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Role of the Limit Directions in the Nonsmooth Dynamics of Towed Wheels

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Introduction The vibrations of towed wheels called *shimmy* appears at several engineering problems such as the nose gears of aeroplanes, front wheels of motorcycles, or the wheels of baby carriages and shopping carts. In this paper, we consider two simple models of the towed wheels consisting of rigid bodies in the presence of dry friction between the wheel and the ground (see Fig. 1). Due to the Coulomb friction model, the system has a codimension2 discontinuity in the phase space. Our purpose is to utilize some recent results of nonsmooth dynamics to understand the discontinuous behaviour of these systems.

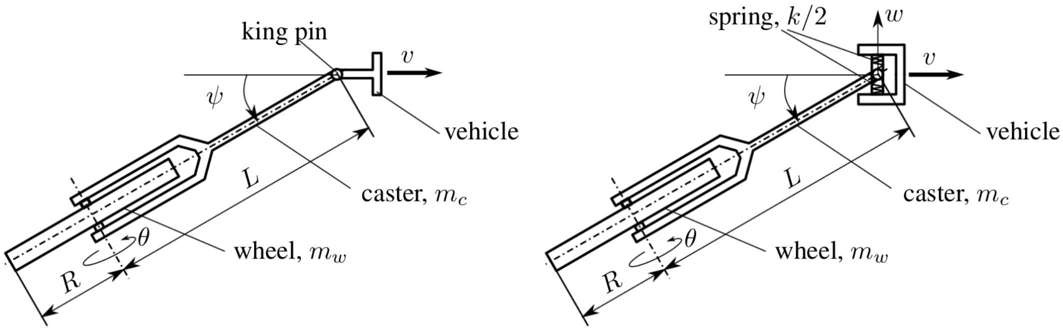

Figure 1: Left panel: the *rigid model* of the towed wheel with two degrees of freedom $(\psi,)$; right panel: the *elastic model* of the towed wheel with three degrees of freedom $(\psi, , w)$. The third degree of freedom is necessary to model the phenomenon of shimmy

The rolling wheel– 2 and 3 DoF models The minimal model for the lateral motion of the towed wheel is a 2 degree-of-freedom (DoF) model, which is referred as the *rigid model.* This model can be seen in the left panel of Fig. 1, where the rotation angle of the wheel is denoted by θ and the angle of the bar is denoted by ψ. Due to the nonholonomic constraint of rolling and the rotational symmetry of the wheel, the dynamics can be reduced to the single kinematic equation $\dot{\psi} = v/L \sin \psi$, where L is the effective length of the caster and v is the constant speed of the vehicle. In this model, the angle ψ of the caster *always tends exponentially to zero without vibrations.*

For the modelling of the vibrations, the elastic deformation w can be introduced as a third degree of freedom, which is related to the finite stiffness of the caster at the king pin. (See the right panel of Fig. 1.) The dynamics of this model leads to a system of three first-order differential equations. It is known from the literature that in this model, *vibrations occur which can be unstable.* (See e.g. [1].) If the geometry or the mass of the bodies changes then the trivial solution $\psi \equiv \omega \equiv w \equiv 0$ can exhibit a subcritical Hopf bifurcation. It can be derived that if the stability condition $m_c L^2 > C m_w R^2$ is satisfied then the trivial solution $\psi \equiv 0$ of the caster angle is asymptotically stable. Here m c and m w is the mass of the caster and the wheel, respectively, R is the radius of the wheel and the dimensionless parameter C depends on the mass distribution of the bodies. For a sufficiently large wheel, $m_c L^2 < C m_w R^2$, and unstable vibrations of the caster appear.

For the linearly unstable parameters, the amplitude of vibrations are increasing exponentially. Moreover, even in the linearly stable case, the amplitudes are increasing outside the unstable limit cycle of the subcritical Hopf bifurcation. In both cases, the growth of the vibrations are limited by different physical effects: For example, nonlinear contact stiffness at the pin becomes important, or the rotating caster can hit other parts of the vehicle, or damage of the parts can occur. A further possible effect is the *slipping* of the wheel on the ground.

Effect of slipping The assumption of a simple Coulomb friction model between rigid bodies can be considered as a limit case of the elastic tyre models when the velocity of the vehicle is very small or the stiffness of the tyre is very large. Moreover, the understanding of the structure of the dynamics can be a basis of the further research on more complicated models.

If the slipping of the wheel is considered then the rolling constraint is released and we have to introduce two further state variables. It is convenient to choose the components u_1 and u_2 of the slipping velocity of the wheel at the contact point in an appropriately chosen coordinate system. Then, the components F_{f1} and F_{f2} of the friction force are determined by the Coulomb friction model in the form

$$F_{f1} = -\mu F_n \frac{u_1}{\sqrt{u_1^2 + u_2^2}}, \qquad F_{f2} = -\mu F_n \frac{u_2}{\sqrt{u_1^2 + u_2^2}} \tag{1}$$

where F_n is the normal force and μ is the friction coefficient. Then the phase space of the *rigid model* is extended from 1 to 3 dimensions, and the phase space of the *elastic model* is extended from 3 to 5 dimensions. In both cases, the rolling condition $u_1 = u_2 = 0$ corresponds to a codimension-2 subset of the phase space where the dynamics of the slipping is discontinuous. The resulting nonsmooth dynamical system is an *extended Filippov system*, and it can be analysed by the mathematical tools published recently [2, 3].

Results It can be shown that the dynamics of the system can change between slipping to rolling strictly along some specific *limit directions*. A limit direction can be described by an angle ϕ in the plane of the phase variables u_1 and u_2 , which angle shows the direction of the slipping velocity at the contact point, as well. It can be proved that in both models of the towed wheel in Fig. 1, four generic scenarios occur with different number and attracting properties (see Fig. 2).

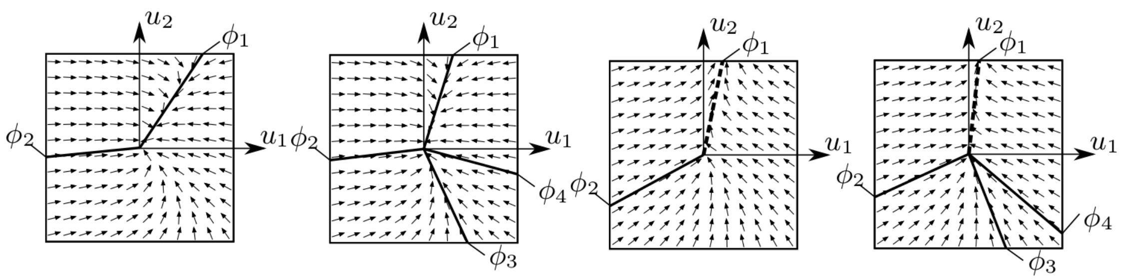

Figure 2: The four generic scenarios of the limit directions in the plane of the slipping velocities u_1 and u_2

In the case of the rigid model, these limit directions determine the number of the possible changes between slipping and rolling before reaching the trivial solution. In the case of the elastic model, the numerical simulations show *stable periodic solutions with slipping and rolling intervals*. Based on the analysis of limit directions, we can understand and categorize the different cases of these vibrations.

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