Discontinuous dynamics of wheels with a towed axis

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Summary. A simple model of wheels with a towed axis is considered. Spatial Coulomb friction with both rolling and slipping behaviour leads to a nonsmooth dynamical system with a codimension–2 discontinuity surface in the phase space, which provides a challenge when calculating the solutions. By using the concept of limit trajectories, attracting or repelling properties of the discontinuity set can be determined, which facilitates the analytical or numerical calculation of switching between rolling and slipping.

Introduction

A simple model of a towed wheel is considered consisting of the wheelset and the towing bar (see Figure 1). It was shown in [1] for a similar model that switching between rolling and slipping can cause chaotic vibrations of the wheelset. When considering the spatial Coulomb friction model, the slipping friction forces become discontinuous as the velocity of the contact point between the wheel and the ground tends to zero. This provides a challenge not only for analytical calculations of the system but also for numerical simulation. In the present work, the rolling-slipping transitions are analysed by the recently developed methods of the authors [2].

Mechanical model

The wheelset and the towing bar are modelled by rigid bodies (see Figure 1). The masses of the wheel and the bar are denoted by m_w and m_b, respectively. The length of the bar is L and the radius of the wheel is R. It is assumed that the connection between the bar and the vehicle is rigid in the longitudinal direction of the bar but it is elastic in the lateral direction with an overall stiffness k at the pin. The vehicle is assumed to have a constant speed v.

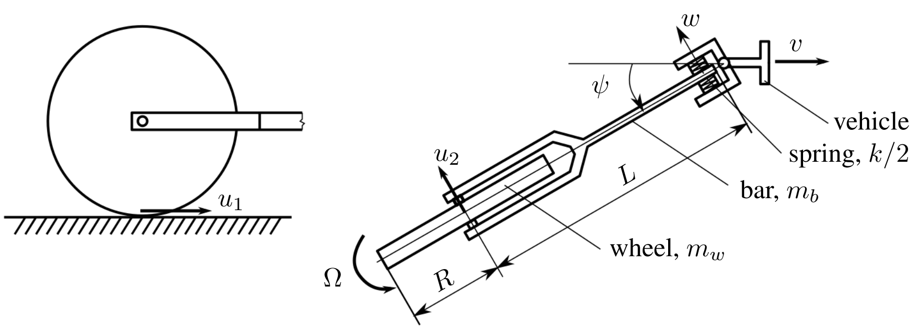

Figure 1: Sketch of the mechanical model.

The angle of the bar around the pin is denoted by ψ, and $\Omega := \dot{\psi}$ is the angular velocity of the bar around the vertical axis. The lateral position of the pin is denoted by w. The longitudinal and lateral components of the velocity of the contact point of the wheel at the ground are denoted by u_1 and u_2, respectively. If the wheel is slipping then Coulomb friction is assumed with a magnitude C of friction force. Then, by applying e.g. the Newton–Euler equations, the dynamics becomes

$$\begin{aligned}
\dot{\psi} &= \Omega, \\
\dot{\Omega} &= -\frac{B_1 k}{mL} \cdot w + -\frac{B_2 C}{mL} \cdot \frac{u_2}{\sqrt{u_1^2 + u_2^2}}, \\
\dot{w} &= v \sin\psi + L\Omega + u_2, \\
\dot{u}_1 &= -v\Omega \sin\psi - \frac{B_3 C}{m} \cdot \frac{u_1}{\sqrt{u_1^2 + u_2^2}}, \\
\dot{u}_2 &= -v\Omega \cos\psi - \frac{B_4 k}{m} \cdot w - \frac{B_5 C}{m} \cdot \frac{u_2}{\sqrt{u_1^2 + u_2^2}},
\end{aligned} \tag{1}$$

where B_1–B_5 are dimensionless parameters depending on m_w, m_b, R and L.

Nonsmooth dynamics

The dynamics of the system is described in the 5 dimensional phase space of the generalised coordinates ψ, w and the pseudo-velocities Ω, u_1, u_2. The slipping dynamics (1) can be written in the form $\dot{x} = F(x)$ with $x = (\psi, \Omega, w, u_1, u_2)$. The spatial Coulomb friction results that (1) is discontinuous in the 3 dimensional set $\Sigma = \{x : u_1 = u_2 = 0\}$, which corresponds to the case when the wheel is rolling. Dynamics of rolling can be determined either directly from (1) by constructing the sliding dynamics by the theory of *extended Filippov systems* [2], or by applying Newtons Second Law to the system with assuming the rolling constraint $u_1 = u_2 = 0$. By both methods, the rolling dynamics becomes

