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Effects of Varying Dynamics of Flexible Workpieces in Milling Operations

In this study, surface error calculations and stability conditions are presented for milling operations in case of slender parts. The dynamic behavior of the flexible beam-type workpiece is modeled by means of finite element method (FEM), while the varying dynamical properties related to the feed motion as well as the material removal process are incorporated in the model. The FEM-generated direct frequency response function is verified through a closed-form solution based on the distributed transfer function method. Relative errors and convergence of the FEM are investigated based on the analytical solutions of the continuum model, from which appropriate element size and mode number can be selected for modal coordinate transformations. The pattern in the variation of the natural frequencies is explored using the analytical model in case of high radial depth of cut relative to the original cross section of the beam-like workpiece. Both the stability conditions and the resulted surface errors are predicted as a function of the tool position. The presented approach and the results are validated by laboratory tests. [DOI: 10.1115/1.4045418]

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1 Introduction

Cutting is a widely used manufacturing method; however, this kind of machining process can be disturbed by harmful vibrations that can reduce the life of the machine tool and the surface quality. It is an important task to provide a reliable prediction of machine tool vibrations, both in terms of increasing productivity and minimizing costs and losses. The most accepted explanation for the large amplitude machine tool vibrations (chatter) is the so-called surface regenerative effect [1], where the position of the previous cutting edge influences the chip cross-section [2], which can be described by delayed differential equations [3].

The range of chatter-free technological parameters is usually represented in so-called stability charts, which are determined by the stability of the stationary solution of the corresponding mathematical model. In the last decades, several numerical methods have been developed to calculate these stability charts in order to support the technological design. Without completeness, we mention:

(i) the methods in the time domain such as the semi-discretization method [4], the full-discretization method [5], the integration method [6], the Chebyshev collocation method [7], and the temporal finite element analysis [8];
(ii) the methods in the frequency domain such as the zero-order approximation [9], the multi-frequency solution (MFS) [10], or the extended multi-frequency solution (EMFS) [11].

In the time domain, the dynamical behavior of the model can be described by means of the identification of the modal parameters, which is a complex engineering procedure itself. An advantage of the frequency domain solutions is that they can directly use the measured frequency response function (FRF) after applying some filtering techniques.

For high-precision machining, for example, the errors of the machined surface are still relevant even in case of stable machining operations. These surface errors may appear during both turning and milling processes. In the case of stable turning, there is an offset error generated by the constant cutting force [12]. Due to the intermittent characteristics of milling processes, relative vibrations are induced between the tool and the workpiece even in case of stable cutting operations. Consequently, not just the so-called surface location error (*SLE*) appears, but also the surface roughness (*R*) [13–15]. In Ref. [16], the surface location errors and the stability lobe diagram are presented together on a so-called superchart.

It is well known that the dynamical characteristics of the CNC machine and the workpiece significantly influence the behavior of the machining process. It is an essential task to predict this dynamical behavior precisely in order to construct a reliable superchart, from which, one can select robust parameter combinations to reach a high-material removal rate [17]. The dynamical properties may vary during machining due to the different configurations of the machine tool structure within the workspace [18–21] and due to the variation of the workpiece geometry according to the material removal [17,22–24]. The latter is dominant for thin-walled structures, so it is essential to take these changes into account. Finite element method (FEM) is a natural choice for modeling thin-walled workpieces and to trace the dynamical properties during material removal [25–27].

Although the finite element method can be used for any complex workpiece geometry, the model and the associated FE mesh have to be updated frequently to make it follow the variation of the geometry due to the material removal process. This requires significant computational time and effort [25]. The modal matrices of the FEM are quite large for an appropriately densed finite element mesh. Thus, it requires significant computational effort to obtain the stationary solutions of the milling process and to determine their stability. One way to decrease the size of the matrices and to save computational cost is the application of modal coordinate transformation [28] by considering only a few dominant vibration modes [29].

