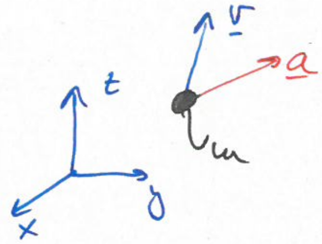
Dinamika 9. gyakorlat

Merev testek kinetikája

Ismétlés: (Anyagi pontra)

$\underline{I} = m \cdot \underline{v}$ – impulzus

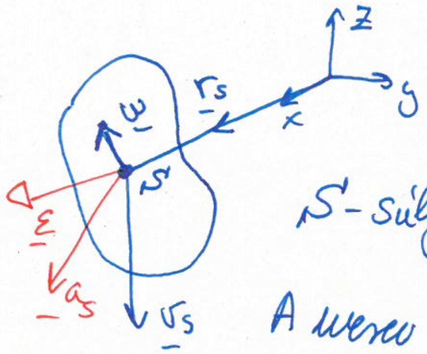
$\underline{\Pi}_A = \underline{r}_{AS} \times \underline{I}$ – perdület



Kinetikus energia: $T = \frac{1}{2} m \underline{v}^2$

Merev test esetén

↳ Impulzus (valamennyi anyagi pont impulzusának összege)
~ integrálmennyiség
DE! kiegyszerűsíthető



S – súlypont

A merev test kinematikai állapotát leírhatjuk a súlyponthoz rendelt mennyiségekkel

$\boxed{\underline{I} = m \cdot \underline{v}_S}$

súlypont sebessége

↳ Dinamika alaptételében: $\dot{\underline{I}} = m \cdot \underline{a}_S$ (ha a tömeg állandó)

↳ Perdület ⇒ Írjuk fel a súlypontra ⇒ hasonlóan integrál mennyiség

$\boxed{\underline{\Pi}_S = \underline{\underline{\Theta}}_S \cdot \underline{\omega}}$

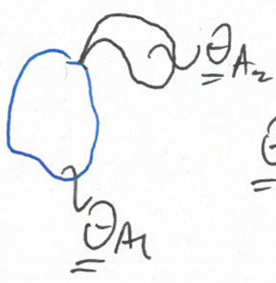
merev test szögsebessége

Tehetetlenségi nyomaték mátrix

$\underline{\underline{\Theta}}_S = \underline{\underline{\Theta}}_S^T$ (szimmetrikus mátrix)

$[kg\, m^2]$

A Tehetetlenségi nyomaték additív



$\underline{\underline{\Theta}}_A = \underline{\underline{\Theta}}_{A_1} + \underline{\underline{\Theta}}_{A_2}$

$$\underline{\underline{\Theta}}_S = \begin{bmatrix} \Theta_\xi & -D_{\xi\eta} & -D_{\xi\zeta} \\ -D_{\xi\eta} & \Theta_\eta & -D_{\eta\zeta} \\ -D_{\xi\zeta} & -D_{\eta\zeta} & \Theta_\zeta \end{bmatrix}$$

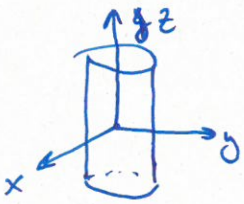
ahol

$$\left.\begin{aligned} \Theta_\xi &= \int_m (\eta^2 + \zeta^2)\,dm \\ \Theta_\eta &= \int_m (\xi^2 + \zeta^2)\,dm \\ \Theta_\zeta &= \int_m (\xi^2 + \eta^2)\,dm \end{aligned}\right\}\text{ az adott tengelytől vett távolság négyzet}$$

$$\left.\begin{aligned} D_{\xi\eta} &= \int_m \xi\eta\,dm \\ D_{\xi\zeta} &= \int_m \xi\zeta\,dm \\ D_{\eta\zeta} &= \int_m \eta\zeta\,dm \end{aligned}\right\}\text{ deviációs nyomatékok}$$

a főirányokban: $\underset{(\hat{\xi},\hat{\eta},\hat{\zeta})}{\underline{\underline{\Theta}}_S} = \begin{bmatrix} \Theta_{\hat{\xi}} & 0 & 0 \\ 0 & \Theta_{\hat{\eta}} & 0 \\ 0 & 0 & \Theta_{\hat{\zeta}} \end{bmatrix}$ főirányok: $D_{ij} = 0$

