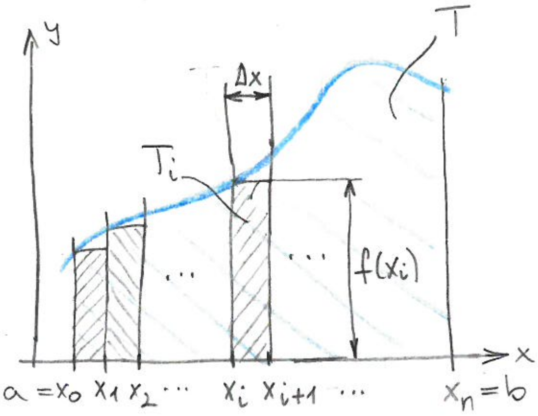
Matek A1 / 13. gyakorlat: Az integrálszámítás alkalmazásai

1. Számítsuk ki a függvény görbéje és az x tengely által határolt területet az adott intervallumon!
   a) $f(x) = x^2,\ x \in [0,1]$
   b) $f(x) = \sin(x),\ x \in [0,\pi]$
   c) $f(x) = \sin(x),\ x \in [0,2\pi]$
2. Számítsuk ki a kör területét!
3. Számítsuk ki az $f(x) = x^2 - 3x - 2$ és a $g(x) = -2x^2 + 18x - 32$ görbék által határolt terület nagyságát!
4. a) $\int\limits_0^1 \frac{1}{x} dx$ b) $\int\limits_0^1 \frac{1}{\sqrt{x}} dx$ c) $\int\limits_0^1 \ln^2(x) dx$
5. a) $\int\limits_0^\infty x \cdot e^{-x^2} dx$ b) $\int\limits_0^\infty \frac{1}{x^2} dx$ c) $\int\limits_{-\infty}^0 e^{2x} dx$
6. $\int\limits_{-\infty}^\infty \frac{1}{1+x^2} dx$
7. Számítsuk ki a függvény ívhosszát az adott intervallumon!
   a) $f(x) = \cosh(x),\ x \in [-1,1]$
   b) $f(x) = x^2,\ x \in [0,1]$
   c) $f(x) = x^{3/2},\ x \in [0,1]$
   d) $f(x) = \ln(\cos(x)),\ x \in [0,\pi/3]$
8. Számítsuk ki az $x(t) = t\cos(t)$ és $y(t) = t\sin(t)$ egyenletekkel adott görbe ívhosszát a $t \in [0,2\pi]$ intervallumon!
9. Számítsuk ki annak a testnek a térfogatát, amelyet a függvény görbéjének x tengely körüli megforgatásával kapunk!
   a) $f(x) = \sin(x),\ x \in [0,\pi]$
   b) $f(x) = e^{-x},\ x \in [0,\infty]$
10. Számítsuk ki annak a testnek a felszínét, amelyet a függvény görbéjének x tengely körüli megforgatásával kapunk!
    a) $f(x) = \sqrt{x},\ x \in [1,3]$
    b) $f(x) = \sin(x),\ x \in [0,\pi]$

13. gyakorlat

A határozott integrál alkalmazásai

1, Területszámítás



$$\int \quad f(x) \quad dx$$

$$\uparrow \quad \uparrow \quad \uparrow$$

$$T \approx \sum \underbrace{T_i}_{f(x_i)\cdot\Delta x} = \sum f(x_i)\cdot\Delta x \xrightarrow{\Delta x \to 0} \int_a^b f(x)\,dx$$

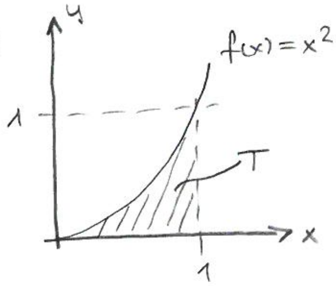
A terület tehát: Newton-Leibniz

$$T = \int_a^b f(x)\,dx = \left[F(x)\right]_a^b = F(b) - F(a)$$

, ha: · $F'(x) = f(x)$

· $f(x)$ folytonos $[a,b]$-n

a,



$$T = \int_0^1 x^2\,dx$$ Kicsit határozatlannak tűnsz

$$\Downarrow$$

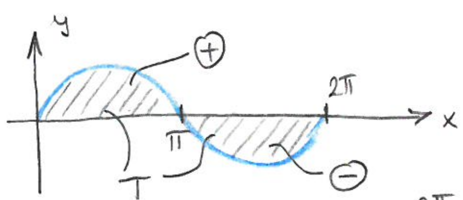
$$\int x^2\,dx = \underbrace{\frac{x^3}{3} + c}_{F(x)}$$

$$T = \int_0^1 x^2\,dx = \Big[\underbrace{\frac{x^3}{3} + c}_{F(x)}\Big]_0^1 = \underbrace{\left(\frac{1^3}{3} + c\right)}_{F(1)} - \overbrace{\left(\frac{0^3}{3} + c\right)}^{F(0)} = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}$$

c mindig kiesik a határozott integrál kiszámításakor.

b, $$\int_0^{\pi} \sin x\,dx = \left[-\cos x\right]_0^{\pi} = (\underbrace{-\cos\pi}_{1}) - (\underbrace{-\cos 0}_{-1}) = 2$$

c, $$\int_0^{2\pi} \sin x\,dx = \left[-\cos x\right]_0^{\pi} = (\underbrace{-\cos 2\pi}_{-1}) - (\underbrace{-\cos 0}_{-1}) = 0 \quad ?$$

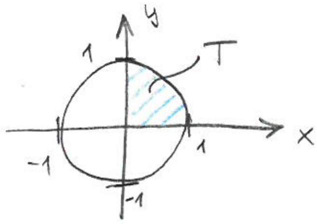


A határozott integrál „előjeles területet" ad!

Terület: $$T = \int_0^{\pi} \sin x\,dx - \int_{\pi}^{2\pi} \sin x\,dx = \ldots = 4$$

2, Kör területe:

a,



A kör egyenlete: $x^2+y^2=1$

$$\Downarrow$$

$$y=\sqrt{1-x^2}$$

$$T=\int_0^1 \sqrt{1-x^2}\,dx=\left(\frac{1}{2}\underbrace{\arcsin 1}_{\pi/2}+1\cdot\underbrace{\sqrt{1-1^2}}_{0}\right)-\left(\frac{1}{2}\underbrace{\arcsin 0}_{0}+0\cdot\sqrt{\ldots}\right)=$$

A határozatlan integrál helyettesítéssel:

$$\int\sqrt{1-x^2}\,dx=\int\underbrace{\sqrt{1-\sin^2 t}}_{\cos t}\cdot\cos t\,dt=\int\cos^2 t\,dt=\int\frac{1+\cos 2t}{2}\,dt=$$