The sufficient number of modes and the resolution of the FE mesh have to be selected appropriately to reach a proper balance between accuracy and computational needs. These parameters are usually determined based on test cases for which an analytical solution exists.

In this study, we analyze the dynamical behavior of a flexible workpiece model that can be used in both for turning and for milling processes. For this reason, we use a beam-type workpiece, which is complex enough to capture all aspects of the varying workpiece dynamics, but simple enough for analytical modeling. Note

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that for complex models such as turbine blade, pocket milling, rib machining, cutting tubular, and thin-walled parts, the change in the dynamic properties are very problem specific and it is difficult to draw a general conclusion.

First, the model based on FEM is generated for Bernoulli beam elements, which is later compared with the exact FRF of the workpiece. Using a suitable Green function, the closed analytical form is based on the distributed transfer function method (DTFM) [30]. This eliminates the need for finite element discretization and the need for solving the large eigenvalue-eigenvector problem, which may require high-computational resources [31]. Then, we examine the FE mesh resolution and the number of considered modes in the modal transformation to obtain the parameters of sufficiently accurate FRF for stability and surface error calculations.

Usually, in the case of a general industrial machining process, the technological parameters are already given, such as tool path relating to workpiece geometry, cutting tool, and other machining parameters, they are technology-dependent. Modifications only in the spindle speed and the feed rate are allowed due to these limitations from the industrial side.

With this model, we examine in detail how dynamic properties change as a function of the tool position according to the material removal process of cutting. In the next step, stability chart and surface property calculations are developed using an FRF-based model for milling operation. The variations of the dynamic properties caused by the material removal and the changing tool position are also considered. Finally, the predicted phenomena are identified by measurement results.

The paper is organized as follows: in Sec. 2, we provide the FRF based on FEM model of the workpiece which is validated through analytical FRF. Then, in Sec. 3, the effects of the feed motion are analyzed with respect to the dynamical parameters. In this section, an extreme case is illustrated, from which, one can get a picture into the changing dynamics of the underlying system. Section 4 gives the governing equation of the milling process together with the corresponding stability and surface error calculations. A case study is conducted in Sec. 5 with numerical and measurement results, where we analyze the machined surface quality and investigate the effects of the feed motion. Finally, in Sec. 6, we summarize conclusions and discuss future research directions.

2 Dynamics of the Workpiece

In this section, we describe the mechanical model of the cutting process, with special attention on the dynamic behavior of the flexible workpiece while the material removal process is also taken into account.

During the machining process, the cutting tool's actual position along its path determines the location of the contact region, that is, the resulting cutting force acts only at the actual contact point identified by the coordinate e in Fig. 1. In the dynamical models, only the relative vibrations between the tool and the workpiece are included. Therefore, only the direct FRF, with excitation and response at the same position, have to be taken into account in the stability and surface error calculations.

From the dynamical point of view, the workpiece dynamics can change significantly during the cutting process in two different ways [32]. One corresponds to the varying contact position, which is taken into account by means of the variation of the modal stiffness through the corresponding mode shapes. The other one relates to the effects of the material removal process. According to the feed motion, the cross section is reduced along the preceding tool path. This leads to changing workpiece geometry and correspondingly varying dynamical parameters, especially in the natural frequencies of the workpiece.

Due to the reason that one of the main scopes of this contribution is to explore the behavior of the varying dynamics, the workpiece is assumed to be much more flexible when compared with the cutting tool together with the whole machine tool structure. This

