Matek A1 / 11. gyakorlat: Integrálszámítás

Számítsuk ki az integrálokat!

1. a) $\int \frac{x^4 - 3x^2}{1+x} dx$ b) $\int \frac{2}{(x-1)(x+3)} dx$ c) $\int \frac{2x^5 + x^4 + 6x^3 + 4x^2 + x + 2}{x^4 + x^2} dx$

   d) $\int \frac{x}{x^4 + 3x^2 + 2} dx$ e) $\int \frac{x^4}{x+2} dx$

2. a) $\int x \sqrt[4]{\frac{x+2}{3}} dx$ b) $\int \frac{1}{\sqrt{x} + \sqrt[4]{x}} dx$ c) $\int \frac{1}{1 + \sqrt[3]{x+1}} dx$

   d) $\int \frac{e^{\sqrt{x}}}{\sqrt{x}} dx$

3. a) $\int \sqrt{4x^2 - 8x + 40} dx$ b) $\int \sqrt{16 - x^2} dx$ c) $\int \frac{x^2}{\sqrt{x^2 - 1}} dx$

4. a) $\int \frac{e^{2x}}{e^x + e^{-x}} dx$ b) $\int \frac{1}{\cosh(x) + 1} dx$

5. a) $\int \frac{1}{2\sin(x) + \cos(x) + 1} dx$ b) $\int \frac{2}{5\cos(x) + 1} dx$

6. a) $\int \sin(x)\cos^5(x) dx$ b) $\int \cos^2(x) dx$ c) $\int \cos^4(x) dx$

   d) $\int \cos^7(x) dx$ e) $\int \sin^2(x)\cos^3(x) dx$ f) $\int \sin^2(x)\cos^4(x) dx$

   g) $\int \sin^3(x)\cos^5(x) dx$

12:30

1, Racionális törtfüggvények integrálásai

a) $\int \frac{x^4-3x^2}{1+x}\,dx = \int x^3 - x^2 - 2x + 2 - \frac{2}{x+1}\,dx = \frac{x^4}{4} - \frac{x^3}{3} - x^2 + 2x - 2\cdot \ln|x+1| + C$ ✓

I.: polinomosztás $\Rightarrow$ cél: számláló $\boxed{\text{ne}}$ legyen magasabb fokú

$(x^4-3x^2):(x+1) = x^3 - x^2 - 2x + 2$ $\longleftrightarrow$ addig csináljuk, amíg itt egy szám áll

$-(x^4+x^3)$

$(0-x^3-3x^2)$

$-(-x^3-x^2)$

$(-2x^2)$

$-(-2x^2-2x)$

$(2x)$

$-(2x+2)$

-2

$\Rightarrow$ jelentés: $x^4-3x^2 = x^3(x+1) + (-x^3-3x^2)$

$353 : 9 = 30 + 9$

-270

83

-81

2

$\Rightarrow 353 = (30+9)\cdot 9 + 2$

Azaz: $x^4-3x^2 = (x+1)(x^3-x^2-2x+2) - 2$

$\frac{2}{x^2+2x-3}$

b) " $\int \frac{2}{(x-1)(x+3)}\,dx \overset{*}{=} \int \frac{1}{2}\cdot\frac{1}{x-1} - \frac{1}{2}\cdot\frac{1}{x+3}\,dx = \frac{1}{2}\ln|x-1| - \frac{1}{2}|x+3| + C$ ✓

II. Parciális törtekre bontás

Nevezőben lehet: • $(x-a) \Rightarrow \frac{A}{x-a}$

• $(x-a)^n \Rightarrow \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \dots + \frac{A_n}{(x-a)^n}$

• $(x^2+bx+c) \Rightarrow \frac{Ax+B}{x^2+bx+c}$

• $(x^2+bx+c)^n \Rightarrow \frac{A_1x+B_1}{x^2+bx+c} + \dots + \frac{A_nx+B_n}{(x^2+bx+c)^n}$

Most:

$$\frac{2}{(x-1)(x+3)} \overset{*}{=} \frac{A}{x-1} + \frac{B}{x+3} = \frac{(Ax+3A)+(Bx-B)}{(x-1)(x+3)}$$

IIa, $(Ax+3A)+(Bx-B) = x(A+B) + (3A-B) = 0x + 2$

$\Rightarrow A+B=0$ és $3A-B=2$

IIb, $x=0$ esetén: $3A-B=2$

$x=1$ esetén: $(A\cdot 1+3A)+(B\cdot 1-B) = 2 \Rightarrow 4A = 2$

tetszőleges értékre felírható

A megoldás: $A=\frac{1}{2}$ $\quad$ (*)

