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Booklet of Main results

from the PhD dissertation submitted to
Géza Pattantyús-Ábrahám Doctoral School of Mechanical Engineering,
entitled:

Micro-chaos in digitally controlled mechanical systems

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Budapest, 2018

1 Introduction

In the past 50 years, with the appearance of digital electronic devices, a new challenge was introduced in the field of control engineering and computational science: dealing with the so-called *digital effects*.

The main digital effects are sampling, delay and quantization. Sampling arises from the fact, that processors operate in a periodic manner, they process one operation per cycle. Since the computation of control feedback takes time, processing delay between signal measurement and control effort output is unavoidable. Integer and floating-point numbers in computers are mostly represented in finite amount of bits, therefore they have a given precision, which leads to rounding (or with a more technical term, quantization) in calculations. Furthermore, many digital components like converters and filters can introduce one or more of these digital effects. For example, an analog-to-digital converter (ADC) can be treated as a composition of a quantizer and a sampler.

In control engineering, one of the most significant books – Widrow and Kollár [10] – provides a sophisticated way to deal with sampling and quantization in the frequency domain. The book develops the *theory of quantization* analogously to the sampling theory, and focuses on uniform quantization. The statistical analysis of quantization leads to the *Pseudo Quantization Noise* (PQN) model. Various properties and application conditions of the PQN model are discussed in details. The recovery of original signal properties from quantized signals and analysis of quantization in feedback systems are discussed. The book also covers floating-point quantization and extends the corresponding quantization theory. Additionally, various examples of quantization in feedback control systems and filters are presented.

While the quantization theory offers a great way to analyse statistical properties of complex systems including analog-to-digital conversions or floating point calculations, chaos and chaotic systems are often easier to analyse in the time domain rather than in the frequency domain.

Mathematicians – Berkolaiko, Boyarsky, Góra, Domokos – studied quantized and sampled systems as piecewise linear or nonlinear maps, often with hysteresis. Many fundamental properties and intricate details of these maps have been elaborated, but corresponding practical applications are often not included and sometimes hard to find.

G. Domokos and D. Szász thoroughly studied the effects of rounding introduced by computers in numerical simulation of chaotic maps [11]. They have proposed an approach to compute invariant measures of chaotic systems, which, in general cannot be preserved due to the fact, that simulations generate only a finite set with finite period. The effect of this secondary quantization was also found by G. Csernák in [12].

Cs. Budai, L. Kovács, J. Kövecses and G. Stépán examined the stabilization effect of Coulomb-friction in an otherwise unstable mechanical system with digital control and sampling. They present a qualitative picture of the time history of the corresponding vibrations (concave envelope curve) which can be recognized in position control applications. Limit cycles, thorough stability analysis and experimental validation is also presented.

The term *micro-chaos* (or μ-chaos) was first introduced by G. Stépán in 1994, then examined by G. Haller [13] and E. Enikov [14] (who were his PhD students at that time). They found that digital effects (sampling, rounding and delay) can lead to very small amplitude – hence the *micro* prefix – chaotic oscillations.

In works of G. Csernák, transient chaos is thoroughly examined when friction is added to the stick-slip model exhibiting micro-chaos. The escape rate and mean lifetime are estimated based on the fractal dimension of the repellor [15]. A recursive procedure for life expectancy (mean kickout number) calculation was also provided in [16].

The PhD dissertation covers the following topics:

Chapter 1 presents the past and current state of the corresponding research area which served as an entry point for the author's research.

Chapter 2 gives an overall picture about the 2D micro-chaos map corresponding to a digitally controlled 1-DoF mechanical oscillator. Various properties are shown and a simple classification of the possible cases is presented. Finally, certain results are generalized to multi-DoF systems.

Chapter 3 introduces an extension to the Simple Cell Mapping method, which allows adaptive expansion of the analysed state space region along with the opportunity of parallel execution.

Chapter 4 analyses the effect of twofold quantization: when both the input and output of the digital controller are affected by rounding.

Chapter 5 formulates the hybrid-switching micro-chaos map that describes the effect of dry friction on the motion of a 1-DoF digitally controlled oscillator. Besides the quantization-related switching events – that happen at the sampling instants – the friction-related switching events are also incorporated in the model that can be extended to the consideration of impact-like events, too.

2 Main results related to the micro-chaos map

I have examined the general behaviour of 2D micro-chaos maps corresponding to a digitally controlled 1 DoF mechanical oscillator with sampling and quantization. The thorough analysis of the case with negative stiffness and quantization at the output revealed the existence of a characteristic pattern in the state space. It was found that chaotic attractors (or repellors) and fixed points are situated alternately along the x coordinate axis.

Various methods were generalized to higher dimensional systems, e.g., the calculation of Lyapunov exponents and the periodic orbits. Special attention was devoted to the determination of the size of the so-called absorbing domain, since this property characterizes the maximal control error $\|\mathbf{y}_\infty\|$. A formula was derived for the estimation of $\|\mathbf{y}_\infty\|$ that was successfully applied to a 4D micro-chaos map.

