PAMM · Proc. Appl. Math. Mech. **14**, 303 – 304 (2014) / **DOI** 10.1002/pamm.201410139

Nonlinear kinematic oscillations of railway wheelsets of general surface geometry

Mate Antali[1,*] and **Gabor Stepan**[1]

[1] Department of Applied Mechanics, Budapest University of Technology and Economics, Budapest, H-1521, Hungary

Kinematic oscillation of railway wheelsets is investigated in the case of a straight track, focusing on the nonlinear effects of the geometry of wheel and rail profiles. Euler angles are used to describe the motion of the wheelset, the contacting bodies are modelled by two-parametric rigid surfaces. The surface parameters of the current contact points are followed during the calculations. The curvatures and their variations are also included in the analysis. From the implicit form of the geometric constraints, the dynamics is determined by means of Taylor expansion. Equations of the local nonlinear motion are derived and the qualitative behaviour of the system is given.

1 Introduction

Kinematic oscillation of railway wheelsets during their rolling along a straight track is one of the fundamental phenomena of railway dynamics, which is induced by the dual-point rolling contact of the wheelset on the rails. This kinematic oscillation is highly effected by the local geometry of the contact surfaces. The first mathematical description of kinematic oscillations was given by Klingel [1] for the case of conical wheelsets and „square" rails. Heumann [2], Joly [3] and others extended Klingel's concept for different geometries, but the nonlinear kinematics related to the curved profiles has not been explored in details yet. The authors of this paper recently determined and analysed the exact nonlinear governing equations of a conical wheelset rolling on cylindrical rails [4]. The formerly applied methods are now generalized for the local nonlinear dynamics of these systems in case of arbitrary wheelset and railway profiles.

2 Mechanical model

The connection between the wheelset and the rails can be characterised by the nominal radius r and the nominal semi-gauge d, which is shown in Fig. 1. The local geometry of the rails can be defined by two-parametric surfaces $\mathbf{f}_r^+$ and $\mathbf{f}_r^-$, which are created by the extrusion of a profile curve c_r along the direction of the rails. In central position (see Fig.1), the local geometry of the wheels can defined by the surfaces $\tilde{\mathbf{f}}_w^+$ and $\tilde{\mathbf{f}}_w^-$, created by revolving a profile curve c_w around the symmetry axis of rotation. The nominal conicity is denoted by $h := c_{1w} = c_{1r}$, where c_{nr} and c_{nw} denotes the n^{th} derivative of the profile at the nominal contact point. Using the coordinate system $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ of Fig. 1 and 2, the surfaces can be written as

$$\mathbf{f}_r^\pm(\gamma,\chi) = \begin{bmatrix} \xi \\ \pm(d+\gamma) \\ -r + c_r(\gamma) \end{bmatrix} \qquad \text{and} \qquad \tilde{\mathbf{f}}_w^\pm(\chi,\delta) = \begin{bmatrix} (-r + c_w(\chi))\sin\delta \\ \pm(d+\chi) \\ (-r + c_w(\chi))\cos\delta \end{bmatrix}. \tag{1}$$

To get an arbitrary position of the wheels, the surfaces are mapped by a general rigid body transformation $\mathbf{u}$, that is, $\mathbf{f}_w^\pm(\chi,\delta) := \mathbf{u} \circ \tilde{\mathbf{f}}_w^\pm(\chi,\delta)$, where $\mathbf{u}$ is defined by

$$\mathbf{u}(\mathbf{r}) := \mathbf{R}\,\mathbf{r} + \mathbf{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\vartheta & -\sin\vartheta \\ 0 & \sin\vartheta & \cos\vartheta \end{bmatrix} \cdot \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos\varphi & 0 & \sin\varphi \\ 0 & 1 & 0 \\ -\sin\varphi & 0 & \cos\varphi \end{bmatrix} \cdot \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} + \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \tag{2}$$

Here x, y, z are the longitudinal, lateral and vertical displacements of the centre of the wheelset, respectively, and the Euler angles ϑ, ψ, φ are called pitch, yaw and roll angles, respectively. The Lagrangian velocity field can be defined simply by $\mathbf{v} := \dot{\mathbf{u}}$, the dot denotes differentiation with respect to time.

3 Derivation of the equation of motion

The corresponding surfaces coincide at the contact points and they have a common tangent plane, that is

$$\mathbf{f}_w^\pm(\chi^\pm,\delta^\pm) = \mathbf{f}_r^\pm(\xi^\pm,\gamma^\pm), \qquad \text{and} \qquad \mathrm{Span}(\partial_1\mathbf{f}_w^\pm, \partial_2\mathbf{f}_w^\pm)(\chi^\pm,\delta^\pm) = \mathrm{Span}(\partial_1\mathbf{f}_r^\pm, \partial_2\mathbf{f}_r^\pm)(\xi^\pm,\gamma^\pm) \tag{3}$$

* Corresponding author: e-mail antali@mm.bme.hu, phone +36 1 463 1369, fax +36 1 463 3471