Alappélda:



$$\underline{\underline{\Theta}}_S = \begin{bmatrix} \frac{1}{12} m h^2 + \frac{1}{4} m R^2 & 0 & 0 \\ 0 & \frac{1}{12} m h^2 + \frac{1}{4} m R^2 & 0 \\ 0 & 0 & \frac{1}{2} m R^2 \end{bmatrix}$$

A szimmetriatengelyek mindig főirányok

↳ $\underline{\underline{\Theta}}_S$ sajátértékei $\Theta_1 > \Theta_2 > \Theta_3$ pozitívak (poz. definit)

- Súlypontra: $\dot{\underline{\Pi}}_S = \underline{D}_S$ → dinamika alaptétele

Álló pontra a perdület: $\boxed{\underline{\underline{\Theta}}_0 \cdot \underline{\omega} = \underline{\Pi}_0}$

Nem álló pontra: $\boxed{\underline{\Pi}_B = \underline{\underline{\Theta}}_B \cdot \underline{\omega} + \underline{r}_{BS} \times m \underline{v}_B}$

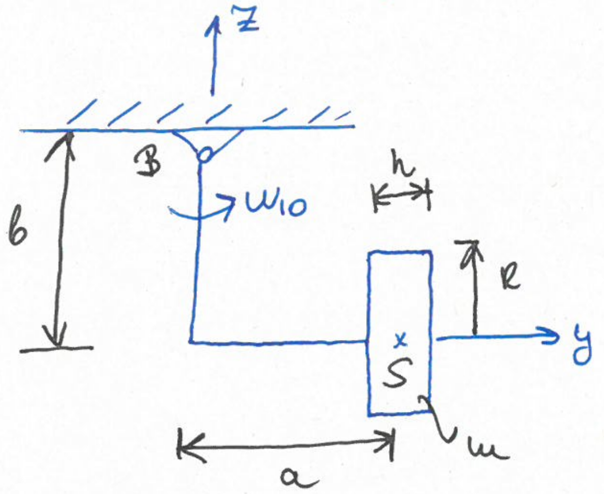
$\underline{\underline{\Theta}}_0 = \underline{\underline{\Theta}}_S + \underline{\underline{\Theta}}_{0S}$ ahol $\underline{\underline{\Theta}}_{0S} = m \begin{bmatrix} y_S^2 + z_S^2 & -x_S y_S & -x_S z_S \\ \cdot & x_S^2 + z_S^2 & -y_S z_S \\ \cdot & \cdot & x_S^2 + y_S^2 \end{bmatrix}$

Steiner tétel

Kinetikus energia

$$T = \underbrace{\frac{1}{2} m v_S^2 + \frac{1}{2} \underline{\omega}^T \underline{\underline{\Theta}}_S \underline{\omega}}_{\text{súlypontra!}} = \underbrace{\frac{1}{2} \underline{\omega}^T \underline{\underline{\Theta}}_0 \underline{\omega}}_{\text{álló pontra}}$$

1. példa Repülőgép futómű



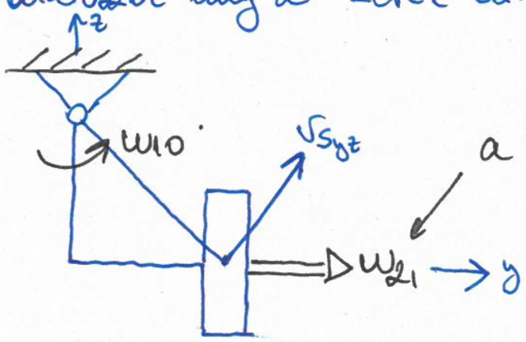
Adott:

v_0; ω_{10}, m, R, h, a, b

A repülőgép v_0 sebesség elérésének pillanatában elemelkedik a talajtól és az első futóművet elkezdi behúzni. ~~az~~
A kerék azzal a szögsebességgel forog, amellyel gördült.