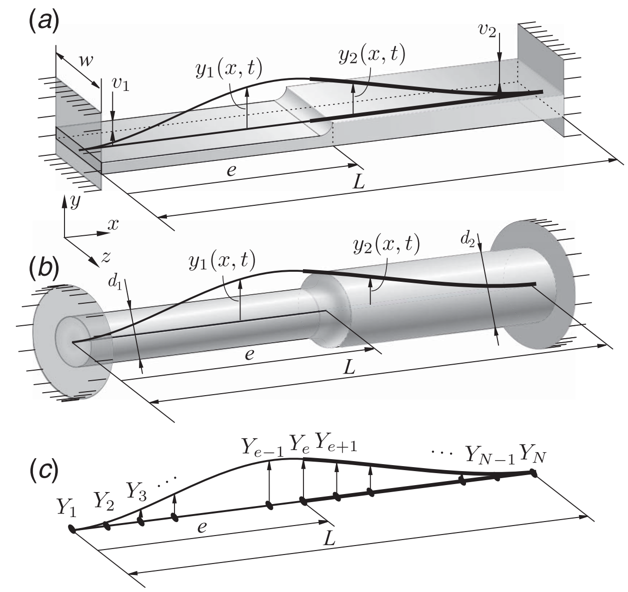

Fig. 1 Schematic figure of the applied beam-type workpiece considering the material removal for (a) milling, (b) turning processes, and (c) discrete model of the flexible workpiece

assumption is valid for thin-walled workpieces [19], as shown for example in Fig. 1. Thus, the cutting tool and the workpiece are assumed to be rigid and flexible, respectively. Note that for a given workpiece geometry, the appropriate modal matrices can be extracted from most of the industrial FEM software. In addition, one can include the dynamic characteristic of the cutting tool and the machine tool and can extend the model by means of measuring and adding the tool tip FRF to the computed workpiece FRF [18,21,33].

2.1 Discrete Model of the Workpiece.

The mechanical model of the flexible workpiece during the machining process is shown in Fig. 1(c). It is assumed that the transverse dimensions are negligible relative to the longitudinal one. In this case, the horizontal (longitudinal) vibration is not modeled, since the structure is more rigid in this direction when compared with the transverse one. According to these assumptions, bending vibration dominates the model, which justifies the consideration of a beam model. For the actual tool position e, the governing equation assumes the form

$$\mathbf{M}(e)\ddot{\mathbf{Y}}(t) + \mathbf{C}(e)\dot{\mathbf{Y}}(t) + \mathbf{K}(e)\mathbf{Y}(t) = \mathbf{v}(e)F_y(t) \tag{1}$$

where the deformation of the workpiece is described by the N-dimensional generalized coordinate vector $\mathbf{Y}$ and the dynamical parameters are defined by means of the mass, damping, and stiffness matrices as $\mathbf{M}(e)$, $\mathbf{C}(e)$, and $\mathbf{K}(e)$, respectively. On the right-hand side, the general force vector is composed as a product. Its first term is the vector $\mathbf{v}(e)$ that represents the actual contact position; all its elements are zero except the unitary one of the element at the $\mathrm{int}(1+(N-1)e/L)^{\mathrm{th}}$ coordinate. The second term $F_y(t)$ describes the cutting-force component in the y-direction, which is time-periodic in case of the milling process. Following the assumption in Refs. [34,35], the dynamics of the feed motion is considered to be slow relative to the fast dynamics of $\mathbf{Y}(t)$ and $F(t)$, that is, the variation of the tool position e is considered to be quasi-static [36].

Up to this point, Eq. (1) can describe both milling and turning processes. The only difference between them is the cutting force on the right-hand side, which is discussed in detail in Sec. 4.

According to the theory of modal analysis [29], in the case of proportional damping, the equation of motion in Eq. (1) can be given as decoupled equations of each modal coordinate ξ_k. After transforming to the frequency domain of ω, it reads as

$$-\omega^2\xi_k(\omega)+\mathrm{i}\omega 2\zeta_k(e)\omega_{\mathrm{n},k}(e)\xi_k(\omega)+\omega_{\mathrm{n},k}^2(e)\xi_k(\omega)=\phi_k(\omega) \tag{2}$$

where $\phi_k(\omega)$ is the modal force of the kth mode shape, $\omega_{\mathrm{n},k}(e)$ is the kth undamped natural angular frequency, and $\zeta_k(e)$ is the modal damping of the kth mode shape ($k=1, 2, \ldots, m$, where m is the number of the modes). In case of proportional damping, the damping coefficients $\zeta_k(e)$ can be calculated as