$B=-\frac{1}{2}$

9 ✓ $\displaystyle\int \frac{2x^5+x^4+6x^3+4x^2+x+2}{x^4+x^2}\,dx \overset{(*)}{=} \int 2x+1+\frac{4x^3-3x^2+x+2}{x^4+x^2}\,dx \overset{(**)}{=}$

I. Polinosztás:

$$\begin{array}{l} (2x^5+x^4+6x^3+4x^2+x+2):(x^4+x^2)=2x+1 \\ -\ (2x^5+2x^3) \\ \hline (x^4+4x^3+4x^2+x+2) \\ -\ (x^4+x^2) \\ \hline 4x^3-3x^2+x+2 \end{array}$$

$\Downarrow$

$$(2x^5+x^4+6x^3+4x^2+x+2)\overset{(*)}{=}(2x+1)(x^4+x^2)+$$
$$+(4x^3-3x^2+x+2)$$

II. Parciális törtekre bontás:

Nevező szorzattá alakítása:

$$x^4+x^2=x^2\underbrace{(x^2+1)}_{\text{tovább nem bontható!}}$$

$$\frac{4x^3+3x^2+x+2}{x^2\cdot(x^2+1)}=\frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2+1}=$$
$$=\frac{A\cdot x\cdot(x^2+1)+B(x^2+1)+(Cx+D)\cdot x^2}{x^2\cdot(x^2+1)}$$

II_b: $x=0$: $2=B$

$x=1$: $4+3+1+2=2A+2B+C+D$

$x=-1$: $-4+3-1+2=-2A+2B-C+D$

$\oplus$ $10=4B+2D \Rightarrow D=1$

$x=2$: $4\cdot 8+3\cdot 4+2+2=10A+5B+8C+4D$ (...) ez hosszabb most

II_a: $4x^3+3x^2+x+2=x^3(A+C)+x^2(B+D)+x(A)+1\cdot(B)$

$2=B$

$1=A$

$3=B+D \Rightarrow D=3-B=1$

$4=A+C \Rightarrow C=4-A=3$

$\overset{(**)}{=} \displaystyle\int 2x+1+\frac{1}{x}+\frac{2}{x^2}+\underbrace{\frac{3x+1}{x^2+1}}\,dx = x^2+x+\ln|x|-\frac{2}{x}+\frac{3}{2}\ln|x^2+1|+\mathrm{arctg}\,x+c$

$$\int\frac{3x+1}{x^2+1}\,dx=\int\frac{3}{2}\cdot\frac{2x}{x^2+1}+\frac{1}{1+x^2}\,dx=\frac{3}{2}\ln|x^2+1|+\mathrm{arctg}\,x+c$$

d) $\int \frac{x}{x^4+3x^2+2}\,dx \overset{*}{=} \int \frac{x}{x^2+1} - \frac{x}{x^2+2}\,dx = \int \frac{1}{2}\frac{2x}{x^2+1} - \frac{1}{2}\frac{2x}{x^2+2}\,dx =$

$= \frac{1}{2}\ln|x^2+1| - \frac{1}{2}\ln|x^2+2| + c$

II. Nevező szorzattá alakítása:

$x^4+3x^2+2 = (x^2+1)(x^2+2)$

Parc. törtek:

$$\frac{x}{(x^2+1)(x^2+2)} \overset{*}{=} \frac{Ax+B}{(x^2+1)} + \frac{Cx+D}{(x^2+2)} = \frac{(Ax+B)(x^2+2)+(Cx+D)(x^2+1)}{(x^2+1)(x^2+2)}$$

x^3: $0 = A+C$

x^2: $0 = B+D$

x: $1 = 2A+C$

1: $0 = 2B+D$

$\ominus$ $A=1,\ C=-1,\ B=0,\ D=0$ $*$

e) $\int \frac{x^4}{x+2}\,dx = \int x^3-2x^2+4x-8+\frac{16}{x+2}\,dx = \frac{x^4}{4} - \frac{2x^3}{3} + 2x^2 - 8x + 16\ln|x+2| + c$

I: Polinom osztás:

