (10)-(11). The dimension of this system is double of the dimension of the open-loop system and it involves two point delays if the internal model is not perfectly accurate. It was pointed out that the corresponding control law (9) involves terms of integrals with respect to the past similarly to the Finite Spectrum Assignment control technique. There is still a significant difference: while Finite Spectrum Assignment employs an integral over the delay period $[t-\tau, t]$, the Smith Predictor uses an integral over the entire past $[0, t]$. Furthermore, Finite Spectrum Assignment results in a system of Neutral Functional Differential Equations (NFDE) [2], [16], the original Smith Predictor is equivalent to a system of RFDE. A similar study for the stabilizability of unstable systems (inverted pendulum) using the Finite Spectrum Assignment in case of parameter uncertainties is presented in [17].

The above observations were demonstrated on a second-order system with delayed feedback subjected to a Smith predictor, for which the stability analysis was performed in time domain using the D-subdivision method and Stepan's formulas [24]. It was shown that in case of a stable plant, the closed loop system is sensitive to the sense of the modeling error. Underestimation of the system parameter does not significantly affect the stability properties, while even the slightest overestimation radically changes the stability diagram. Modeling error in the feedback delay also affects the stability properties, but it is not sensitive to the direction (under- or overestimation) of the error.

The Smith predictor was also applied to an unstable plant. In this case, the system describes the inverted pendulum with feedback delay, which is a paradigm in control theory [22], but this model also has a high importance in understanding human balancing and human motor control [18], [10], [14], [25]. It is known that traditional PD or proportional-derivative-acceleration (PDA) controllers cannot stabilize an unstable equilibrium if the system parameter is less than a critical value. For PD controllers, this critical value is $a_{\text{crit,PD}} = -2/\tau^2$, while for PDA controllers, it is $a_{\text{crit,PDA}} = -4/\tau^2$ (see [22], [5]). Here, it was shown that the original Smith Predictor with extremely detuned model parameters (see Eq. 29.) may stabilize the unstable plant. In this case the predicted state can be neglected compared to the actual state and the stabilization mechanism is practically equivalent to a delayed state feedback. However, if the system cannot be stabilized by the conventional delayed state feedback (for instance, because $a < a_{\text{crit,PD}} = -2/\tau^2$), then the original Smith Predictor cannot stabilize the system either. Still, this example shows that the general concept that the Smith Predictor is not capable to stabilize unstable systems is technically not true.

ACKNOWLEDGMENTS

This work was supported by the Hungarian National Science Foundation under grant OTKA–K105433.

The work reported in the paper has been developed in the framework of the project „Talent care and cultivation in the scientific workshops of BME" project. This project is supported by the grant TÁMOP-4.2.2.B-10/1--2010-0009.

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