Feladat: Adjuk meg $\underline{\underline{\Theta}}_S$; $\underline{\Pi}_S$; $\underline{\underline{\Theta}}_B$, $\underline{\Pi}_B$, T

Határozzuk meg a kerék szögsebességét:



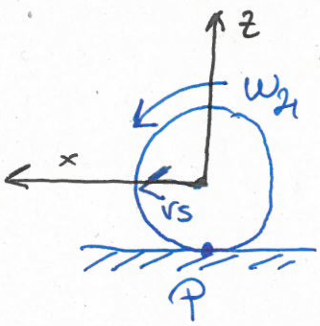
a kerék forgási szögsebessége

$$v_S = \sqrt{v_0^2 + \omega_{10}^2(a^2+b^2)}$$

$$v_{Sx} = v_0$$

$$v_{Syz} = \omega_{10}\sqrt{a^2+b^2}$$

Az elemelkedés pillanatában a kerék gördül



$$\omega_{21} = \frac{v_S}{R} = \frac{v_0}{R}$$

y-irányú

$$\underline{\omega}_{21} = \begin{bmatrix} 0 \\ \omega_{21} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ v_0/R \\ 0 \end{bmatrix}$$

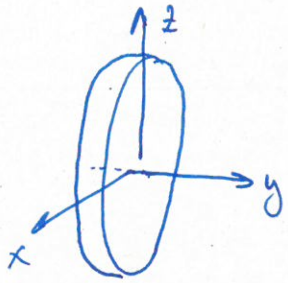
$$\underline{\omega}_{10} = \begin{bmatrix} \omega_{10} \\ 0 \\ 0 \end{bmatrix}$$

x-irányú

$$\underline{\omega}_{20} = \underline{\omega}_{10} + \underline{\omega}_{21} = \begin{bmatrix} \omega_{10} \\ v_0/R \\ 0 \end{bmatrix}$$

A perdülethez kell még $\underline{\underline{\Theta}}_S$ mátrixot a (2)es testre!

$\downarrow$ korong $\Rightarrow$ a



$$\underline{\underline{\Theta}}_S = \begin{bmatrix} \frac{1}{4}mR^2 + \frac{1}{12}mh^2 & 0 & 0 \\ 0 & \frac{1}{2}mR^2 & 0 \\ 0 & 0 & \frac{1}{4}mR^2 + \frac{1}{12}mh^2 \end{bmatrix}$$

az (x,y,z) KR
a főirányokkal megegyező irányba mutat!

Ebből:

$$\underline{\Pi}_S = \underline{\underline{\Theta}}_S \cdot \underline{\omega} = \begin{bmatrix} \frac{1}{4}mR^2 + \frac{1}{12}mh^2 & 0 & 0 \\ 0 & \frac{1}{2}mR^2 & 0 \\ 0 & 0 & \frac{1}{4}mR^2 + \frac{1}{12}mh^2 \end{bmatrix} \begin{bmatrix} \omega_{10} \\ v_0/R \\ 0 \end{bmatrix} =$$

$$= \begin{bmatrix} \Theta_{Sx} & 0 & 0 \\ 0 & \Theta_{Sy} & 0 \\ 0 & 0 & \Theta_{Sz} \end{bmatrix} \begin{bmatrix} \omega_{10} \\ v_0/R \\ 0 \end{bmatrix} = \underline{\underline{\begin{bmatrix} \Theta_{Sx}\,\omega_{10} \\ \Theta_{Sy}\, v_0/R \\ 0 \end{bmatrix}}}$$

Ha tudjuk $\underline{\Pi}_S$-t és $\underline{v}_S$-t akkor lehet kinetikus energiát is számolni:

$$T = \frac{1}{2} m \underline{v}_S^2 + \frac{1}{2} \underline{\omega}_2^T \underbrace{\underline{\underline{\Theta}}_S\, \underline{\omega}}_{\underline{\Pi}_S} = \frac{1}{2} m v_S^2 + \frac{1}{2} [\omega_{10} \;\; v_0/R \;\; 0] \begin{bmatrix} \Theta_{Sx}\,\omega_{10} \\ \Theta_{Sy}\, v_0/R \\ 0 \end{bmatrix} =$$

$$= \underline{\underline{\frac{1}{2} m \left[v_0^2 + \omega_{10}^2 (a^2+b^2)\right] + \frac{1}{2}\left(\Theta_{Sx}\,\omega_{10}^2 + \Theta_{Sy} \frac{v_0^2}{R^2}\right)}}$$