$(x^4\quad) : (x+2) = x^3-2x^2+4x-8$

$-(x^4+2x^3)$

$(-2x^3)$

$-(-2x^3-4x^2)$

$(4x^2)$

$-(4x^2+8x)$

$(-8x)$

$-(-8x-16)$

16

- Integrálandó résztörtek típusai:

a) $\int \frac{1}{x-a}\,dx$ pl: $\int \frac{1}{x+3}\,dx = \ln|x+3| + c$

b) $\int \frac{1}{(x-a)^n}\,dx$ pl: $\int \frac{1}{(x+3)^2}\,dx = \int \underbrace{1}_{1}\cdot\underbrace{(x+3)^{-2}}_{1-2}\,dx = -(x+3)^{-1} + c = \frac{-1}{x+3} + c$

c) $\int \frac{1}{ax^2+bx+c}\,dx$ pl: $\int \frac{1}{x^2-6x+10}\,dx = \int \frac{1}{(x-3)^2+1}\,dx = \operatorname{arctg}(x-3) + c$

tovább nem bontható most

d, $\int \frac{1}{(ax^2+bx+c)^n}\,dx$

pl: $\int \frac{1}{(x^2+1)^2}\,dx = \int \frac{x^2+1 \quad -x^2}{(x^2+1)^2}\,dx =$

$= \int \frac{1}{x^2+1} + \frac{-x^2}{(x^2+1)^2}\,dx =$

$= \mathrm{arctg}(x) - \int \underbrace{\frac{x}{2}}_{f} \cdot \underbrace{\frac{2x}{(x^2+1)^2}}_{g'}\,dx =$

$g' \Rightarrow g = \frac{-1}{x^2+1}$

$= \mathrm{arctg}x - \left(\frac{x}{2} \cdot \frac{-1}{x^2+1} - \int \frac{1}{2} \cdot \frac{-1}{x^2+1}\,dx\right) =$

$= \mathrm{arctg}x + \frac{1}{2} \cdot \frac{1}{x^2+1} - \frac{1}{2} \cdot \mathrm{arctg}x + c =$

$= \frac{1}{2}\left(\mathrm{arctg}x + \frac{1}{1+x^2}\right) + c$

e, $\int \frac{px+q}{ax^2+bx+c}\,dx$

pl: $\int \frac{2x+3}{x^2+2x+3}\,dx = \int \frac{2x+2}{x^2+2x+3} + \frac{1}{x^2+2x+3}\,dx =$

$= \ln(x^2+2x+3) + \underbrace{\int \frac{1}{x^2+2x+3}\,dx}_{\text{d, eset}}$

f, $\int \frac{px+q}{(ax^2+bx+c)^n}\,dx$

35p. 10:05

Helyettesítéses integrálás:

$\int f'(\underbrace{g(x)}_{u}) \cdot \underbrace{\underbrace{g'(x)}_{\frac{du}{dx}}\,dx}_{\frac{du}{dx} \cdot dx = du} = \int f'(u)\,du = f(u) + c = f(\underbrace{g(x)}_{u}) + c$

Példa: $\int \cos(\underbrace{2x-1}_{u})\,dx \overset{*}{=} \int \cos(u) \cdot \frac{1}{2} \cdot du = \frac{1}{2} \cdot \sin(u) + c = \frac{1}{2} \cdot \sin(2x-1) + c$

$u = 2x-1$

$\frac{du}{dx} = 2 \Rightarrow dx = \frac{du}{2}$

$\Rightarrow \frac{du}{dx} \cdot dx = 2\,dx$

$\overset{*}{=} \int \cos(u) \cdot \frac{1}{2} \cdot \overbrace{2\,dx}^{du} = \int \cos(u) \cdot \frac{1}{2} \cdot du = (\ldots)$

1, $R(x, \sqrt[n]{\underbrace{ax+b}_{\text{lineáris cucc}}})$ pl.: $\int x \cdot \sqrt[4]{\frac{x+2}{3}}\,dx$

helyettesítés: $t = \sqrt[n]{ax+b}$ $\int \frac{1+\sqrt[3]{x+3}}{2x}$

a, $\int x \cdot \underbrace{\sqrt[4]{\frac{x+2}{3}}}_{t}\,dx = \int (3t^4-2)\cdot t \cdot 12t^3\,dt = 12\int 3t^8 - 2t^4\,dt = 12\left(\frac{3t^9}{9} - \frac{2t^5}{5}\right) + c \overset{*}{=}$

$t = \sqrt[4]{\frac{x+2}{3}} = \left(\frac{x+2}{3}\right)^{\frac{1}{4}} \Rightarrow x = 3t^4 - 2$

