Nonsmooth analysis of a simple rolling-sliding mechanical system with Coulomb friction

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Abstract

If both rolling and sliding is considered between rigid bodies, simple Coulomb friction can cause discontinuities in the dynamics (see e.g. [2]). Conditions of rolling and rolling-sliding transitions can be easily identified in the nonsmooth dynamical system, using the notions of Filippov systems. These correspondences are demonstrated on a simple example of a rolling disk, also considering global behaviour and symbolic dynamics.

1. Mechanical model

Let us consider a rigid disk in 2D, which can roll or slide on the ground according to Coulomb's law. The centre of the disk is connected to the wall by a spring. The displacement and velocity of the centre of the disk is described by u and v, respectively, and the velocity of the contact point is described by w.

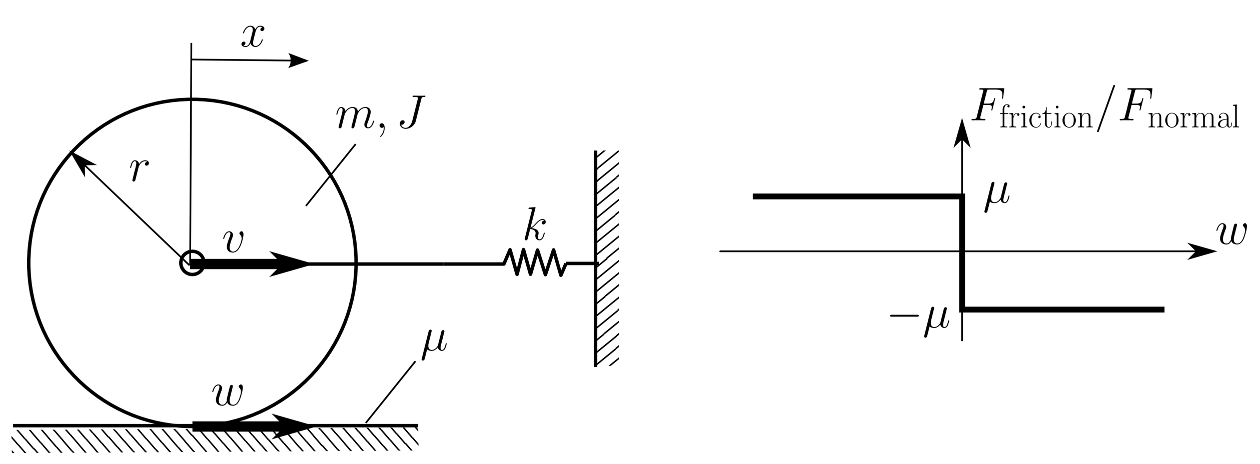

Figure 1: Mechanical model (left) with the characteristic curve of Coulomb's law (right). Notations: m: mass; J: mass moment of inertia; k: stiffness; μ: friction coefficient

The equations of motion of the disk can be transformed into a system of first-order differential equations in the form

$$\dot u = v; \qquad \dot v = -\frac{k}{m}u - \mu g \operatorname{sgn} w; \qquad \dot w = -\frac{k}{m}u - \mu g \frac{1+j}{j}\operatorname{sgn} w \tag{1}$$

for the sliding case ($w \neq 0$) and in the form

$$\dot u = v \qquad \dot v = -\frac{k}{m(1+j)}u \qquad \dot w = 0 \tag{2}$$

for the rolling case ($w \equiv 0$). Here $j := J/(mr^2)$ describes the mass distribution of the disk, $j = 0$ for a particle, $j = 1$ for a circle and $j = 0.4$ for a solid disk. In the rolling case, we also require

$$kj|u| \leq \mu g m(1+j), \tag{3}$$

according to Coulomb's law.