$$\zeta_k(e)=\frac{\alpha_\mathrm{M}}{2\omega_{\mathrm{n},k}(e)}+\frac{\alpha_\mathrm{K}}{2}\omega_{\mathrm{n},k}(e) \tag{3}$$

where α_M and α_K are the proportional damping coefficients in $\mathbf{C}=\alpha_\mathrm{M}\mathbf{M}+\alpha_\mathrm{K}\mathbf{K}$ (see Ref. [29]). With a FEM-based model, the slow variation of the FRF can be given for each mode. The corresponding scalar-valued direct FRF $H(\omega, e)$ is given as a function of the frequency ω and the tool position e:

$$\tilde{H}(\omega, e)=\sum_{k=1}^{\tilde{m}}\frac{\tilde{T}_{\mathrm{e},k}^2(e)}{-\omega^2+\mathrm{i}\omega 2\tilde{\zeta}_k(e)\tilde{\omega}_{\mathrm{n},k}(e)+\tilde{\omega}_{\mathrm{n},k}^2(e)} \tag{4}$$

where $\tilde{T}_{\mathrm{e},k}(e)$ is the element of the kth mass normalized mode shape vector at the contact point and $\tilde{m}\le m$ is the number of relevant modes. Tilde denotes the approximation of the exact FRF. The discrete model tends to the exact FRF only if the element size of the corresponding FE mesh tends to zero and the number of considered modes tends to infinity.

2.2 Distributed Parameter Model. In order to validate the FEM model, we present the mathematical background of the closed-form expression of FRF through the derivation of the bending vibration of continuum beams based on DTFM [30]. Since there is a sharp change in the geometry of the cross section at the tool position e, the total length L of the workpiece is distinguished with index i along the x coordinate of the beam. In the displacement function $y_i(x, t)$, the subscript i refers to the corresponding section: $i=1$ for the part that has already been machined ($x\in[0, e]$), while $i=2$ for the intact part ($x\in[e, L]$). The linearized equations of motion form the partial differential equations [37–39]

$$\begin{aligned}&\rho A_i\ddot{y}_i(x,t)+c_{\mathrm{M}i}\dot{y}_i(x,t)+c_{\mathrm{K}i}\dot{y}_i''''(x,t)+I_{zi}Ey_i''''(x,t)\\&=\frac{1}{2}F_y(x,t)\delta(x-e),\quad i=1,2\end{aligned} \tag{5}$$

with the corresponding boundary conditions

$$\begin{aligned}y_1(0,t)=0,\quad y_1'(0,t)=0\\ y_2(L,t)=0,\quad y_2'(L,t)=0\end{aligned} \tag{6}$$

at the clamped ends, and the interface conditions

$$\begin{aligned}y_1(e,t)=y_2(e,t),\qquad y_1'(e,t)=y_2'(e,t)\\ I_{z1}y_1''(e,t)=I_{z2}y_2''(e,t),\quad I_{z1}y_1'''(e,t)=I_{z2}y_2'''(e,t)\end{aligned} \tag{7}$$

that describe the smooth connection between the two segments. Prime stands for differentiation with respect to the spatial coordinate x. The material parameters ρ and E are the density and the young modulus, respectively. Also, $c_{\mathrm{M}i}$ and $c_{\mathrm{K}i}$ are the proportional damping coefficients, which are related to the proportional damping coefficients (see in Eq. (3)): $c_{\mathrm{M}i}=\alpha_\mathrm{M}\rho A_i$ and $c_{\mathrm{K}i}=\alpha_\mathrm{K}I_{zi}E$.

The cross sections are characterized with area A_i and area moment of inertia (or second moment of area) I_{zi}. Note that the governing equation (5) is suitable to model the beam-like workpiece during milling and turning processes (see Figs. 1(*a*) and 1(*b*)), but the parameters A_i and I_{zi} are calculated with different formulas.