$\frac{dx}{dt} = 12t^3 \Rightarrow dx = 12t^3\,dt$

$\overset{*}{=} 4\cdot\left(\frac{x+2}{3}\right)^{9/4} - \frac{24}{5}\cdot\left(\frac{x+2}{3}\right)^{5/4} + c$

b, $\int \frac{1}{\sqrt{x}+\sqrt[4]{x}}\,dx = \int \frac{4t^3}{t^2+t}\,dt = 4\int \frac{t^2}{t+1}\,dt \overset{*}{=} 4\cdot\int t-1+\frac{1}{t+1}\,dt = 4\cdot\left(\frac{t^2}{2} - t + \ln|t+1|\right) + c =$

$= 2\sqrt{x} - 4\sqrt[4]{x} + 4\ln(\sqrt[4]{x}+1) + c$

$t = \sqrt[4]{x} \Rightarrow x = t^4 \Rightarrow \frac{dx}{dt} = 4t^3 \Rightarrow dx = 4t^3\,dt$

$t^2 = \sqrt{x}$

I. $(t^2) : (t+1) = t-1$

$-(t^2+t)$

$(-t)$

$-(-t-1)$

$+1$

13:10

c, $\int \frac{1}{1+\sqrt[3]{x+1}}\,dx = \int \frac{1}{1+t}\cdot 3t^2\,dt = \int \frac{3t^2}{1+t}\,dx \overset{*}{=} \int 3t - 3 + \frac{3}{1+t}\,dt =$

$t = \sqrt[3]{x+1} \Rightarrow x = t^3 - 1 \Rightarrow \frac{dx}{dt} = 3t^2 \Rightarrow dx = 3t^2\,dt$

$= \frac{3t^2}{2} - 3t + 3\ln|t+1| + c$

$= \frac{3(x+1)^{2/3}}{2} - 3(x+1)^{1/3}$

$+ 3\ln|\sqrt[3]{x+1} + 1| + c$

I.: $(3t^2) : (t+1) = 3t - 3$

$-(3t^2+3t)$

$(-3t)$

$-(-3t-3)$

$+3$

$*$

d, $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = \int \frac{e^t}{t}\cdot 2t\,dt = 2\cdot\int e^t\,dt = 2e^t + c = 2\cdot e^{\sqrt{x}} + c$

$t = \sqrt{x} \Rightarrow x = t^2 \Rightarrow \frac{dx}{dt} = 2t \Rightarrow dx = 2t\,dt$

13:20

3, $R(x, \sqrt{ax^2+bx+c})$

I, Teljes négyzetté alakítás a gyök alatt

$\Downarrow$

I, a, $\sqrt{(\underbrace{\dots}_{ch\,t})^2-1} = \sqrt{ch^2t-1} = \sqrt{sh^2t} = sh\,t$

b, $\sqrt{(\underbrace{\dots}_{sh\,t})^2+1} = \sqrt{sh^2t+1} = \sqrt{ch^2t} = ch\,t$

$\Big\}\ ch^2t - sh^2t = 1$

c, $\sqrt{1-(\underbrace{\dots}_{sin\,t \text{ vagy } cos\,t})^2} = \sqrt{1-\sin^2t} = \sqrt{\cos^2 t} = \cos t$

$\}\ \sin^2t + \cos^2 t = 1$

$$\int \sqrt{4x^2-8x+40}\,dx = 2\cdot\int\sqrt{x^2-2x+1+9}\,dx = 2\cdot\int\sqrt{(x-1)^2+9}\,dx =$$

$$= 2\cdot 3\cdot \int\sqrt{\Big(\underbrace{\frac{x-1}{3}}\Big)^2+1}\,dx = 2\cdot 3\cdot\int\underbrace{\sqrt{sh^2t+1}}_{ch\,t}\cdot 3\,ch\,t\,dt = 2\cdot 9\cdot\int ch^2t\,dt \overset{*}{=}$$

- $sh\,t = \frac{x-1}{3} \Rightarrow x = 3\,sh\,t + 1 \Rightarrow \frac{dx}{dt} = 3\,ch\,t \Rightarrow dx = 3\,ch\,t\,dt$

- ch^2t linearizálása:

$ch\,2t = ch^2t + sh^2t$

$1 = ch^2t - sh^2t$

$\Big\}\oplus \quad ch^2t = \frac{ch\,2t+1}{2}$

$$\overset{*}{=} 2\cdot 9\cdot\int\frac{ch\,2t+1}{2}\,dt = 9\cdot\int 1 + ch\,2t\,dt = 9t + \frac{9\,sh\,2t}{2} \overset{*}{=}$$