$$\dot{\psi} = \Omega, \qquad \dot{\Omega} = -\frac{A_1 k}{mL} \cdot w - \frac{A_2 v}{L} \Omega \cos \psi, \qquad \dot{w} = v \sin \psi + L\Omega, \qquad \dot{u}_1 = 0, \qquad \dot{u}_2 = 0, \qquad (2)$$

where $A_1 = B_1 + B_4 B_2 / B_5$ and $A_2 = B_2 / B_5$. The trivial solution of (2) is $\psi \equiv \Omega \equiv w \equiv 0$, which corresponds to the rolling of the wheel without any vibrations. Linear stability analysis shows that for $A_2 > 0$, the trivial solution is asymptotically stable; and for $A_2 < 1$, the trivial solution is unstable. In the unstable case, the effect of small perturbations results that the amplitude of the vibration in (2) is increasing until the magnitude of the contact force between the wheel and the ground reaches C. Then, the wheel starts slipping and the system switches from (2) to (1). During slipping, the amplitude of the vibration is moderated by the strong dissipation of the Coulomb friction, and after a while, the wheel starts rolling again. The two effects lead to a sequence of switches of the system between (1) and (2), which result in chaotic behaviour. To follow the changes between rolling and slipping, one should know the directions of the trajectories when leaving or arriving to the discontinuity set Σ. An effective method for doing that is the concept of limit trajectories.

Analyis of rolling-slipping transitions by the concept of limit directions

The unit vectors that are orthogonal to the discontinuity set Σ are denoted by $n(\phi) = (0, 0, 0, \cos \phi, \sin \phi)$, where the parameter $\phi \in [0, 2\pi)$ denotes the angle in the phase space between $n(\phi)$ and the coordinate direction u_1. For any point $x_0 \in \Sigma$, the functions

$$R(\phi) = \lim_{\varepsilon \to 0^+} \left\langle F\big(x_0 + \varepsilon n(\phi)\big), n(\phi) \right\rangle, \qquad V(\phi) = \lim_{\varepsilon \to 0^+} \left\langle F\big(x_0 + \varepsilon n(\phi + \pi/2)\big), n(\phi + \pi/2) \right\rangle \qquad (3)$$

are defined. Roots of the equation $V(\phi) = 0$ give the *limit directions* along which trajectories of the slipping system (1) reach the discontinuity set Σ in forward or backward time. The sign of $R(\phi)$ at the limit directions show whether the limit directions are attracting or repelling, that is, whether the trajectories point towards or away from Σ (see Figure 2.). In the left panel, there are two attracting limit directions, and the wheel preserves rolling. In the right panel, there is an attracting and a repelling limit direction, the wheel starts slipping. These two cases are analogous to the *sliding* and *crossing* regions of simple Filippov systems (see e.g. [3], p. 76).

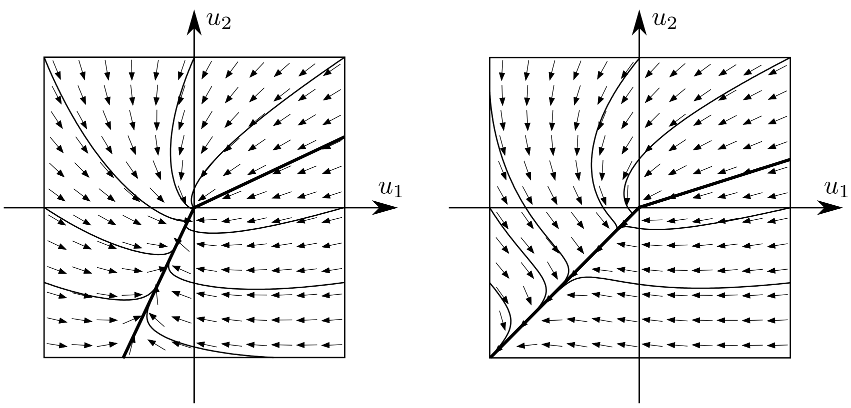

Figure 2: Typical types of behaviour of the dynamics near a point $x_0 \in \Sigma$ visualised by the projection of the phase portrait to the plane $u_1 - u_2$. The point x_0 is located in the origin. The limit directions are denoted by thick lines.

By the limit directions, the condition of rolling can be calculated directly from (1) without calculating the magnitude of the contact force. Limit directions show the direction of trajectories at transition from rolling to slipping. Moreover, limit directions provide the directions of trajectories when they approach the discontinuity set at transition from slipping to rolling. These properties can be used effectively when calculating or simulating the dynamics of the towed wheel.

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