- Ugyanezt a mennyiséget a B pontra

Perdület átredukálásával

1. út

$$\underline{\Pi}_B = \underline{\Pi}_S + \underline{r}_{BS} \times \underline{I}$$

$$\underline{I} = m \cdot \underline{v}_S = m \begin{bmatrix} v_0 \\ \omega_{10} b \\ \omega_{10} \cdot a \end{bmatrix}$$

$$\underline{\Pi}_B = \underline{\Pi}_S + m \begin{vmatrix} i & j & k \\ 0 & a & -b \\ v_0 & \omega_{10} b & \omega_{10} a \end{vmatrix} = \underline{\underline{\begin{bmatrix} \Theta_{Sx}\,\omega_{10} + m\,\omega_{10}(a^2+b^2) \\ \Theta_{Sy}\, v_0/R - m v_0 \cdot b \\ - m v_0 a \end{bmatrix}}}$$

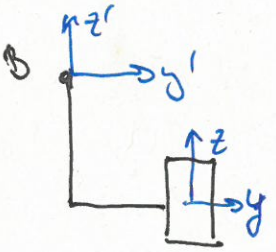
2. út B NEM álló pont!

a kettes test sebesség áll. átredukálva!

$$\underline{\Pi}_B = \underline{\underline{\Theta}}_B \cdot \underline{\omega}_2 + \underline{r}_{BS} \times (m\, \underline{v}_{B_2})$$

$$\underline{\underline{\Theta}}_B = \underline{\underline{\Theta}}_S + \underline{\underline{\Theta}}_{SB} = \begin{bmatrix} \Theta_{Sx} + m(a^2+b^2) & 0 & 0 \\ 0 & \Theta_{Sy} + mb^2 & m \cdot ab \\ 0 & mab & \Theta_{Sz} + ma^2 \end{bmatrix}$$

Steiner



$$\underline{\underline{\Theta}}_B \cdot \underline{\omega}_2 = \begin{bmatrix} \Theta_{Bx} & 0 & 0 \\ 0 & \Theta_{By} & D_{Byz} \\ 0 & D_{Byz} & \Theta_{Bz} \end{bmatrix} \begin{bmatrix} \omega_{10} \\ v_0/R \\ 0 \end{bmatrix} = \begin{bmatrix} \Theta_{Bx}\,\omega_{10} \\ \Theta_{By} \cdot v_0/R \\ D_{Byz} \cdot \frac{v_0}{R} \end{bmatrix}$$

mintha a 2es testen lenne

- $$\underline{v}_{B,\text{Steiner}} = \underline{v}_{B_2} = \underline{v}_S + \underline{\omega}_2 \times \underline{r}_{SB} = \begin{bmatrix} v_0 \\ \omega_{10} b \\ \omega_{10} a \end{bmatrix} + \begin{vmatrix} i & j & k \\ \omega_{10} & v_0/R & 0 \\ 0 & -a & b \end{vmatrix} = \begin{bmatrix} v_0 + \frac{v_0 \cdot b}{R} \\ 0 \\ 0 \end{bmatrix} = \underline{\underline{\begin{bmatrix} \left(1+\frac{b}{R}\right) v_0 \\ 0 \\ 0 \end{bmatrix}}}$$

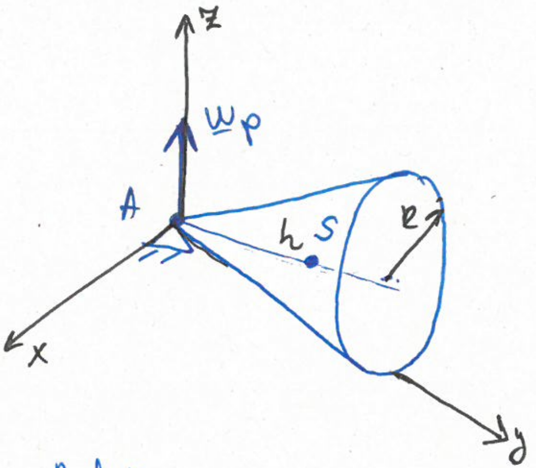
- $$\underline{r}_{BS} \times m\, \underline{v}_{B_2} = m \begin{vmatrix} i & j & k \\ 0 & a & -b \\ \left(1+\frac{b}{R}\right) v_0 & 0 & 0 \end{vmatrix} = \begin{bmatrix} 0 \\ -bm\left(1+\frac{b}{R}\right) v_0 \\ -am\left(1+\frac{b}{R}\right) v_0 \end{bmatrix}$$