- $t = arsh\frac{x-1}{3}$
- $sh\left(2\cdot arsh\frac{x-1}{3}\right) = 2\cdot \overbrace{sh\left(arsh\frac{x-1}{3}\right)}^{\frac{x-1}{3}}\cdot ch\left(arsh\frac{x-1}{3}\right)$

$ch^2x - sh^2x = 1$

$ch^2x = 1 + sh^2x$

$ch\,x = \sqrt{1+sh^2x}$

$\Downarrow$

$ch\left(arsh\frac{x-1}{3}\right) = \sqrt{1+\left(\frac{x-1}{3}\right)^2} = \frac{1}{3}\cdot\sqrt{x^2-2x+10}$

$$\overset{*}{=} 9\,arsh\frac{x-1}{3} + (x-1)\cdot\sqrt{x^2-2x+10} + C$$

b, $\int \sqrt{16-x^2}\,dx = \int 4\cdot\sqrt{1-\left(\underbrace{\frac{x}{4}}_{\sin t}\right)^2}\,dx = \int 4\cdot \underbrace{\sqrt{1-\sin^2 t}}_{\cos t}\cdot 4\cdot \cos t\,dt =$

$\sin t = \frac{x}{4} \Rightarrow x = 4\sin t \Rightarrow \frac{dx}{dt} = 4\cos t \Rightarrow dx = 4\cos t\,dt$

$= 16\cdot\int \cos^2 t\,dt = 16\cdot\int \frac{1+\cos 2t}{2}\,dt = 8\cdot\int 1+\cos 2t\,dt = 8\cdot\left(t+\frac{\sin 2t}{2}\right)+c$

∘ $\cos^2 t$ linearizálása:

$\sin^2 t + \cos^2 t = 1$

$\cos^2 t - \sin^2 t = \cos 2t$ } ⊕

$2\cos^2 t = 1+\cos 2t \Rightarrow \cos^2 t = \frac{1+\cos 2t}{2}$

∘ $t = \arcsin\frac{x}{4}$

$= 8\cdot\arcsin\frac{x}{4} + \cancel{4}\cdot 2\cdot\frac{x}{4}\cdot\sqrt{1-\frac{x^2}{16}} = 8\cdot\arcsin\frac{x}{4} - \frac{1}{2}x\cdot\sqrt{16-x^2}$

$\sin(2\cdot\arcsin u) = 2\cdot\underbrace{\sin(\arcsin u)}_{u}\cdot\underbrace{\cos(\arcsin u)}_{\sqrt{1-\sin^2(\arcsin u)}\,=\,\sqrt{1-u^2}} = 2\cdot u\cdot\sqrt{1-u^2}$

$\cos u = \sqrt{1-\sin^2 u}$

ugyanaz mint előbb

c) $\int \frac{x^2}{\sqrt{x^2-1}}\,dx = \int \frac{ch^2 t}{\underbrace{\sqrt{ch^2 t-1}}_{sh t}}\cdot sh t\,dt = \int ch^2 t\,dt = (\ldots) =$

$x = ch t \Rightarrow \frac{dx}{dt} = sh t \Rightarrow dx = sh t\,dt$

$= \frac{t}{2} + \frac{sh 2t}{4} \underset{\uparrow}{=} \frac{t}{2} + \frac{sh(2 arch x)}{4} = \frac{arch x}{2} + \frac{2\cdot \overbrace{sh(arch x)}^{\sqrt{ch^2(arch x)-1}\,=\,\sqrt{x^2-1}}\cdot ch(arch x)}{4} =$

$t = arch x$

$= \frac{arch x}{2} + \frac{x\cdot\sqrt{x^2-1}}{2}$

13:35

4, $R(\underbrace{e^{ax}})$

$t = e^{ax}$

a, $\int \frac{e^{2x}}{e^x + e^{-x}}\,dx = \int \frac{(e^x)^2}{e^x + \frac{1}{e^x}}\,dx = \int \frac{t^2}{t+\frac{1}{t}} \cdot \frac{dt}{t} = \int \frac{t^2}{t^2+1}\,dt \overset{*}{=}$

$e^x = t \Rightarrow \frac{dt}{dx} = e^x \Rightarrow dx = \frac{dt}{t}$

$\overset{*}{=} \int +1 + \frac{-1}{1+t^2}\,dt =$

$* \quad (t^2) : (t^2+1) = 1$

$\underline{-(t^2+1)}$

-1

$= +t - \operatorname{arctg} t + c =$

$= e^x - \operatorname{arctg}(e^x) + c$

b) $\int \frac{1}{\operatorname{ch} x + 1}\,dx = \int \frac{1}{\frac{e^x+e^{-x}}{2}+1}\,dx = \int \frac{2}{e^x+e^{-x}+2}\,dx = \int \frac{2}{t+\frac{1}{t}+2} \cdot \frac{dt}{t} =$