$x=\sin t$

$\frac{dx}{dt}=\cos t \Rightarrow dx=\cos t\,dt$

$$=\frac{1}{2}t+\frac{\sin 2t}{4}+c=\frac{1}{2}t+\frac{2\sin t\cos t}{4}+c$$

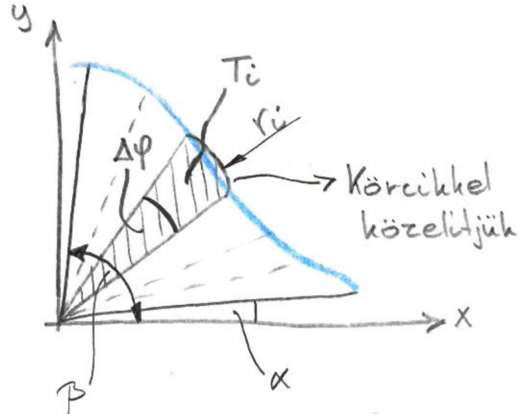
$$=\frac{1}{2}\left(t+\sin t\cdot\sqrt{1-\cos^2 t}\right)+c$$

$$=\frac{1}{2}\left(\arcsin x+x\cdot\sqrt{1-x^2}\right)+c$$

A kör területe:

$T_k=4\cdot T=4\cdot\frac{\pi}{4}=\pi$ $\quad\rightarrow$ megj.: $T_k=r^2\pi=1^2\cdot\pi=\underline{\underline{\pi}}$

b, Polárkoordinátákkal: (szektorterület)



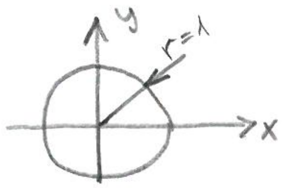
- r_i sugarú kör területe: $r_i^2\cdot\pi$
- r_i sugarú, $\Delta\varphi$ (radiánban) középponti szög
  körcikk területe: $r_i^2\cdot\pi\cdot\frac{\Delta\varphi}{\underbrace{2\pi}_{\to 360^\circ}}=\frac{r_i^2\cdot\Delta\varphi}{2}$

A szektor terület:

$$T\approx\sum\frac{r_i^2\cdot\Delta\varphi}{2}\xrightarrow{\max(\Delta\varphi)\to 0}\frac{1}{2}\int_\alpha^\beta r^2\,d\varphi$$

Azaz: $T=\frac{1}{2}\cdot\int_\alpha^\beta(r(\varphi))^2\,d\varphi$

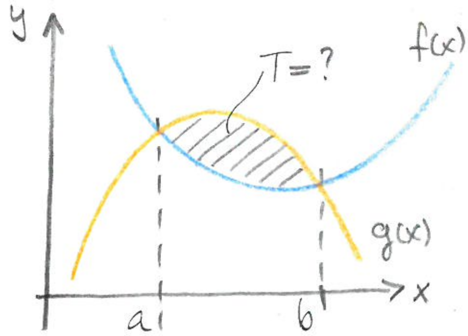
Kör esetén: $r(\varphi)=1$ $\quad\Rightarrow T=\frac{1}{2}\int_0^{2\pi}(r(\varphi))^2\,d\varphi=\frac{1}{2}\int_0^{2\pi}1\,d\varphi=\frac{1}{2}\left[\varphi\right]_0^{2\pi}=\frac{1}{2}(2\pi-0)=\underline{\pi}$

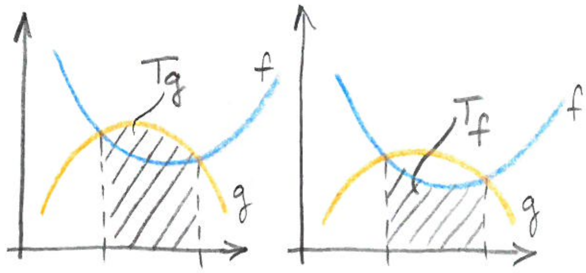


, Görbék által határolt terület

$f(x) = x^2 - 3x - 2$

$g(x) = -2x^2 + 18x - 32$





I. Metszéspontok meghatározása: a és b

$f(x) = g(x)$

$x^2 - 3x - 2 = -2x^2 + 18x - 32$

$3x^2 - 21x + 30 = 0$

$x^2 - 7x + 10 = 0$

$(x-5)(x-2) = 0$

$x_1 = 5 = b \qquad x_2 = 2 = a$

II. A terület:

- $T = T_g - T_f = \int_a^b g(x)\,dx - \int_a^b f(x)\,dx$

  $= \int_a^b g(x) - f(x)\,dx$

- Ha nem tudjuk melyik görbe „van felül":

  $T = \left| \int_a^b f(x) - g(x)\,dx \right|$

$$T = \left| \int_2^5 (x^2 - 3x - 2) - (-2x^2 + 18x - 32)\,dx \right| =$$

$$= \left| 3\int_2^5 x^2 - 7x + 10\,dx \right| = \left| 3 \cdot \left[\frac{x^3}{3} - \frac{7x^2}{2} + 10x \right]_2^5 \right| = \left| 3 \cdot \underbrace{\left(\left(\frac{5^3}{3} - \frac{7 \cdot 5^2}{2} + 10 \cdot 5 \right) - \left(\frac{2^3}{3} - \frac{7 \cdot 2^2}{2} + 10 \cdot 2 \right) \right)}_{-\frac{9}{2}} \right|$$

$$= \ldots = \left| 3 \cdot \frac{-9}{2} \right| = \frac{27}{2}$$

4. Improprius integrálok

1. típus: A függvény nem korlátos az adott intervallumon

a, $\int_0^1 \frac{1}{x}\,dx = [\ln x]_0^1 = \ln 1 - \overbrace{\ln 0}^{-\infty ?} = ?$

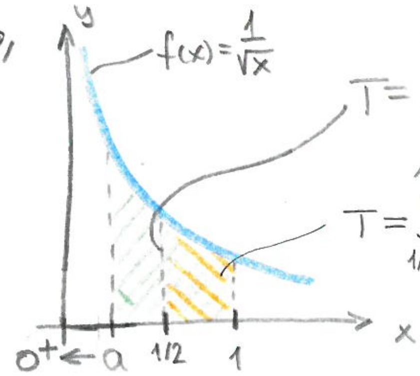
A Newton-Leibniz formula nem alkalmazható, mert $f(x) = \frac{1}{x}$ nem korlátos $[0, 1]$ <u>zárt</u> intervallumon.