Main Result 1: Topological pattern

An alternating pattern of chaotic attractors or transient chaotic repellors and fixed points is present in the state space of the digitally controlled 1 DoF mechanical oscillator if proportional-derivative control scheme is applied with sampling, zero-order-hold and quantized output. Depending on the parameters, border collision bifurcations of fixed points at the switching lines can change this pattern. Moreover, crisis bifurcations can turn attractors to repellors.

Related publications: [5, 1]

Main Result 2: Absorbing cuboid

An upper bound was given for the control error of the micro-chaos map, by reformulating it as a stabilized system without quantization and with additional correction terms corresponding to the neglected fractional parts.
In case of output-quantization, the farthest possible point of the invariant set is expressed in the form:

$$\mathbf{y}_\infty = \lim_{j\to\infty} \sum_{k=0}^{j} \mathbf{S}^k\, \mathbf{b}\, \chi_k = \cdots = \begin{bmatrix} \sum_{k=0}^{\infty} \sigma_{1,k}\, \chi_k \\ \vdots \\ \sum_{k=0}^{\infty} \sigma_{n,k}\, \chi_k \end{bmatrix}.$$

The choice of the infinite sequence of fractional parts χ_k that maximize the i^{th} component of $\mathbf{y}_\infty$, is $\chi^i = \{\chi_0, \chi_1, \ldots, \chi_k, \ldots\} = \{\mathrm{sign}(\sigma_{i,0}), \mathrm{sign}(\sigma_{i,1}), \ldots, \mathrm{sign}(\sigma_{i,k}), \ldots\}$, which yields a close upper bound to the control error.
This approach can be adapted to the case of input quantization, where multiple fractional part sets correspond to the quantization of state variables.
By taking the separately calculated maxima for each component of $\mathbf{y}_\infty$, an *absorbing cuboid* was expressed which can be used to provide an absorbing region in the state space. A practically usable algorithm was also developed for the determination of periodic orbits. This algorithm is based on a symbolic dynamics-based description of the phase-space and can be utilized to verify the control error estimation provided by the absorbing cuboid.

Related publications: [1, 7, 8]

It should be noted, that the upper bound corresponding to other norms can be given based on the separately calculated maximized components, as well, but these estimations will be excessive due to the fact that every component was maximized with a different choice of χ_k.

3 Main results related to Clustered Simple Cell Mapping

Main Result 3: Clustered Simple Cell Mapping

In order to adaptively discover state space objects with cell mapping approach, an extension to the Simple Cell Mapping (SCM) method was proposed. The Clustered Simple Cell Mapping method is the procedure of joining two Simple Cell Mapping solutions, thus creating a cluster of SCMs. Initially, two separate SCM solutions are present with non-overlapping and not necessarily adjacent domains of interest. The procedure consists of two stages:

- The first stage updates transient cell sequences, which lead from one SCM domain to a known object in the other domain.
- The second stage examines cell sequences, which lead to the other domain, but to an unclassified state. The idea of *cell tree mapping* is used to discover new periodic groups situated at the boundary of the two SCM domains.

After the second stage, all cells either correspond to a known state space object or lead to the *reduced sink cell*, the state space region outside the cluster. A simple way to select an adjacent state space region to be added to the cluster is also described, enabling one to carry out Clustered SCM in an adaptive and automatic manner.
The computational effort of the method is linear in terms of the total number of cells.

Related publications: [2].

The proposed method may have an impact in various fields of application, because it offers the following advantages:

- The method allows the continuation of the SCM solution after human assessment in cases when automatic state space extension is not used, but human supervision is conducted. Solving an SCM for a new region and incorporating it into the cluster is computationally cheaper than solving an SCM over the whole extended state space (see Table 3.1).
- Parallelization of the method is trivial, as separate SCM solutions can be generated independently before the joining procedure. Also, Stage 1 of the joining procedure (for each previously calculated SCM solution) can be done in parallel.
- The method is useful in real-time situations, where the region of interest is changing as a parameter is varied. Clustered Simple Cell Mapping handles screen panning well, as a separate SCM solution at the (narrow) state space region entering into the computer's screen can be calculated quickly and joined to the already existing cluster (see Table 3.2).
- The proposed approach helps to overcome memory limitations by dividing large problems into smaller ones. During the generation of a Clustered SCM solution,

if all adjacent regions of a cluster have already been examined, the SCM solution corresponding to the inner (fully surrounded) cluster can be written to disk and freed from memory. (Later, if any dynamics maps to this region, it can be reloaded from the disk.)

4 Main results related to Twofold Quantization

I have analysed the effect of twofold quantization in digital control, i.e., when both the input and the output of the controller are quantized. I have formulated the corresponding micro-chaos map, and determined how the single quantization cases can be derived from the twofold quantization formalism. I pointed out that the transition between these cases is not trivial.