Most lehet összegezni:

$$\underline{\Pi}_B = \underline{\Theta}_B \cdot \underline{\omega}_2 + \underline{r}_{BS} \times m\,\underline{v}_{B_2} = \begin{bmatrix} \Theta_{Sx} \cdot \omega_{10} + m\omega_{10}(a^2+b^2) \\ \Theta_{Sy} \cdot \frac{v_0}{R} + \cancel{mb^2 \cdot \frac{v_0}{R}} - bmv_0 \cancel{- mb^2 \frac{v_0}{R}} \\ \cancel{m \cdot a \cdot b \cdot \frac{v_0}{R}} - amv_0 \cancel{- mab\frac{v_0}{R}} \end{bmatrix}$$

$$\underline{\Pi}_B = \begin{bmatrix} \Theta_{Sx}\,\omega_{10} + m\omega_{10}(a^2+b^2) \\ \Theta_{Sy}\frac{v_0}{R} - bmv_0 \\ -amv_0 \end{bmatrix}$$

2. példa Gördülő kúp



Adatok:

$m = 6\,[kg]$

$R = 0{,}2\,[m]$

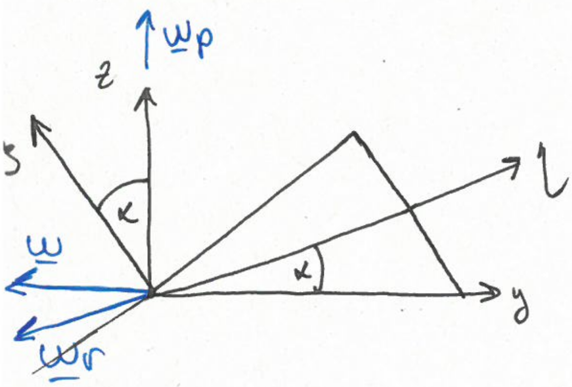
$h = 0{,}5\,[m]$

$\omega_p = 2\,\left[\frac{rad}{s}\right]$

A kúp a palástján gördül. Adjuk meg a kinetikus energiát (T) $\underline{\Pi}_S$-t és $\underline{\Pi}_A$-t

$$T = \frac{1}{2} m v_S^2 + \frac{1}{2} \underline{\omega}^T \underline{\underline{\Theta}}_S \underline{\omega} = \cancel{\frac{1}{2} m v_A^2} + \frac{1}{2} \underline{\omega}^T \underline{\underline{\Theta}}_A \underline{\omega} = \frac{1}{2} \underline{\omega}^T \underline{\underline{\Theta}}_A \underline{\omega}$$

Oldalnézet:



lokálisan az y-tengely körül forog

$$\underset{(\xi,\eta,\zeta)}{\underline{\omega}} = \begin{bmatrix} 0 \\ -\omega\cos\alpha \\ \omega\sin\alpha \end{bmatrix}$$

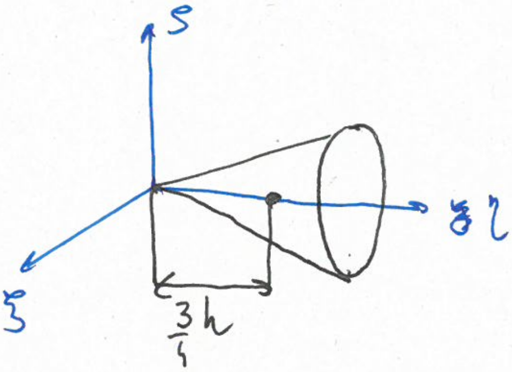
$$\underset{(\xi,\eta,\zeta)}{\underline{\omega}_p} = \begin{bmatrix} 0 \\ \omega_p\sin\alpha \\ \omega_p\cos\alpha \end{bmatrix} \qquad \underline{\omega}_r = \begin{bmatrix} 0 \\ -\omega_r \\ 0 \end{bmatrix}$$

$$\underline{\omega} = \underline{\omega}_p + \underline{\omega}_r$$

$$\begin{bmatrix} 0 \\ -\omega\cos\alpha \\ \omega\sin\alpha \end{bmatrix} = \begin{bmatrix} 0 \\ \omega_p\sin\alpha - \omega_r \\ \omega_p\cos\alpha \end{bmatrix} \rightarrow \omega = \omega_p \frac{\cos\alpha}{\sin\alpha} - \omega_p \mathrm{tg}\,\alpha = \underline{\underline{\omega_p \frac{h}{R}}}$$