↳ $x = 0$-ban nem is értelmezett

$$\int_0^1 \frac{1}{x}\,dx = \lim_{a \to 0^+} \int_a^1 \frac{1}{x}\,dx = \lim_{a \to 0^+} [\ln x]_a^1 = \lim_{a \to 0^+} (\underbrace{\ln 1}_{0} - \underbrace{\ln a}_{\to -\infty}) = „1 - (-\infty)" = +\infty$$

Impr. int. | Improprius integrálok:

4.b,



$T=\int_a^1 \frac{1}{\sqrt{x}}\,dx$

$T=\int_{1/2}^1 \frac{1}{\sqrt{x}}\,dx$

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0^+} \int_a^1 \frac{1}{\sqrt{x}}\,dx =$$

$$= \lim_{a\to 0^+} \left[2\sqrt{x}\right]_a^1 = \lim_{a\to 0^+} \left(2\cdot\sqrt{1} - 2\cdot\sqrt{a}\right) =$$

$$= 2 - 0 = \underline{\underline{2}}$$

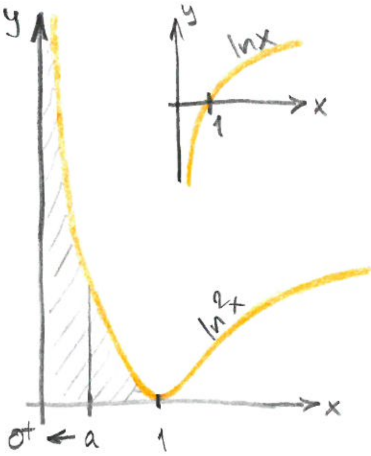
c, gond: „ln0 = −∞"

$$\int_0^1 \ln^2 x\,dx = \lim_{a\to 0^+} \int_a^1 \ln^2 x\,dx = \lim_{a\to 0^+} \left[x\cdot(\ln^2 x - 2\ln x + 2)\right]_a^1 \overset{*}{=}$$

$$\int \ln^2 x\,dx = \int \underbrace{1}_{f'}\cdot \underbrace{\ln^2 x}_{g}\,dx = \underbrace{x}_{f}\cdot\underbrace{\ln^2 x}_{g} - \int \underbrace{x}_{f}\cdot \underbrace{2\ln x\cdot\frac{1}{x}}_{g'}\,dx = x\cdot\ln^2 x - 2\cdot\int \underbrace{1}_{f'}\cdot\underbrace{\ln x}_{g}\,dx =$$

$$= x\cdot\ln^2 x - 2\cdot\left(\underbrace{x}_{f}\cdot\underbrace{\ln x}_{g} - \int \underbrace{x}_{f}\cdot\underbrace{\frac{1}{x}}_{g'}\,dx\right) = x\cdot\ln^2 x - 2x\cdot\ln x + 2x + c =$$

$$= x\cdot\left(\ln^2 x - 2\ln x + 2\right) + c$$



$$\overset{*}{=} \lim_{a\to 0^+}\left(1\cdot(\underbrace{\ln^2 1}_{0} - 2\cdot\underbrace{\ln 1}_{0} + 2) - a\cdot(\underbrace{\ln^2 a - 2\cdot\ln a}_{\ln a(\ln a - 2)} + 2)\right) =$$

$$= 2 - \lim_{a\to 0^+}\left(\frac{\overset{-\infty}{\ln a}\,(\overset{-\infty}{\ln a - 2})}{\underset{+\infty}{\frac{1}{a}}} - \underset{0}{2a}\right) \overset{LH}{=}$$

$$= 2 - \lim_{a\to 0^+} \frac{(\ln a\cdot(\ln a - 2))'}{\left(\frac{1}{a}\right)'} = 2 - \lim_{a\to 0^+} \frac{\frac{1}{a}\cdot(\ln a - 2) + \frac{1}{a}\cdot\ln a}{-\frac{1}{a^2}} =$$

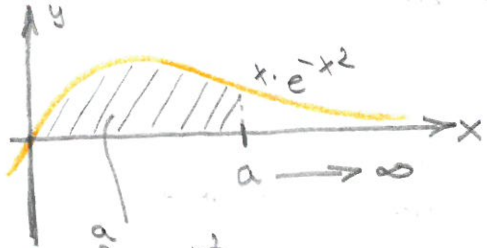
$$= 2 + \lim_{a\to 0^+} \frac{\overset{-\infty}{2\ln a - 2}}{\underset{+\infty}{\frac{1}{a}}} \overset{LH}{=} 2 + \lim_{a\to 0^+} \frac{2\cdot\frac{1}{a}}{-\frac{1}{a^2}} = 2 + \lim_{a\to 0^+} \frac{2}{a}\cdot(-a^2) =$$

$$= 2 + \lim_{a\to 0^+} -2a = \underline{\underline{2}}$$

Impropriusz integrálok: 2. típus: Az integrálási tartomány nem véges

a)
$$\int_0^{\infty} x\cdot e^{-x^2}\,dx = \lim_{a\to\infty}\left[-\frac{1}{2}\cdot e^{-x^2}\right]_0^a = \lim_{a\to\infty}\Big(-\frac{1}{2}\underbrace{e^{-a^2}}_{\downarrow}+\frac{1}{2}\underbrace{e^{-0^2}}_{1}\Big) =$$

„$e^{-\infty}=0$”



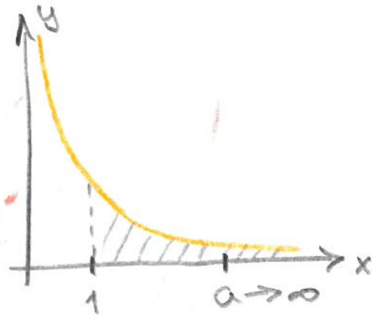
$$\int_0^a x\cdot e^{-x^2}\,dx$$

$$= 0+\frac{1}{2}=\frac{1}{2}$$

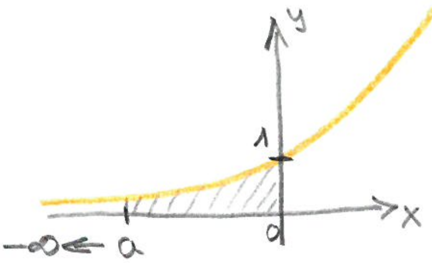
A határozatlan integrál:

$$\int x\cdot e^{-x^2}\,dx = -\frac{1}{2}\int \underbrace{-2x}_{g'(x)}\cdot \underbrace{e^{-x^2}}_{f'(g(x))}\,dx = -\frac{1}{2}\cdot e^{-x^2}+c$$

b)
$$\int_1^{\infty}\frac{1}{x^2}\,dx = \lim_{a\to\infty}\left[\frac{-1}{x}\right]_1^a = \lim_{a\to\infty}\Big(\underbrace{\frac{-1}{a}}_{\to 0}-\frac{-1}{1}\Big) = 1$$



c)
$$\int_{-\infty}^{0} e^{2x}\,dx = \lim_{a\to-\infty}\left[\frac{e^{2x}}{2}\right]_a^0 = \lim_{a\to-\infty}\Big(\frac{e^{2\cdot 0}}{2}-\underbrace{\frac{e^{2a}}{2}}_{\to 0}\Big) = \frac{1}{2}-0=\frac{1}{2}$$