$t = e^x \Rightarrow \frac{dt}{dx} = e^x = t \Rightarrow dx = \frac{dt}{t}$

$= \int \frac{2}{t^2+2t+1}\,dt = \int 2\cdot(t+1)^{-2}\,dt = -2\cdot(t+1)^{-1} + c = \frac{-2}{t+1} + c = \frac{-2}{e^x+1} + c$

13:40

5, a) $\int \frac{1}{2\sin x + \cos x + 1}\,dx = \int \frac{1}{2\cdot\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2} + 1} \cdot \frac{2\,dt}{1+t^2} = \int \frac{2}{4t+1-t^2+1+t^2}\,dt =$

$t = \operatorname{tg}\frac{x}{2} \quad \frac{dt}{dx} = \underbrace{\left(1+\operatorname{tg}^2\frac{x}{2}\right)}_{1+t^2}\cdot\frac{1}{2} \Rightarrow dx = \frac{2\,dt}{1+t^2}$

$= \int \frac{2}{2+4t}\,dt = \frac{1}{2}\cdot\int \frac{2}{2t+1}\,dt = \frac{1}{2}\cdot\ln|2t+1| = \frac{1}{2}\cdot\ln\left|2\operatorname{tg}\frac{x}{2}+1\right|$

13:45

b, $\int \frac{\cos x}{\cos x + 2\sin x}\,dx = \int \frac{\frac{1-t^2}{1+t^2}}{\frac{1-t^2}{1+t^2}+2\frac{2t}{1+t^2}} \cdot \frac{2\,dt}{1+t^2} = 2\int \frac{\frac{1-t^2}{1+t^2}}{1-t^2+4t}\,dt =$

$t = \operatorname{tg}\frac{x}{2} \Rightarrow dx = \frac{2\,dt}{1+t^2}$

$= \int \frac{1-t^2}{(1+t^2)\cdot(t^2-4t-1)}\,dt$

$t_{1,2} = \frac{4\pm\sqrt{16+4}}{2} = 2\pm\sqrt{5}$

$$\int \frac{2}{5\cos x+1}\,dx = \int \frac{2}{5\cdot\frac{1-t^2}{1+t^2}+1}\cdot\frac{2\,dt}{1+t^2} = 4\cdot\int \frac{1}{5-5t^2+1+t^2}\,dt = 4\cdot\int\frac{1}{6-4t^2}\,dt =$$

$$t=\mathrm{tg}\frac{x}{2} \Rightarrow dx=\frac{2\,dt}{1+t^2}$$

$$= 2\cdot\int\frac{1}{3-2t^2}\,dt = 2\cdot\int\frac{1}{-2\left(t+\sqrt{\frac{3}{2}}\right)\left(t-\sqrt{\frac{3}{2}}\right)}\,dt \overset{(*)}{=} -\int \frac{\sqrt{1/6}}{t-\sqrt{\frac{3}{2}}}-\frac{\sqrt{1/6}}{t+\sqrt{\frac{3}{2}}}\,dt \overset{(**)}{=}$$

$2t^2-3=0$

$t^2=\pm\frac{3}{2}$

$t=\sqrt{\frac{3}{2}}$

$$(*)\quad \frac{1}{\left(t-\sqrt{\frac{3}{2}}\right)\left(t+\sqrt{\frac{3}{2}}\right)} = \frac{A}{t-\sqrt{\frac{3}{2}}}+\frac{B}{t+\sqrt{\frac{3}{2}}} = \frac{A\cdot\left(t+\sqrt{\frac{3}{2}}\right)+B\left(t-\sqrt{\frac{3}{2}}\right)}{(\dots)(\dots)}$$

$t:\quad 0=A+B$

$1:\quad 1=\sqrt{\frac{3}{2}}A-\sqrt{\frac{3}{2}}B \Rightarrow \sqrt{\frac{2}{3}}=A-B$ $\quad\oplus\quad \sqrt{\frac{2}{3}}=2A \Rightarrow A=\sqrt{\frac{1}{6}}$

$$\overset{(**)}{=} -\sqrt{\frac{1}{6}}\cdot\left(\ln\left|t-\sqrt{\frac{3}{2}}\right|-\ln\left|t+\sqrt{\frac{3}{2}}\right|\right)+c = -\sqrt{\frac{1}{6}}\cdot\left(\ln\left|\mathrm{tg}\frac{x}{2}-\sqrt{\frac{3}{2}}\right|-\ln\left|\mathrm{tg}\frac{x}{2}+\sqrt{\frac{3}{2}}\right|\right)+c$$