$$\hookrightarrow \underline{\omega} = \begin{bmatrix} 0 \\ -\omega_p \frac{h}{R} \cdot \frac{h}{\sqrt{h^2+R^2}} \\ \omega_p \frac{h}{R} \cdot \frac{R}{\sqrt{h^2+R^2}} \end{bmatrix} = \begin{bmatrix} 0 \\ -\omega_p \frac{h^2}{R\sqrt{h^2+R^2}} \\ \omega_p \frac{h}{\sqrt{h^2+R^2}} \end{bmatrix} = \begin{bmatrix} 0 \\ \omega_\eta \\ \omega_\zeta \end{bmatrix}$$

$$\underline{\underline{\Theta}}_S = \begin{bmatrix} \frac{3}{80} m\left(4R^2+h^2\right) & 0 & 0 \\ 0 & \frac{3}{10} mR^2 & 0 \\ 0 & 0 & \frac{3}{80} m\left(4R^2+h^2\right) \end{bmatrix}$$



$$\underline{\Pi}_S = \underline{\underline{\Theta}}_S \cdot \underline{\omega} = \begin{bmatrix} \Theta_\xi & 0 & 0 \\ 0 & \Theta_\eta & 0 \\ 0 & 0 & \Theta_\zeta \end{bmatrix} \cdot \begin{bmatrix} 0 \\ \omega_\eta \\ \omega_\zeta \end{bmatrix} = \begin{bmatrix} 0 \\ \Theta_\eta \omega_\eta \\ \Theta_\zeta \omega_\zeta \end{bmatrix} = \begin{bmatrix} 0 \\ -\omega_p \frac{h^2}{R} \frac{1}{\sqrt{h^2+R^2}} \cdot \frac{3}{10} mR^2 \\ \omega_p \frac{h}{\sqrt{h^2+R^2}} \cdot \frac{3}{80} m\left(4R^2+h^2\right) \end{bmatrix}$$

$$\underline{\Pi}_S = \begin{bmatrix} 0 \\ -\omega_p \frac{h^2}{\sqrt{h^2+R^2}} \cdot \frac{3}{10} mR \\ \omega_p \frac{h}{\sqrt{h^2+R^2}} \cdot \frac{3}{80} m\left(4R^2+h^2\right) \end{bmatrix} = \begin{bmatrix} 0 \\ -0{,}334 \\ 0{,}171 \end{bmatrix} \left(\frac{kg\, m^2}{s}\right)$$

$$\underline{\Pi}_A = \underline{\underline{\Theta}}_A \cdot \underline{\omega}$$

$$\hookrightarrow \underline{\underline{\Theta}}_A = \underline{\underline{\Theta}}_S + \begin{bmatrix} m\left(\frac{3}{4}h\right)^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & m\left(\frac{3}{4}h\right)^2 \end{bmatrix}$$

$$\underline{\Pi}_A = \begin{bmatrix} 0 \\ \Theta_\eta \cdot \omega_\eta \\ \Theta_\zeta \cdot \omega_\zeta + \left(\frac{3}{4}h\right)^2 \cdot m \cdot \omega_\zeta \end{bmatrix} = \begin{bmatrix} 0 \\ -0{,}334 \\ 1{,}737 \end{bmatrix} \left(\frac{kg\, m^2}{s}\right)$$

Kinetikus energia

$$T = \frac{1}{2}\,\underline{\omega}^T \underbrace{\underline{\underline{\Theta}}_A\,\underline{\omega}}_{\underline{\pi}_A} = \frac{1}{2}\left(\Theta_\eta\,\omega_\eta^2 + \Theta_\xi\,\omega_\xi^2 + \left(\frac{3}{4}h\right)^2 m\,\omega_\xi^2\right) =$$

$$= \frac{1}{2}\left(\frac{3}{10}\,m R^2\left(\omega_p\,\frac{h^2}{R}\,\frac{1}{\sqrt{h^2+R^2}}\right)^2 + \left(\frac{3}{80}\,m\left(4R^2+h^2\right) + m\left(\frac{3}{4}h\right)^2\right)\cdot\omega_p\,\frac{h}{\sqrt{h^2+R^2}}\right)$$

$$= \underline{\underline{2{,}39\ \mathrm{J}}}$$