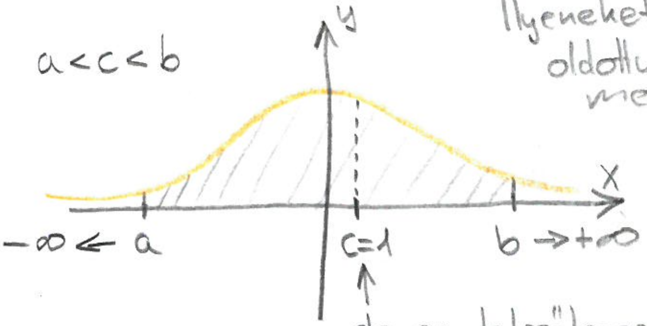


6., Improprius integrál

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{1} \frac{1}{1+x^2}\,dx + \int_{1}^{\infty} \frac{1}{1+x^2}\,dx = \lim_{a\to-\infty}\left[F(x)\right]_a^1 + \lim_{b\to\infty}\left[F(x)\right]_1^b =$$

Ilyeneket már oldottunk meg

$a<c<b$



de ez tetszőleges!

$$= F(1) - \lim_{a\to\infty} F(a) + \lim_{b\to\infty} F(b) - \cancel{F(1)}$$

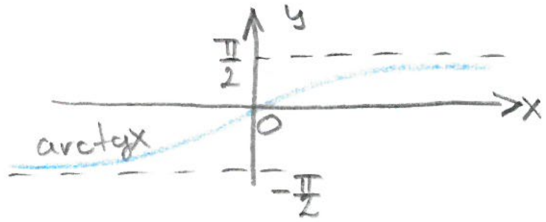
$$= \lim_{b\to\infty} F(b) - \lim_{a\to-\infty} F(a) =$$

$$= \lim_{b\to\infty} \underbrace{\operatorname{arctg} b}_{\nearrow \frac{\pi}{2}} - \lim_{a\to-\infty} \underbrace{\operatorname{arctg} a}_{\nearrow -\frac{\pi}{2}} =$$

$$= \frac{\pi}{2} - \frac{-\pi}{2} = \underline{\underline{\pi}}$$

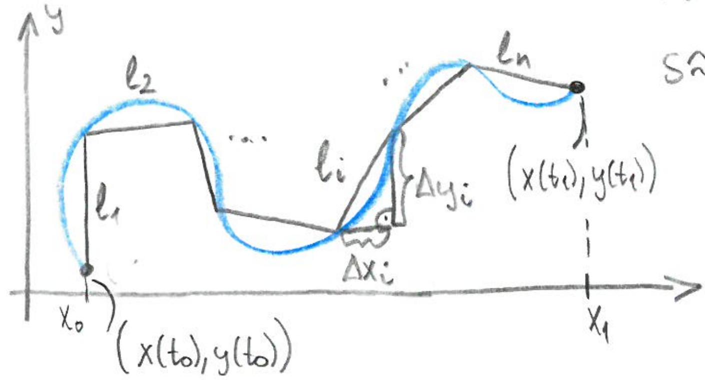
A határozatlan integrál:

$$\int \frac{1}{1+x^2}\,dx = \underbrace{\operatorname{arctg} x}_{F(x)} + c$$



7., Ívhossz számítás:

Egy töröttvonallal közelítjük a görbét



$$s \approx \sum_{i=1}^{n} l_i = \sum_{i=1}^{n} \sqrt{\Delta x_i^2 + \Delta y_i^2}$$

$$l_i = \sqrt{\Delta x_i^2 + \Delta y_i^2}$$

$\max(\Delta x_i) \to 0$, $\max(\Delta y_i) \to 0$, $n \to \infty$

$$\text{"}\int_{t_0}^{t_1} \sqrt{dx^2+dy^2}\text{"} = s$$

- $s = \text{"}\int_{t_0}^{t_1} \sqrt{dx^2+dy^2}\text{"} \quad / \cdot \text{"}\frac{dt}{dt}\text{"}$

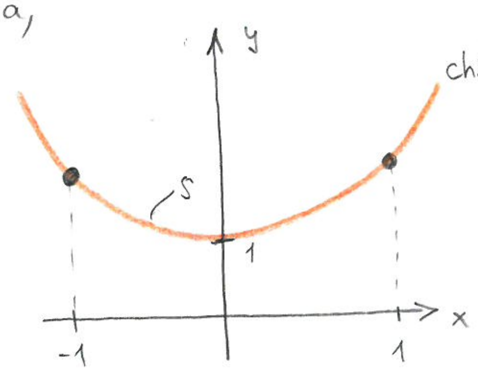
$$s = \int_{t_0}^{t_1} \sqrt{dx^2+dy^2} \cdot \frac{1}{dt} \cdot dt = \int_{t_0}^{t_1} \sqrt{\Big(\underbrace{\frac{dx}{dt}}_{\dot{x}}\Big)^2 + \Big(\underbrace{\frac{dy}{dt}}_{\dot{y}}\Big)^2}\,dt = \int_{t_0}^{t_1} \sqrt{\dot{x}(t) + \dot{y}(t)}\,dt$$

- $s = \int_{x_0}^{x_1} \sqrt{dx^2+dy^2} \quad / \cdot \text{"}\frac{dx}{dx}\text{"}$

$$s = \int_{x_0}^{x_1} \sqrt{dx^2+dy^2} \cdot \frac{1}{dx}\,dx = \int_{x_0}^{x_1} \sqrt{\Big(\underbrace{\frac{dx}{dx}}_{1}\Big)^2 + \Big(\underbrace{\frac{dy}{dx}}_{f'(x)}\Big)^2}\,dx = \int_{x_0}^{x_1} \sqrt{1+(f'(x))^2}\,dx$$