Laplace transform of Eq. (5) leads to

$$\begin{aligned}&(s^2\rho A_i+sc_{\mathrm{M}i})\bar{y}_i(x,s)+(sc_{\mathrm{K}i}+I_{zi}E)\bar{y}_i''''(x,s)\\&=\frac{1}{2}\overline{F}_y(x,s)\delta(x-e),\quad i=1,2\end{aligned} \tag{8}$$

where $\bar{y}(x,s)=\mathcal{L}(y(x,t))$ and $\overline{F}_y(x,s)=\mathcal{L}(F_y(x,t))$ are the Laplace transforms of $y(x, t)$ and $F_y(x, t)$, respectively, and $s\in\mathbb{C}$ represents the complex Laplace domain.

We introduce the following state space vector:

$$\boldsymbol{\eta}_i(x,s)=\begin{pmatrix}\bar{y}_i(x,s) & \bar{y}_i'(x,s) & \bar{y}_i''(x,s) & \bar{y}_i'''(x,s)\end{pmatrix}^\mathrm{T} \tag{9}$$

and the corresponding excitation vector

$$\boldsymbol{\Phi}_i(x,s)=\begin{pmatrix}0 & 0 & 0 & \dfrac{\overline{F}_y(x,s)\delta(x-e)}{2(sc_{\mathrm{K}i}+I_{zi}E)}\end{pmatrix}^\mathrm{T} \tag{10}$$

Therefore, the governing equation and the corresponding boundary conditions in Eqs. (5) and (6)–(7) can be written as a first-order ordinary differential equation

$$\boldsymbol{\eta}'(x,s)=\mathbf{A}(s)\boldsymbol{\eta}(x,s)+\boldsymbol{\Phi}(x,s) \tag{11}$$

with boundary conditions

$$\mathbf{P}\boldsymbol{\eta}(0,s)+\mathbf{Q}\boldsymbol{\eta}(e,s)+\mathbf{R}\boldsymbol{\eta}(L,s)=\mathbf{0} \tag{12}$$

where $\boldsymbol{\eta}(x,s)=(\boldsymbol{\eta}_1(x,s)^\mathrm{T}\ \boldsymbol{\eta}_2(x,s)^\mathrm{T})^\mathrm{T}$ and $\boldsymbol{\Phi}(x,s)=(\boldsymbol{\Phi}_1(x,s)^\mathrm{T}\ \boldsymbol{\Phi}_2(x,s)^\mathrm{T})^\mathrm{T}$. The coefficient matrix of Eq. (11) can be given as

$$\mathbf{A}(s)=\begin{pmatrix}\mathbf{A}_1(s) & \mathbf{0}\\ \mathbf{0} & \mathbf{A}_2(s)\end{pmatrix} \tag{13}$$

where

$$\mathbf{A}_i(s)=\begin{pmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -\dfrac{s^2\rho A_i+sc_{\mathrm{M}i}}{sc_{\mathrm{K}i}+I_{zi}E} & 0 & 0 & 0\end{pmatrix} \tag{14}$$

and $\mathbf{0}$ stands for the 4×4 zero matrix. The coefficient matrices

$$\mathbf{P}=\begin{pmatrix}\mathbf{I}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\end{pmatrix},\quad \mathbf{R}=\begin{pmatrix}\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{I}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\end{pmatrix} \tag{15}$$

in Eq. (12) are defined to fulfill the boundary conditions Eq. (6), while coefficient matrix

$$\mathbf{Q}=\begin{pmatrix}\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{I}&\mathbf{0}&-\mathbf{I}&\mathbf{0}\\ \mathbf{0}&I_{z1}E\mathbf{I}&\mathbf{0}&-I_{z2}E\mathbf{I}\end{pmatrix} \tag{16}$$

comes from the interface condition Eq. (7). In Eqs. (15) and (16), where $\mathbf{I}$ and $\mathbf{0}$ stand for 2×2 identity and zero matrices, respectively.