6., Trigonometrikus hatványok szorzata

a) $\int \sin x\cdot\cos^5 x\,dx = \frac{-\cos^6 x}{6}+c$

b) $\int \cos^2 x\,dx = \int\frac{1+\cos 2x}{2}\,dx = \frac{x}{2}+\frac{\sin(2x)}{4}+c$

$\cos^2 2x = \frac{1+\cos 4x}{2}$

c) $\int\cos^4 x\,dx = \int(\cos^2 x)^2\,dx = \int\left(\frac{1+\cos 2x}{2}\right)^2 dx = \int\frac{\cos^2 2x+2\cos 2x+1}{4}\,dx$

$$=\frac{1}{4}\cdot\int \cos^2 2x+2\cos 2x+1\,dx = \frac{1}{4}\cdot\int\frac{1+\cos 4x}{2}+2\cos 2x+1\,dx =$$

$$=\frac{1}{8}\cdot\int\cos 4x+4\cos 2x+3\,dx = \frac{1}{8}\left(\frac{\sin 4x}{4}+2\sin 2x+3x\right)+c$$

d) $\int\cos^7 x\,dx = \int\cos^6 x\cdot\cos x\,dx = \int(\cos^2 x)^3\cdot\cos x\,dx =$

$$=\int\underbrace{(1-\sin^2 x)^3}_{f(\sin x)}\cdot\underbrace{\cos x}_{(\sin x)'}\,dx = \int(1-t^2)^3\,dt = \int 1-3t^2+3t^4-t^6\,dt = t-t^3+\frac{3t^4}{4}-\frac{t^7}{7}+c \overset{*}{=}$$

$t=\sin x \Rightarrow \frac{dt}{dx}=\cos x \Rightarrow \cos x\,dx = dt$

$\overset{*}{=} \sin x-\sin^3 x+\frac{3\sin^4 x}{4}-\frac{\sin^7 x}{7}+c$

e, $\int \sin^2 x \cdot \cos^3 x \, dx = \int \underbrace{\sin^2 x \cdot (1-\sin^2 x)}_{f(\sin(x))} \cdot \underbrace{\cos x}_{(\sin x)'} \, dx = \int t^2 \cdot (1-t^2) \, dt = \int t^2 - t^4 \, dt \overset{*}{=}$

$t = \sin x \Rightarrow \frac{dt}{dx} = \cos x \Rightarrow \cos x \, dx = dt$

$\overset{*}{=} \frac{t^3}{3} - \frac{t^5}{5} + c = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + c$

f, $\int \sin^2 x \cdot \cos^4 x \, dx = \int \frac{1}{4} (2 \sin x \cos x)^2 \cdot \cos^2 x \, dx = \frac{1}{4} \cdot \int \overbrace{\sin^2 2x}^{\frac{1-\cos 4x}{2}} \cdot \frac{1+\cos 2x}{2} \, dx$

$= \frac{1}{4} \cdot \int \frac{1-\cos 4x}{4} + \frac{\cos 2x \cdot \sin^2 2x}{2} \, dx = \frac{1}{4} \left(\frac{1}{4} x - \frac{\sin 4x}{16} + \frac{\sin^3 2x}{2 \cdot 2 \cdot 3} \right) + c$

VAGY:

$\int \sin^2 x \cdot \cos^4 x \, dx = \int (1-\cos^2 x) \cos^4 x \, dx = \underbrace{\int \cos^4 x \, dx}_{5.c. \text{ feladat}} - \int \cos^6 x \, dx$

csak sin / csak cos

$\int \cos^6 x \, dx = \int (\cos^2 x)^3 \, dx = \int \left(\frac{1+\cos 2x}{2} \right)^3 dx = \dots$

g)

$\int \sin^3 x \cdot \cos^5 x \, dx = \int \frac{1}{2} \cdot \underbrace{2 \sin x \cdot \cos x}_{\sin 2x} \cdot \sin^2 x \cdot \cos^4 x \, dx =$