$f'(x)$, ha $y = f(x)$

a)



$$ch\,x=\frac{e^x+e^{-x}}{2}$$

$$s=\int_{-1}^{1}\sqrt{1+[(ch\,x)']^2}\,dx=\int_{-1}^{1}\sqrt{1+sh^2x}\,dx=$$

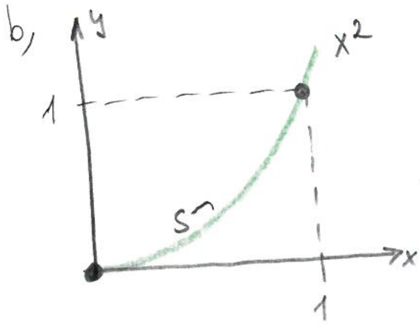
$$ch^2x-sh^2x=1 \Rightarrow 1+sh^2x=ch^2x$$

$$=\int_{-1}^{1}ch\,x\,dx=[sh\,x]_{-1}^{1}=sh1-sh(-1)$$

$$=2\cdot sh1=2\cdot\frac{e^1-e^{-1}}{2}=e-\frac{1}{e}\approx 2{,}35$$

shx páratlan:
$sh(-1)=-sh(1)$

b)



$$s=\int_0^1\sqrt{1+[(x^2)']^2}\,dx=\int_0^1\sqrt{1+(2x)^2}\,dx\overset{*}{=}$$

helyettesítés: $2x=sh\,t \Rightarrow t=arsh(2x)$

$$\frac{dx}{dt}=\frac{1}{2}ch\,t \Rightarrow dx=\frac{1}{2}ch\,t\,dt$$

$$\int\sqrt{1+(2x)^2}\,dx=\int\sqrt{1+sh^2t}\cdot\frac{1}{2}ch\,t\,dt=\frac{1}{2}\int ch^2t\,dt=$$

$$=\frac{1}{2}\cdot\int\frac{1+ch2t}{2}\,dt=\frac{1}{4}\cdot\int 1+ch2t\,dt=\frac{1}{4}\left(t+\frac{sh2t}{2}\right)=\frac{1}{4}\left(t+\frac{sh2t}{2}\right)\overset{**}{=}$$

$$sh2t=2\,sh\,x\cdot ch\,x=2\,sh\,x\cdot\underbrace{\sqrt{1+sh^2x}}_{ch\,x}$$

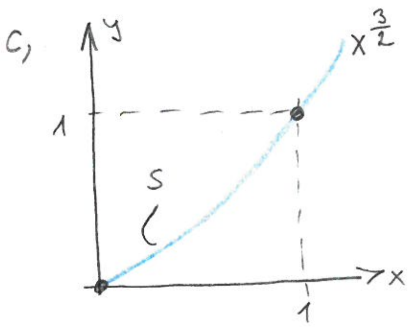
$$ch^2x+sh^2x=ch2x$$
$$ch^2x-sh^2x=1$$
$$\oplus\quad 2ch^2x=1+ch2x \Rightarrow ch^2x=\frac{1+ch2x}{2}$$

$$\overset{**}{=}\frac{1}{4}\left(t+sh\,t\cdot\sqrt{1+sh^2t}\right)=\frac{1}{4}\cdot\left(arsh(2x)+2x\cdot\sqrt{1+(2x)^2}\right)+C$$

$$s\overset{*}{=}\int_0^1\sqrt{1+(2x)^2}\,dx=\left[\frac{1}{4}\left(arsh(2x)+2x\cdot\sqrt{1+(2x)^2}\right)\right]_0^1=$$

$$=\frac{1}{4}\left(\left(arsh2+2\cdot\sqrt{1+4}\right)-\left(\underbrace{arsh0}_{0}+\underbrace{0\cdot\sqrt{\dots}}_{0}\right)\right)=\frac{arsh2}{4}+\frac{\sqrt{5}}{2}\approx 1{,}48$$

c)



$$s=\int_0^1\sqrt{1+[(x^{3/2})']^2}\,dx=\int_0^1\sqrt{1+\left(\frac{3}{2}\sqrt{x}\right)^2}\,dx=$$

$$=\int_0^1\sqrt{1+\frac{9x}{4}}\,dx=\left[\frac{8}{27}\cdot\left(1+\frac{9}{4}x\right)^{3/2}\right]_0^1=\frac{8}{27}\left(\left(1+\frac{9}{4}\right)^{3/2}-1^{3/2}\right)=$$

$$=\frac{8}{27}\left(\left(\frac{13}{4}\right)^{3/2}-1\right)=$$

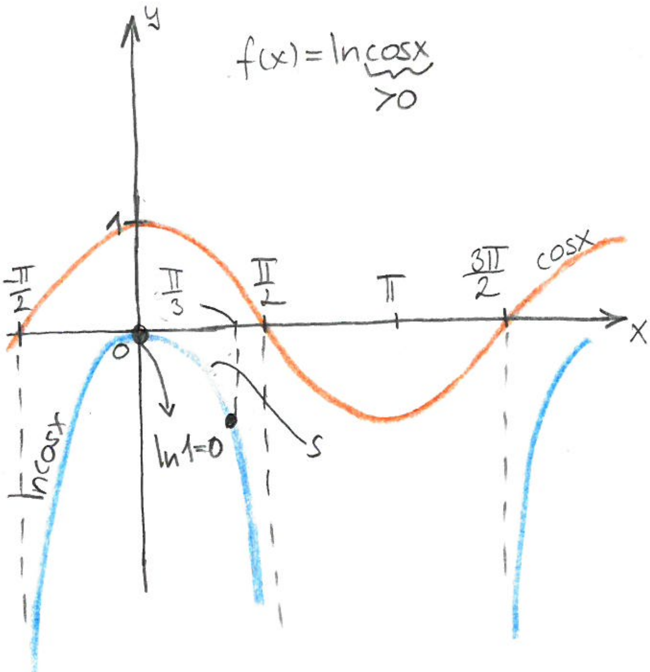
$$=\frac{8}{27}\left(\sqrt{\left(\frac{13}{4}\right)^3}-1\right)\approx 1{,}44$$

$$\int\underbrace{\sqrt{1+\frac{9x}{4}}}_{t}\,dx=\int\frac{8t^2}{9}\,dt=\frac{8t^3}{27}+C=\frac{8\cdot\left(1+\frac{9}{4}x\right)^{3/2}}{27}+C$$

$$t=\sqrt{1+\frac{9x}{4}} \Rightarrow t^2=1+\frac{9x}{4}$$

$$\frac{4}{9}(t^2-1)=x \Rightarrow \frac{dx}{dt}=\frac{4}{9}\cdot 2t=\frac{8t}{9} \Rightarrow dx=\frac{8t}{9}\,dt$$

7d.,



„ln 0 = −∞"

$$s=\int_0^{\pi/3}\sqrt{1+\left[(\ln\cos x)'\right]^2}\,dx=$$

$$=\int_0^{\pi/3}\sqrt{1+\left(\frac{-\sin x}{\cos x}\right)^2}\,dx=\int_0^{\pi/3}\sqrt{1+\frac{\sin^2 x}{\cos^2 x}}\,dx=$$

$$=\int_0^{\pi/3}\sqrt{\frac{\cos^2 x+\sin^2 x}{\cos^2 x}}\,dx=\int_0^{\pi/3}\frac{1}{\cos x}\,dx\overset{*}{=}$$

- - - - - -

$$\int\frac{1}{\cos x}\,dx=\int\frac{1}{\frac{1-t^2}{1+t^2}}\cdot\frac{2}{1+t^2}\,dt\overset{**}{=}$$

$$t=\operatorname{tg}\frac{x}{2}\Rightarrow x=2\operatorname{arctg}t$$

$$\frac{dx}{dt}=\frac{2}{1+t^2}\Rightarrow dx=\frac{2}{1+t^2}\,dt$$

parc. tört.