The homogeneous solution of Eq. (11) is

$$\boldsymbol{\eta}(x,s)=\mathrm{e}^{\mathbf{A}(s)x}\boldsymbol{\eta}(0,s) \tag{17}$$

where $\mathrm{e}^{\mathbf{A}(s)x}$ is the exponential of matrix $\mathbf{A}(s)x$ also called fundamental matrix. Substitute Eq. (17) into the boundary conditions Eq. (12):

$$\left(\mathbf{P}+\mathbf{Q}\mathrm{e}^{\mathbf{A}(s)e}+\mathbf{R}\mathrm{e}^{\mathbf{A}(s)L}\right)\boldsymbol{\eta}(0,s)=\mathbf{0} \tag{18}$$

The eigenvalues of the continuum model are the roots s_k of the transcendent characteristic equation

$$\det\left(\mathbf{P}+\mathbf{Q}\mathrm{e}^{\mathbf{A}(s)e}+\mathbf{R}\mathrm{e}^{\mathbf{A}(s)L}\right)=0 \tag{19}$$

which can be used to extract the modal parameters

$$s_k = -\zeta_k \omega_{n,k} \pm i\omega_{n,k}\sqrt{1-\zeta_k^2}, \quad k = 1, \ldots, \tilde{m} \tag{20}$$

Note that this result can be used to validate the parameters of the FEM; however, the challenging root-finding problem of Eq. (19) is not necessary to determine the exact FRF.

According to Ref. [30], the unique solution of Eq. (11), which satisfies the boundary conditions Eq. (12), can be written in the following form:

$$\boldsymbol{\eta}(x, s) = \int_0^L \mathbf{G}(x, \vartheta, s)\mathbf{q}(\vartheta, s)\,\mathrm{d}\vartheta \tag{21}$$

where the matrix Green function is given as

$$\mathbf{G}(x, \vartheta, s) = \begin{cases} \mathbf{W}(x, s)\mathbf{P}e^{-\mathbf{A}(s)\vartheta} & \text{if} \quad x \ge \vartheta \\ -\mathbf{W}(x, s)\mathbf{R}e^{\mathbf{A}(s)(L-\vartheta)} & \text{if} \quad x < \vartheta \end{cases} \tag{22}$$

and

$$\mathbf{W}(x, s) = e^{\mathbf{A}(s)x}\left(\mathbf{P} + \mathbf{Q}e^{\mathbf{A}(s)e} + \mathbf{R}e^{\mathbf{A}(s)L}\right)^{-1} \tag{23}$$

Finally, the exact scalar-valued direct FRF $H(\omega, e)$ at the contact point of the tool can be calculated by means of the substitution of $x = \vartheta = e$ and $s = i\omega$ into Eq. (22) and by selecting the matrix element 1,4

$$H(\omega, e) = \frac{G_{1,4}(e, e, i\omega)}{I_{z1}E(1 + i\omega\alpha_K)} \tag{24}$$

The indexes in $G_{1,4}(e, e, i\omega)$ refer to the corresponding first element of the state space vector in Eq. (9) and the fourth element of the excitation vector Eq. (10).

It should be noted that the DTFM can give exact and closed-form solution of the FRF without any truncation or approximation by using only the inverse and the exponent of 8×8 matrices.

2.3 Validation of the Finite Element Model. The convergence of natural frequencies and mode shapes regarding the element size of FE model is well-established. Since for the stability calculation of the milling process, a full FRF function is needed, and the effects of the FE model parameters on the FRF function is not so common; therefore, in this subsection, the convergence analysis of the FEM is based on the exact analytical solution of the continuum model. First, the resolution of the FE mesh is investigated through the comparison of the essential natural frequencies in order to determine sufficient element size. The relative error between the exact natural frequencies calculated with DTFM and the ones obtained from FEM is

$$\epsilon_{\omega_{n,k}} = \frac{|\omega_{n,k} - \tilde{\omega}_{n,k}|}{\omega_{n,k}} \tag{25}$$

Figure 2 shows the fourth-order convergence of each natural frequency that belongs to the Bernoulli beam-type finite elements. The convergence is limited by the floating-point arithmetic in two ways. On the one hand, numerical noise appears below the relative error around 10^{-8} for both DTFM and FEM. On the other hand, numerical errors accumulate in the iterative method applied for the solution of the eigenvalue problem, which is typical for higher frequencies. From Fig. 2, one can select an appropriate element resolution to keep the relative error within a desired range. For further investigation, we select 100 elements along the workpiece.