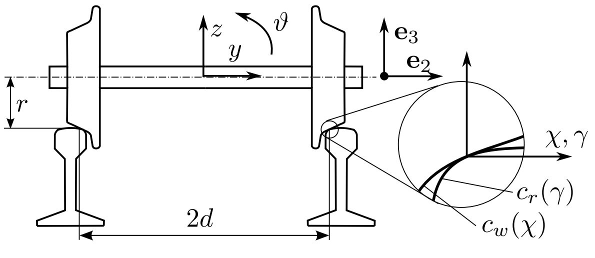

Fig. 1: Front view of the model

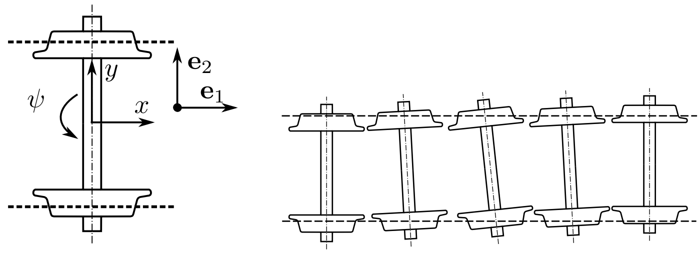

Fig. 2: Top view of the model and mechanism of kinematic oscillation

hold, where $\chi^{\pm}$, $\delta^{\pm}$, $\xi^{\pm}$ and $\gamma^{\pm}$ are the coordinates of the contact points on the surfaces, and ∂_1 and ∂_2 denotes partial differentiation with respect to the first and second variable. These coordinates of the contact points and the six state variables of the rigid body result a 14-dimensional configuration space, which is restricted by 10 scalar holonomic constraints included in the two equations of (3). If we choose $(\psi, \vartheta, x, \varphi)$ for the set of generalised coordinates, then these constraints can be expressed formally as

$$g_i(\psi, \vartheta, x, \varphi, X) = 0, \qquad\qquad i = 1 \ldots 10, \tag{4}$$

where $X := (y, z, \chi^+, \chi^-, \delta^+, \delta^-, \xi^+, \xi^-, \gamma^+, \gamma^-)$. The dependence of X on the generalised coordinates cannot be determined explicitly, but Taylor expansion around the trivial solution can be calculated by implicit differentiation of (4).

In the case of single-point rolling, the velocity of the wheelset is zero at the contact points, thus $\mathbf{v} \circ \mathbf{f}_w^{\pm}(\chi^{\pm}, \delta^{\pm}) = 0$ is satisfied. This condition does not provide 6 independent scalar equations, but it results 3 scalar nonholonomic constraints, which can be written formally in the form of

$$\dot{\psi} + f_\psi(\psi, \vartheta, X)\,\dot{\varphi} = 0, \qquad \dot{\vartheta} + f_\vartheta(\psi, \vartheta, X)\,\dot{\varphi} = 0, \qquad \dot{x} + f_x(\psi, \vartheta, X)\,\dot{\varphi} = 0. \tag{5}$$

In the case of constant rolling velocity, the rolling angle can also be restricted by $\dot{\varphi} \equiv \omega$, and so we have a first-order differential equation for each generalized coordinate. By performing third-order Taylor expansion, the system can be written in the form

$$\begin{aligned} \dot{\psi} &= a_{01}\vartheta + a_{03}\vartheta^3 + a_{21}\vartheta\psi^2 + \mathcal{O}^5(\psi, \vartheta) \qquad & \dot{x} &= c_{00} + c_{20}\psi^2 + c_{02}\vartheta^2 + \mathcal{O}^4(\psi, \vartheta) \\ \dot{\vartheta} &= b_{10}\psi + b_{30}\psi^3 + b_{12}\psi\vartheta^2 + \mathcal{O}^5(\psi, \vartheta) \qquad & \dot{\varphi} &= \omega \end{aligned} \tag{6}$$

4 Linear and nonlinear behaviour of the system

The trivial solution of (6) is $(\psi, \vartheta, x, \varphi) = (0, 0, x_0 + c_{00}t, \varphi_0 + \omega t)$. The first two equations of (6) can be investigated separately, and after an appropriate nonlinear transformation $(\psi, \vartheta) \to (u, v)$, they can be written in the form

$$\begin{bmatrix} \dot{u} \\ \dot{v} \end{bmatrix} = \begin{bmatrix} 0 & -\alpha \\ \alpha & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \beta\,(u^2 + v^2) \begin{bmatrix} 0 & -\alpha \\ \alpha & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \mathcal{O}^5(u, v), \tag{7}$$

where the connections between the parameters are

$$\alpha = \omega \sqrt{\frac{c_{2r}}{c_{2r} - c_{2w}} \cdot \frac{hr}{d} \left(1 + \frac{h}{c_{2r}d}(1 + h^2)\right)} \qquad \text{and} \qquad \beta = \beta\,(d, r, h, c_{2r}, c_{3r}, c_{4r}, c_{2w}, c_{3w}, c_{4w})\,. \tag{8}$$

The origin of the linear system is a centre, the angular frequency of small oscillations is given by α. We obtained the general formula for α, from which the results of [1], [2], [3] and [4] can be obtained as limit cases. From the symmetries of (6), it can be proven that the full nonlinear system has also a centre at the origin, i.e., the trajectories of finite oscillations are also closed, the trivial solution is neutrally stable (see also [4]). The nonlinear parameter β characterises the dependence of the angular frequency on the amplitude of finite oscillations.

The linear angular frequency α is effected by the profiles up to their second derivatives at the contact points; for obtaining the nonlinearity parameter β, the third and fourth derivatives are also required. The exploration of the exact parameter dependence of β in (8) is a further task.

References

[1] J. Klingel, Organ für die Fortschritte des Eisenbahnwesens **38**, 113–123 (1883).
[2] H. Heumann, Organ für die Fortschritte des Eisenbahnwesens **92**, 149–173 (1937).
[3] R. Joly, Revue Francaise de Mecanique **36**, 5–26 (1970).
[4] M. Antali, G. Stepan, and J. S. Hogan, Multibody System Dynamics (2014), accepted, under publication.