$$\overset{**}{=}\int\frac{2}{1-t^2}\,dt\overset{\downarrow}{=}\int\frac{1}{1-t}+\frac{1}{1+t}\,dt\overset{**}{=}$$

$$\frac{2}{1-t^2}=\frac{0t+2\cdot 1}{(1-t)(1+t)}=\frac{A}{1-t}+\frac{B}{1+t}=\frac{A(1+t)+B(1-t)}{1-t^2}=\frac{t(A-B)+1\cdot(A+B)}{1-t^2}$$

$$\left.\begin{aligned}A-B&=0\\A+B&=2\end{aligned}\right\}\quad A=B=1$$

$$\overset{**}{=}-\ln|1-t|+\ln|1+t|+c=\ln\left|1+\operatorname{tg}\frac{x}{2}\right|-\ln\left|1-\operatorname{tg}\frac{x}{2}\right|+c$$

- - - -

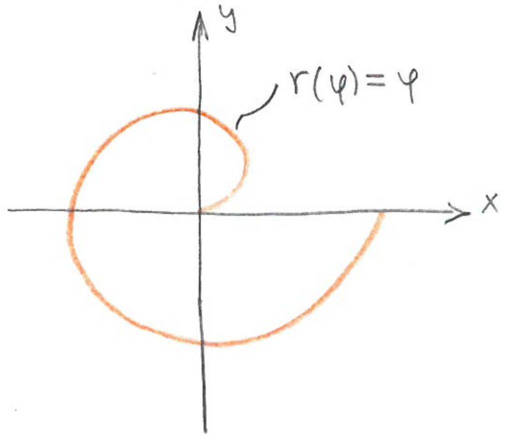
Az eredeti integrál:

$$s\overset{*}{=}\left[\ln\left|1+\operatorname{tg}\frac{x}{2}\right|-\ln\left|1-\operatorname{tg}\frac{x}{2}\right|\right]_0^{\pi/3}\overset{\uparrow}{=}\left(\ln\left|1+\frac{\sqrt3}{3}\right|-\ln\left|1-\frac{\sqrt3}{3}\right|\right)-\left(\underbrace{\ln 1}_{0}-\underbrace{\ln 1}_{0}\right)=$$

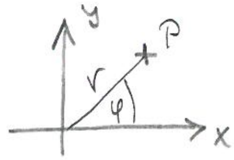
$$\operatorname{tg}\frac{\pi/3}{2}=\operatorname{tg}\frac{\pi}{6}=\frac{\sqrt3}{3}$$

$$=\ln\left(1+\frac{\sqrt3}{3}\right)-\ln\left(1-\frac{\sqrt3}{3}\right)\approx 1{,}32$$

1, Spirál ívhossza



Polár koordináták:



$x_P = r\cdot\cos\varphi$

$y_P = r\cdot\sin\varphi$

A görbe paraméteres egyenete:

$x(\varphi)=\varphi\cdot\cos\varphi$ vagy: $x(t)=t\cdot\cos t$

$y(\varphi)=\varphi\cdot\sin\varphi$ $\quad$ $y(t)=t\cdot\sin t$

$\varphi\in[0,2\pi]$ $\quad$ $t\in[0,2\pi]$

Az ívhossz:

$$S=\int_0^{2\pi}\overbrace{\sqrt{(\dot{x}(t))^2+(\dot{y}(t))^2}}^{\text{sebességvektor hossza}}\,dt=\int_0^{2\pi}\sqrt{\underbrace{\sin^2t+\cos^2t}_{1}+t^2\cdot\underbrace{(\sin^2t+\cos^2t)}_{1}}\,dt\overset{*}{=}$$

Megjegyzés: $\underline{v}=(\dot{x}(t),\dot{y}(t))$

sebességvektor $\quad$ x irányú $\quad$ y irányú sebességek

Most:

$\dot{x}(t)=1\cdot\cos t-t\cdot\sin t$

$\dot{y}(t)=1\cdot\sin t+t\cdot\cos t$

$\Downarrow$

$(\dot{x}(t))^2=\cos^2t-2t\cos t\sin t+t^2\sin^2t$

$(\dot{y}(t))^2=\sin^2t+2t\cos t\sin t+t^2\cos^2t$

$$\overset{*}{=}\int_0^{2\pi}\sqrt{1+t^2}\,dt=\left[\frac{\operatorname{arsh}t}{2}+\frac{t\cdot\sqrt{1+t^2}}{2}\right]_0^{2\pi}\overset{**}{=}$$

Helyettesítés: $\underline{1+t^2}$ hasonló $\operatorname{ch}^2x=\underline{1+\operatorname{sh}^2x}$ -hez

$t=\operatorname{sh}x \Rightarrow x=\operatorname{arsh}t$

$\frac{dt}{dx}=\operatorname{ch}x \Rightarrow dt=\operatorname{ch}x\,dx$

$\operatorname{ch}^2x-\operatorname{sh}^2x=1$

$\operatorname{ch}^2x+\operatorname{sh}^2x=\operatorname{ch}2x$ $\quad\oplus$

$2\operatorname{ch}^2x=1+\operatorname{ch}2x$

$\operatorname{ch}^2x=\frac{1+\operatorname{ch}2x}{2}$

A határozott integrál:

$$\int\sqrt{1+t^2}\,dt=\int\sqrt{1+\operatorname{sh}^2x}\cdot\operatorname{ch}x\,dx=\int\operatorname{ch}^2x\,dx=\int\frac{1+\operatorname{ch}2x}{2}\,dx=$$

$$=\frac{x}{2}+\frac{\operatorname{sh}2x}{4}+c=\frac{\operatorname{arsh}t}{2}+\frac{t\cdot\sqrt{1+t^2}}{2}+c$$

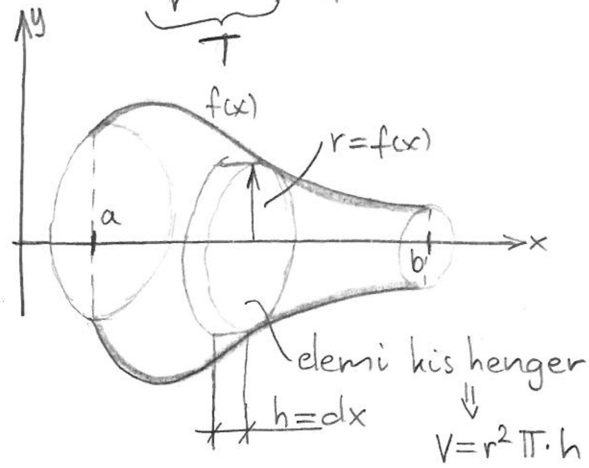
$$\operatorname{sh}2x=2\operatorname{sh}x\operatorname{ch}x=2\operatorname{sh}x\cdot\sqrt{1+\operatorname{sh}^2x}=2\cdot t\sqrt{1+t^2}$$

cél: csak $\operatorname{sh}x=t$ legyen benne

$$\overset{**}{=}\left(\frac{\operatorname{arsh}2\pi}{2}+\frac{2\pi\cdot\sqrt{1+4\pi^2}}{2}\right)-\left(\underbrace{\frac{\operatorname{arsh}0}{2}}_{0}+\underbrace{\frac{0\cdot\sqrt{\ldots}}{2}}_{0}\right)=\frac{\operatorname{arsh}(2\pi)}{2}+\pi\cdot\sqrt{1+4\pi^2}\approx 21{,}26$$