As a next step, the reduction of the number $\tilde{m}$ of the considered modes in Eq. (4) is investigated based on the deviation in the

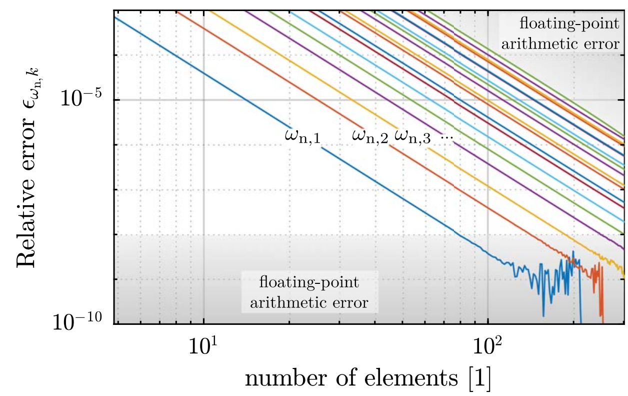

Fig. 2 Relative error of the natural frequencies as a function of the number of elements at $e = L/2$. Parameters are defined in Table 1 for Workpiece I.

resulted FRF. The relative error between the exact direct FRF $H(\omega, e)$ and the one computed by FEM $\tilde{H}(\omega, e)$ is given by means of the formula

$$\epsilon_H = \frac{\int_0^L \int_0^{\omega_{\max}} |H(\omega, e) - \tilde{H}(\omega, e)|\,\mathrm{d}\omega\,\mathrm{d}e}{\int_0^L \int_0^{\omega_{\max}} |H(\omega, e)|\,\mathrm{d}\omega\,\mathrm{d}e} \tag{26}$$

applied for the investigated frequency domain $[0, \omega_{\max}]$. Figure 3 represents the relative errors in a logarithmic scale for different domains, which cover a couple of essential natural frequencies. It is easy to show that the relative error decreases by increasing the number of considered modes. Note that even if we liked to model the first two natural frequencies only in the direct FRF with $\omega_{\max} = 4500$ Hz, then at least 20 modes ($\tilde{m} \ge 20$) should be considered to achieve a high accuracy like $\epsilon_H = 10^{-4}$.

Also, the achievable accuracy in the case of more than 50 modes is strongly limited by the errors generated by the floating-point arithmetic. If $\omega_{\max}$ is selected for the bandwidth of the given excitation, then an appropriate number of modes can be selected with the help of the diagram in Fig. 3.

As a conclusion, to capture the chatter frequency, usually the first or the second "lobe rows" play a role, which is closely related to the first- and second-natural frequencies. So, one might choose only one or two natural frequencies, however, not only the first few modes are relevant to create the proper FRF. Therefore, a high number of modes have to be considered in this type of modeling; consequently, it requires a good resolution of the FE mesh, as well.

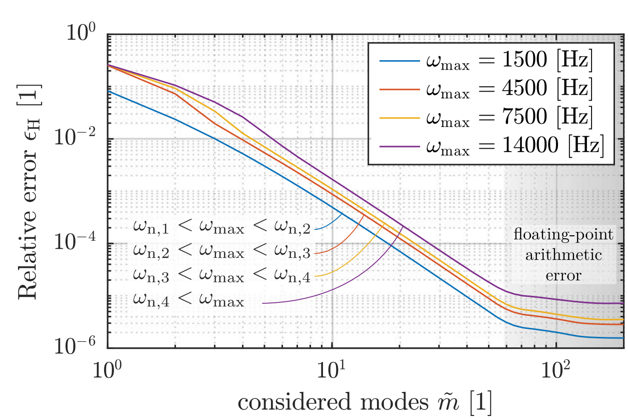

Fig. 3 The convergence of the relative error in the function of considered mode number for the different frequency domains $[0, \omega_{\max}]$. Parameters are defined in Table 1 for Workpiece I.