2. Nonsmooth analysis by means of a Filippov system

By transforming the sliding equations (1) into a dimensionless form, we get the Filippov system

$$\begin{pmatrix} \dot x \\ \dot y \\ \dot z \end{pmatrix} = F(x,y,z) := \begin{cases} F^+(x,y,z) & \text{if } H(x,y,z) > 0 \\ F^-(x,y,z) & \text{if } H(x,y,z) < 0 \end{cases} \tag{4}$$

on the phase space $\{(x,y,z) \in \mathbb{R}^3\}$, where the switching function is $H(x,y,z) := z$ and the vector fields are defined by

$$F_+(x,y,z) := \begin{pmatrix} y \\ -(1+j)\left(x + \frac{j}{1+j}\right) \\ -(1+j)(x+1) \end{pmatrix}; \quad F_-(x,y,w) := \begin{pmatrix} y \\ -(1+j)\left(x - \frac{j}{1+j}\right) \\ -(1+j)(x-1) \end{pmatrix} \tag{5}$$

The switching manifold $\Sigma \subset \mathbb{R}^3$, for which $H(\Sigma) = \{0\}$, is now the $z = 0$ plane. In Σ, F is not determined, but an induced "sliding" vector field F^0 can be defined by using for example, Utkin's equivalent method (see [1], p.77.), and we get

$$F^0(x,y,z) = \begin{pmatrix} y \\ -x \\ 0 \end{pmatrix} \tag{6}$$

The induced field F^0 has a sense only in the "sliding region" $\Sigma^0 \subset \Sigma$, where the switching surface is attracting from both sides. In our case, $\Sigma^0 = \{(x,y,0), |x| < 1\}$, which means an infinite strip in Σ, around the $x = 0$ axis.

One can check that the induced dynamics F^0 coincides with the dynamics given from the rolling equations (2). Moreover, the derived Σ^0 set precisely satisfies the rolling condition (3). Thus, (mechanical) sliding motion is the generic behaviour of the system, and rolling motion can be considered as the induced sliding dynamics (in Filippov sense) on the attracting portion of the switching surface.

3. Local behaviour and rolling-sliding transitions

Transitions between the possible dynamic cases (sliding in both directions and rolling) can be explained in the nonsmooth system as follows, see also Figure 2.

- if a (mechanical) sliding trajectory intersects the $z = 0$ plane outside Σ^0, the direction of sliding changes without rolling ($F^+ \to F^-$ or $F^- \to F^+$) (Fig. 2/a)
- if a (mechanical) sliding trajectory intersects the $z = 0$ plane inside Σ^0, it "sticks" into the switching surface and the disk starts rolling ($F^+ \to F^0$ or $F^- \to F^0$) (Fig. 2/b)
- if a rolling trajectory reaches the boundary of Σ^0, it separates from the switching surface, and (mechanical) sliding starts ($F^0 \to F^-$ or $F^0 \to F^+$) (Fig. 2/c)

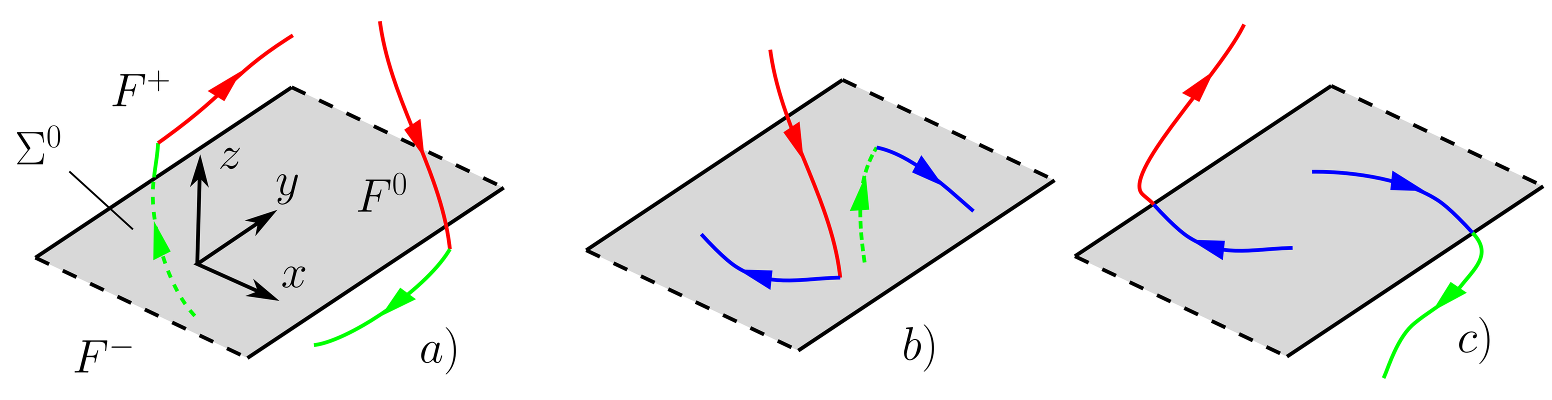

Figure 2: Different types of rolling-sliding transitions in the nonsmooth system

4. Global behaviour and symbolic dynamics

The trajectories of the system are unique in forward direction, but they are not unique in backward direction, which is a usual effect caused by the discontinuity. Due to the energy loss of sliding, the limit set of the system is the unit circle on Σ^0 around the origin, which contains all the circular trajectories of the rolling motion without passing the boundary of Σ_0.