9, Forgástest térfogata:

$$V=\int_a^b \underbrace{\underbrace{f^2(x)}_{r^2}\cdot\pi}_{T}\,\underbrace{dx}_{h} \approx \sum \underbrace{r_i^2\cdot\pi\cdot\Delta x}_{V_i}$$ ↗ kis hengerek térfogatának összege



a, $f(x)=\sin x \quad , x\in[0,\pi]$

$$V=\int_0^\pi \sin^2 x\cdot\pi\,dx=\pi\cdot\int_0^\pi \sin^2 x\,dx=\pi\cdot\left[\frac{x}{2}+\frac{\sin 2x}{4}\right]_0^\pi=\pi\cdot\left(\frac{\pi}{2}+\underbrace{\frac{\sin 2\pi}{4}}_{0}-0\right)=\frac{\pi^2}{2}\approx 4{,}93$$

↓

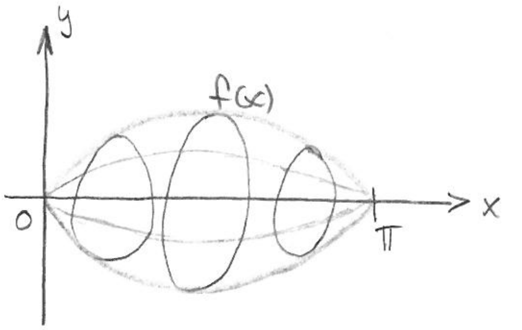
A határozatlan integrál:

$$\int \sin^2 x\,dx=\int\frac{1-\cos 2x}{2}\,dx=\frac{x}{2}+\frac{\sin 2x}{4}+c$$

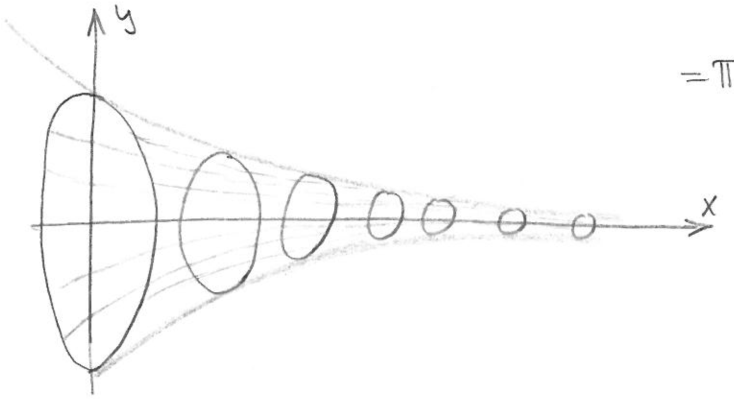
↑

$\sin^2 x+\cos^2 x=1$

$\cos^2 x-\sin^2 x=\cos 2x$ $\Big\downarrow\ominus$ $2\sin^2 x=1-\cos 2x$



b, $f(x)=e^{-x} \quad , x\in[0,\infty]$



$$V=\pi\cdot\int_0^\infty (e^{-x})^2\,dx=\pi\cdot\int_0^\infty e^{-2x}\,dx=$$

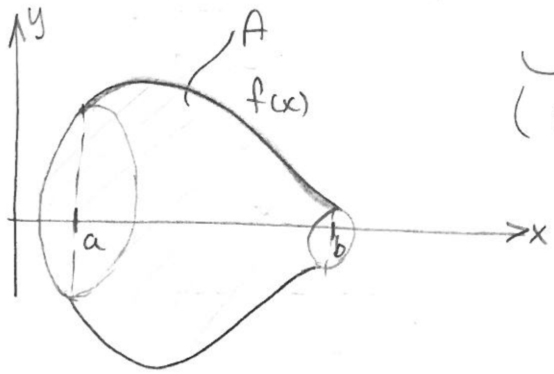
$$=\pi\cdot\lim_{a\to\infty}\int_0^a e^{-2x}\,dx=\pi\cdot\lim_{a\to\infty}\left[\frac{e^{-2x}}{-2}\right]_0^a=$$

$$=\pi\cdot\lim_{a\to\infty}\left(\underbrace{\frac{e^{-2a}}{-2}}_{0}-\underbrace{\frac{e^{-2\cdot 0}}{-2}}_{-1/2}\right)=\frac{\pi}{2}$$

1, Forgástest felszíne

$$A = \underbrace{2\pi \cdot}_{} \int_a^b f(x) \cdot \underbrace{\sqrt{1+[f'(x)]^2}\, dx}_{}$$

($2\pi r$, ívhossz képlet)



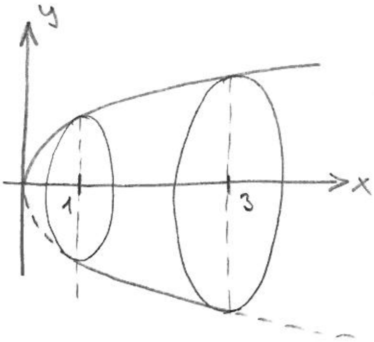
a, $\underbrace{y = \sqrt{x}}_{f(x)}$ $\quad x \in [1,3]$

$$A = 2\pi \cdot \int_1^3 \sqrt{x} \cdot \sqrt{1 + \left[\frac{1}{2\sqrt{x}}\right]^2} \; dx =$$

$$= 2\pi \cdot \int_1^3 \sqrt{x \cdot \left(1 + \frac{1}{4x}\right)} \; dx =$$

$$= 2\pi \cdot \int_1^3 \sqrt{x + \frac{1}{4}} \, dx = 2\pi \cdot \left[\frac{2 \cdot \left(x + \frac{1}{4}\right)^{3/2}}{3} \right]_1^3 =$$

$$= 2\pi \left(\frac{2 \cdot \left(3 + \frac{1}{4}\right)^{3/2}}{3} - \frac{2\left(1 + \frac{1}{4}\right)^{3/2}}{3} \right) \approx$$