I have found two new bifurcation phenomena that can occur in the case of twofold quantization only. One of them is the *deadzone crisis* when the variation of the quantization parameter changes the interaction of the input and output deadzones, leading to a crisis event, that turns the attractor to a chaotic repellor exhibiting transient chaos.

Another phenomenon is the *switching line collision*, when neighbouring switching lines touch each other, significantly changing the state-space topology.

> **Main Result 4: Quantization Ratio**
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> Twofold quantization – when both the input and the output of the controller are quantized – can be characterised by the so-called *quantization ratio* parameter, corresponding to the ratio of input and output quantizers' resolution. Twofold quantization can be reduced to a single quantization case (input-only or output-only quantization) if an appropriate quantization ratio ρ is used and its limit $\rho \to 0$ is analysed.
>
> It is impossible to analyse both kinds of twofold-to-single quantization transitions with a single quantization ratio, because parameter ρ appears in an integer-part function in the governing equations of the dynamical system. Consequently, the upper limit corresponding to one of the transitions is zero: $\lim_{\rho\to\infty} \rho \, \mathrm{Int}(x/\rho) = 0$, therefore, the control effort is turned off for finite values of x, where x is a linear combination of state variables.
>
> Hence, two different quantization ratios are necessary to be used for the two transitions, and they are inversely proportional to each other: $\rho' \sim 1/\rho$.

Related publications: [4, 9].

> **Main Result 5: Switching Line Collision and Deadzone Crisis**
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> In the case of twofold quantization – when both the input and the output of the controller are quantized – the state space of the controlled system can be divided into domains, each corresponding to a certain value of the control effort, separated by switching lines.
>
> Two new bifurcation phenomena were introduced that can occur only in the case of twofold quantization:

- *Switching Line Collision* is the event, when piecewise smooth switching lines touch each other, that is, collide in the state space. This phenomenon can induce qualitative changes in the state space of continuous *flows*, but – since the trajectories of *maps* are allowed to "jump" in the phase-space – the effects of Switching Line Collision are less pronounced in the case of maps.
The condition of the existence of first order switching line collisions is provided for maps where the neighbouring switching lines are originating from twofold quantization and proportional-derivative (PD) control scheme. Conditions for the collision between any switching line and its k-th neighbour were determined as well, along with critical quantization ratios which correspond to special cases, when all switching lines collide at all possible locations.
- *Deadzone Crisis* is an event, when a chaotic attractor turns to a transient chaotic repellor due to the change of the corresponding switching line's shape. The term *deadzone crisis* is originated from the observation, that this event is strongly related to the variation of input-deadzones in case of the 2D micro-chaos map. It was shown that this crisis event can strongly influence the maximal possible control error in the system. In some cases, even the increase of a resolution parameter r_I or r_O can lead to smaller control error.

Related publications: [4, 9].

From practical point of view, it is possible to improve the properties of the control for a given application, by carrying out an analysis of the quantization ratio and selecting a favourable range as illustrated in Section 4.2.2. Doing so, one can also find out how to improve a certain controlled system, i.e., which quantizer should be replaced by a higher-resolution one. In some cases one can even arrive to an unnatural conclusion, that using lower-resolution output quantizer or larger sampling time will actually result in lower control error. Similar results were found in [17, 18], where the quantization improved the stability properties of the controlled system.

5 Main results related to the Hybrid-switching micro-chaos map

I have introduced the *hybrid-switching micro-chaos map* that describes the behavior of a PD-controlled inverted pendulum with sampling and dry friction. Without fricton, this system can have multiple separated chaotic attractors in its state space. By the generalisation of the micro chaos map, the effect of friction was analysed.
I have shown, that chaotic attractors can turn to chaotic repellers via a crisis event, when a chaotic attractors collides with a boundaries of a sticking zone. The conditions for this crisis were formulated. There is a wide range of parameters, where chaotic attractors coexist with sticking zones originating from dry friction.
Using the proposed methodology, other switching phenomena, e.g., impact, could be taken into account, as well.

> **Main Result 6: Hybrid-switching Micro-Chaos Map**
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> The notion of *micro-chaos* can be generalized to incorporate switching events – e.g., related to dry friction or impact – which are independent from sampling. The map obtained this way is called *hybrid-switching micro-chaos map*.
> Chaotic attractors and sticking zones of friction force can coexist in the state-space of the hybrid-switching micro-chaos map, therefore it has been proven, that the *micro-chaos* phenomenon can persists in the presence of Coulomb-friction.

Related publications: [3, 6].

The practical relevance of this result is the fact, that it is possible to detect micro-chaos in measurements even in non-ideal circumstances, e.g., in the presence of unwanted Coulomb-friction. One attempt to experimentally detect micro-chaos was presented in [6], although the chaotic nature of the measured trajectories was not proven.

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