Let us characterise the trajectories by a series of +, 0 and − symbols, according to the rolling or sliding states (F^+, F^0 or F^-) of the dynamics in time. This symbolic dynamical description is useful to distinguish the typical global behaviours of the trajectories. Numerical analysis shows, that the possible long-term behaviour of the trajectories can be either (0) (type A), reaching a continuous rolling state of any circular trajectory, or $(+0-0+0-0\ldots)$ (type B), infinite transitions between sliding and rolling.

Any circular trajectory has a basin of attraction containing two one-parameter families of type A trajectories, characterised by $(+0)$ and (-0). Moreover, the basin of attraction of the unit circle contains all the type B trajectories. Before reaching the periodic $(+0-0)$ cycle, type B trajectories have a wide variety of behaviour, containing rolling and sliding.

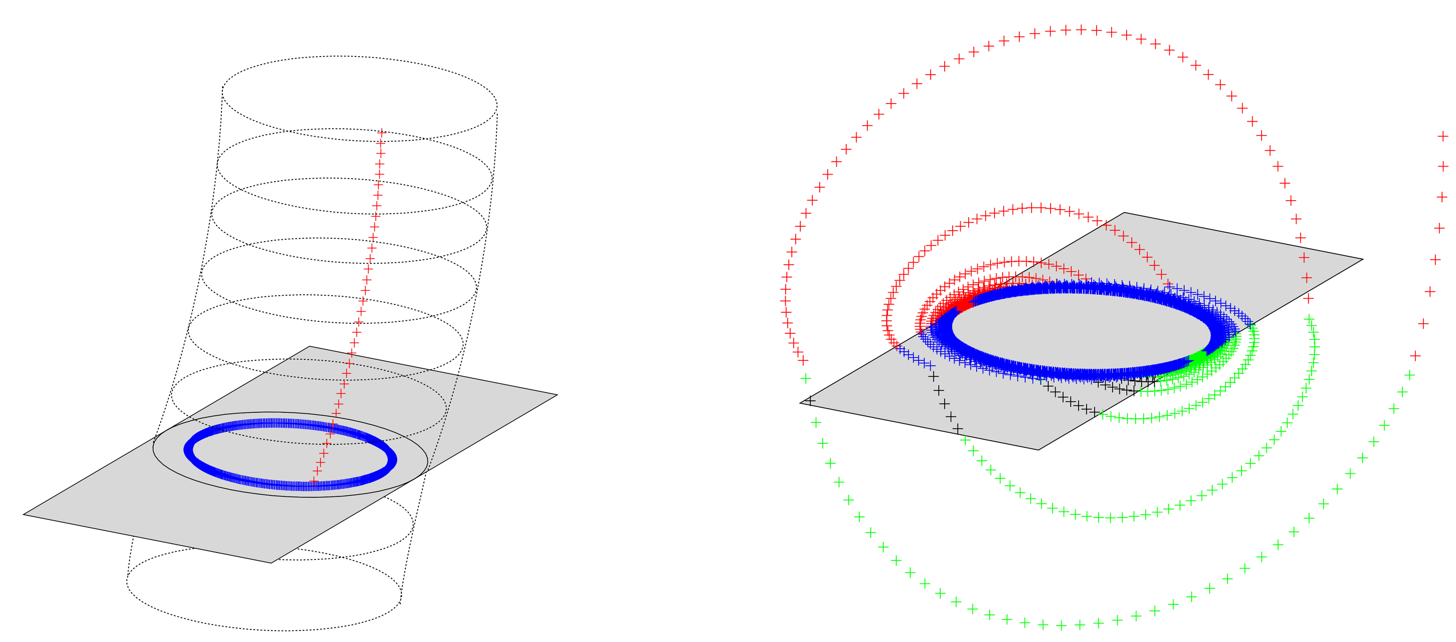

Figure 3: A typical type A trajectory can be seen on the left with the boundary of the region type A and B trajectories. Outside this region, all trajectories tend to the unit circle through infinite number of rolling-sliding transitions. A typical example for these type B trajectories can be seen on the right.