A határozott integrál:

$$\int \underbrace{\sqrt{x + \frac{1}{4}}}_{} \, dx = \int t \cdot 2t\,dt = \int 2t^2 dt = \frac{2t^3}{3} + c = \frac{2 \cdot \left(x + \frac{1}{4}\right)^{3/2}}{3} + c$$

$$t = \sqrt{x + \frac{1}{4}} \Rightarrow x = t^2 - \frac{1}{4}$$

$$\frac{dx}{dt} = 2t \Rightarrow dx = 2t\,dt$$

b, $\underbrace{y = \sin x}_{f(x)}$ $\quad , x \in [0, \pi]$

$$A = 2\pi \cdot \int_0^\pi \underbrace{\sin x}_{-g'(x)} \cdot \underbrace{\sqrt{1 + \cos^2 x}}_{f'(g(x))} \; dx \overset{*}{=}$$

$\rightarrow$ helyettesítsük cosx -et

A határozatlan integrál:

8. feladatban $\uparrow$ megcsináltuk

$$\int \sin x \cdot \sqrt{1 + \cos^2 x} \, dx = -\int \sqrt{1 + t^2} = \ldots = -\left(\frac{\operatorname{arsh} t}{2} + \frac{t \cdot \sqrt{1 + t^2}}{2} + c \right) \overset{**}{=}$$

$$\Leftarrow t = \cos x \Rightarrow \frac{dt}{dx} = -\sin x \Rightarrow dt = -\sin x\, dx$$

$$\overset{**}{=} -\left(\frac{\operatorname{arsh}(\cos x)}{2} + \frac{\cos x \cdot \sqrt{1 + \cos^2 x}}{2} + c \right)$$

10.b/ $\overset{*}{=} -2\pi\cdot\left[\frac{\operatorname{arsh}(\cos x)}{2} + \frac{\cos x\cdot\sqrt{1+\cos^2 x}}{2}\right]_0^{\pi} =$

$$= -2\pi\left(\left(\underbrace{\frac{\operatorname{arsh}(-1)}{2}}_{-\frac{\operatorname{arsh}(1)}{2}} + \underbrace{\frac{-1\cdot\sqrt{1+1^2}}{2}}_{-\frac{\sqrt{2}}{2}}\right) - \left(\frac{\operatorname{arsh}(1)}{2} + \underbrace{\frac{1\cdot\sqrt{1+1^2}}{2}}_{\frac{\sqrt{2}}{2}}\right)\right) = -2\pi\cdot\left(\underbrace{-\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}}_{-\sqrt{2}} - \operatorname{arsh}(1)\right)$$

$\cos\pi = -1$

$\cos 0 = 1$

mert $\operatorname{arsh}(x)$ páratlan

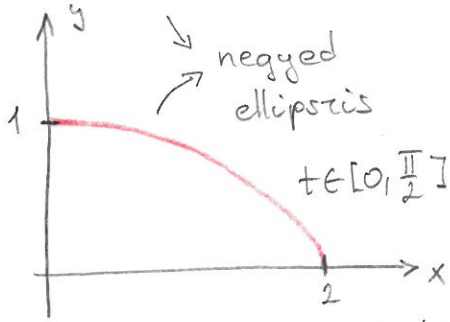
$$= 2\sqrt{2}\pi + 2\pi \operatorname{arsh}(1) \approx 14{,}42$$

11./ $x(t) = 2\cos t$

$y(t) = \sin t$

negyed ellipszis

$t \in [0, \frac{\pi}{2}]$



$$F = 2\pi\cdot\int_0^{\pi/2} y(t)\cdot\sqrt{[\dot{x}(t)]^2+[\dot{y}(t)]^2}\, dt =$$

$$= 2\pi\cdot\int_0^{\pi/2} \sin t\cdot\sqrt{[-2\sin t]^2+[\cos t]^2}\, dt =$$

$$= 2\pi\cdot\int_0^{\pi/2} \sin t\cdot\sqrt{4\sin^2 t+\cos^2 t}\, dt \overset{*}{=}$$

$\dot{x}(t) = -2\sin t$

$\dot{y}(t) = \cos t$

A határozatlan integrál:

cél: csak $\cos t$ legyen a $\sqrt{\ }$ alatt

$$\int \sin t\cdot\sqrt{4\underbrace{\sin^2 t}_{1-\cos^2 t}+\cos^2 t}\, dt = \int \underbrace{\sin t}_{-(\cos t)'}\cdot\underbrace{\sqrt{4-3\cos^2 t}}_{f(\cos t)}\, dt =$$

$$= -\int\sqrt{4-3x^2}\, dx = -2\int\sqrt{1-\Big(\underbrace{\tfrac{\sqrt{3}}{2}x}_{\cos u}\Big)^2}\, dx = -2\int \underbrace{\sqrt{1-\cos^2 u}}_{\sin u}\cdot\frac{-2\sin u}{\sqrt{3}}\, du \overset{**}{=}$$

$x = \cos t$

$\frac{dx}{dt} = -\sin t \Rightarrow dx = -\sin t\, dt$

$\cos u = \frac{\sqrt{3}}{2}x \Rightarrow \frac{2\cos u}{\sqrt{3}} = x \Rightarrow \frac{dx}{du} = \frac{-2\sin u}{\sqrt{3}}$

$\Downarrow$ $u = \arccos\left(\frac{\sqrt{3}}{2}x\right)$

$$\overset{**}{=} \frac{4}{\sqrt{3}}\int\sin^2 u\, du = \frac{4}{\sqrt{3}}\int\frac{1-\cos 2u}{2}\, du = \frac{2}{\sqrt{3}}\int 1-\cos 2u\, du = \frac{2}{\sqrt{3}}\cdot\Big(u - \overbrace{\frac{\sin 2u}{2}}^{\sin u\cdot\cos u = \sqrt{1-c^2}\cdot c}\Big) + c =$$

$$= \frac{2}{\sqrt{3}}\left(\arccos\left(\frac{\sqrt{3}}{2}x\right) - \sqrt{1-\left(\frac{\sqrt{3}}{2}x\right)^2}\cdot\left(\frac{\sqrt{3}}{2}x\right)\right) + c =$$

$$= \underbrace{\frac{2}{\sqrt{3}}\cdot\left(\arccos\left(\frac{\sqrt{3}}{2}\cdot\cos t\right) - \sqrt{1-\left(\frac{\sqrt{3}\cos t}{2}\right)^2}\cdot\frac{\sqrt{3}\cos t}{2}\right)}_{F(t)} + c$$

$$\overset{*}{=} 2\pi\cdot\left[F(t)\right]_0^{\pi/2} = 2\pi\left(F(\tfrac{\pi}{2}) - F(0)\right) = 2\pi\cdot\Big(\underbrace{\left(\frac{\pi}{2}-0\right)}_{F(\frac{\pi}{2})} - \Big(\frac{\pi}{3} -$$