$= \frac{1}{2} \cdot \int \underbrace{\sin 2x}_{-(\cos 2x)'} \cdot \underbrace{\underbrace{\frac{1-\cos 2x}{2}}_{\sin^2 x} \cdot \underbrace{\left(\frac{1+\cos 2x}{2} \right)^2}_{\cos^2 x}}_{f(\cos 2x)} \, dx = \frac{1}{2} \cdot \int \frac{1-t}{2} \cdot \left(\frac{1+t}{2} \right)^2 \cdot \left(-\frac{1}{2} \right) dt =$

$\Rightarrow$ ötlet: helyettesítés:

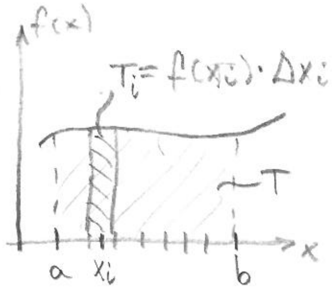
$t = \cos 2x \Rightarrow \frac{dt}{dx} = -2 \sin 2x$

$-\frac{1}{2} dt = \sin 2x \, dx$

$= -\frac{1}{2^5} \cdot \int \overbrace{(1-t) \cdot (1+2t+t^2)}^{-t^3 - t^2 + t + 1} \, dt = -\frac{1}{2^5} \cdot \left(-\frac{t^4}{4} - \frac{t^3}{3} + \frac{t^2}{2} + t \right) + c =$

$= -\frac{1}{2^5} \cdot \left(-\frac{\cos^4 2x}{4} - \frac{\cos^3 2x}{3} + \frac{\cos^2 2x}{2} + \cos 2x \right) + c$

6, Területszámítás:



$$T = \sum f(x_i)\cdot \Delta x_i \xrightarrow{\max(\Delta x_i)\to 0} \int_a^b f(x)\,dx$$

$$\underset{\int}{\sum} \quad \underset{f(x)}{f(x_i)} \quad \underset{dx}{\Delta x_i}$$

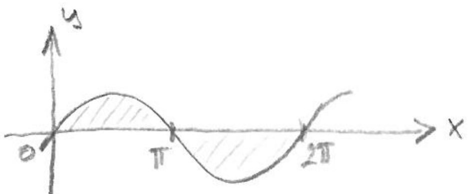
Newton - Leibniz: $\int_a^b f(x)\,dx = F(b) - F(a)$, $\int f(x)\,dx = F(x) + C$

$$= \left[F(x)\right]_a^b$$

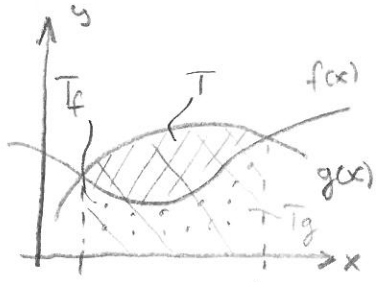
a, $\int_1^2 x^2 - 3\,dx = \Big[\underbrace{\frac{x^3}{3} - 3x}_{F(x)}\Big]_1^2 = \left(\frac{2^3}{3} - 3\cdot 2\right) - \left(\frac{1^3}{3} - 3\cdot 1\right) = \frac{8}{3} - 6 - \frac{1}{3} + 3 = \frac{7}{3} - \frac{9}{3} = -\frac{2}{3}$

b, $\int_0^{\pi} \sin x\,dx = \left[-\cos x\right]_0^{\pi} = (-\cos\pi) - (-\cos 0) = 1 + 1 = 2$

$\int_0^{2\pi} \sin x\,dx = \left[-\cos x\right]_0^{2\pi} = (-\cos 2\pi) - (-\cos 0) = -1 + 1 = 0 \Rightarrow$ ok: $\int_a^b f(x)\,dx$ előjeles „területet" jelent



7, Bezárt terület



$T = \left| \int_a^b f(x) - g(x)\,dx \right|$

$T = T_f - T_g$

Metszéspont:

$f(x) = g(x)$

$x^2 + 4x + 3 = 5x - 3$

$x^2 - x + 6 = 0$

$(x-3)(x+2) = 0$

$x_1 = 3$

$x_2 = -2$

$f(x) = x^2 + 4x + 3$

$g(x) = 5x - 3$

$$\int_{-2}^{3} f(x) - g(x) = \int_{-2}^{3} x^2 - x + 6\,dx = \left[\frac{x^3}{3} - \frac{x^2}{2} + 6x\right]_{-2}^{3} =$$

$$= \left(\frac{3^3}{3} - \frac{3^2}{2} + 6\cdot 3\right) - \left(\frac{-8}{3} - \frac{4}{2} - 12\right) =$$

$$= \frac{1}{6}\big(54 - 27 + {}'$$