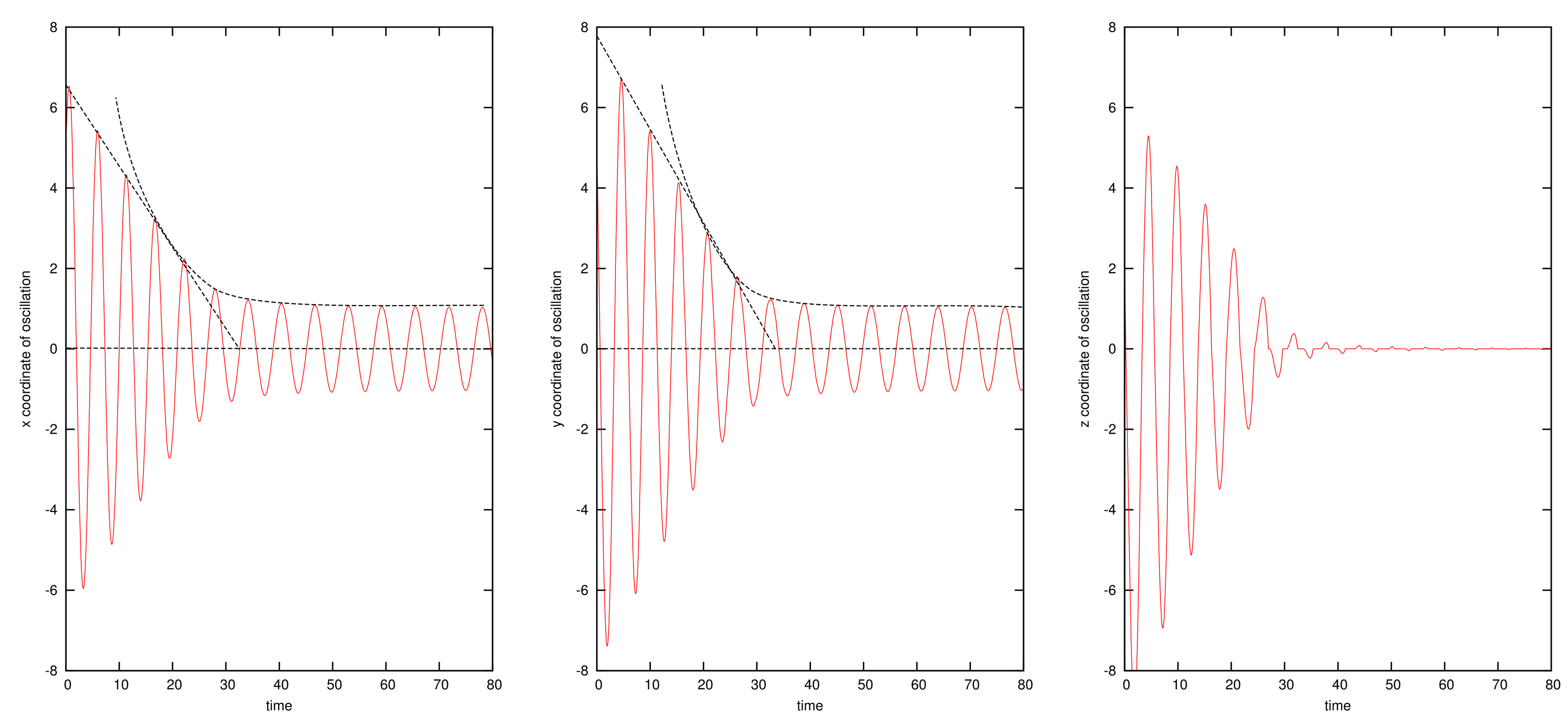

Figure 4: Time histories of a typical type B oscillation. The decrease of the amplitude in state variables x and y is linear at the beginning, and it becomes exponential after a while. On the graph of z, the rolling and sliding intervals can be easily identified.

References

[1] M. di Bernardo et al. (2008) Piecewise-smooth Dynamical Systems. Springer

[2] G. Stepan (1991) Chaotic motion of wheels *Vehicle System Dynamics* **20**:341-351

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