3 Analysis of Varying Dynamics

In this section, we analyze the variation of the dynamic properties along the tool path during the material removal.

In Fig. 4, the amplitude of the direct FRF (colormap) and the natural frequencies (dashed curves) is presented for different tool positions. The natural frequencies of a prismatic beam are linearly proportional to the thickness of the cross section, which is v for rectangular and d for cylindrical cross-section. Thus, the natural frequencies at the end position ($e=L$) of the tool are 16.7% smaller than they are at its starting position ($e=0$), for example, $\omega_1(0)=1155$ Hz $\rightarrow \omega_1(L)=958$ Hz. However, during the material removal process, the natural frequencies change in an intricate way; it can increase or decrease whether the mass or the stiffness reduction has a larger influence. In the colormap of Fig. 4, the variation of the abs(FRF) peaks is proportional to the corresponding mode shapes. It also visualizes that the workpiece is ideally stiff at both ends; moreover, it is dynamically stiff at the nodes of the corresponding mode shapes.

The effects of material removal can be visualized better in Fig. 5, where the relative frequency change for first- and second-natural frequencies is represented along the tool path for different amounts of material removal. The larger the cross-section reduction is, the more the natural frequency values fluctuate. In an extreme reduction (close to 100%) as an illustrative example, it tends to a specific characteristic, referred to as the backbone curve of the material removal.

This limit case can be given if the system is decomposed into two separate beam segments while the connection between them is

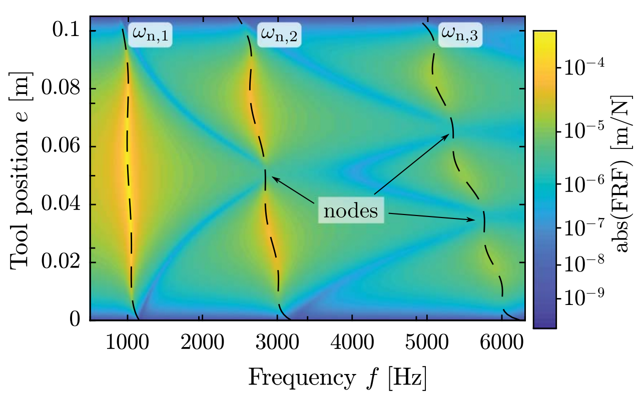

Fig. 4 Direct FRF of the workpiece at the actual tool position, where the thickness of the plate was reduced by 16.7%. The amplitude and the natural frequencies are denoted by colormap and dashed black curves, respectively. Parameters are presented for Workpiece I in Table 1.

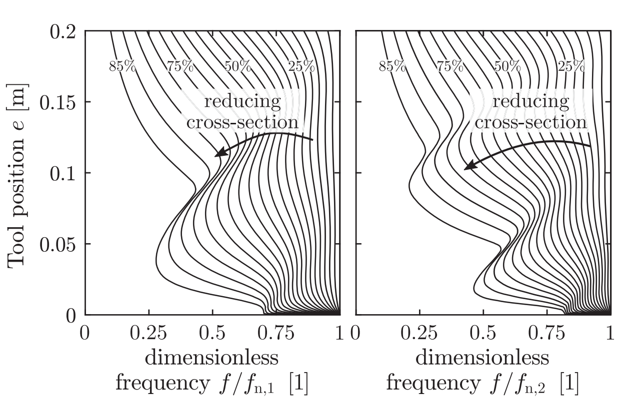

Fig. 5 Characteristics of the dimensionless natural frequencies as a function of the tool position for different cross-section removal rate

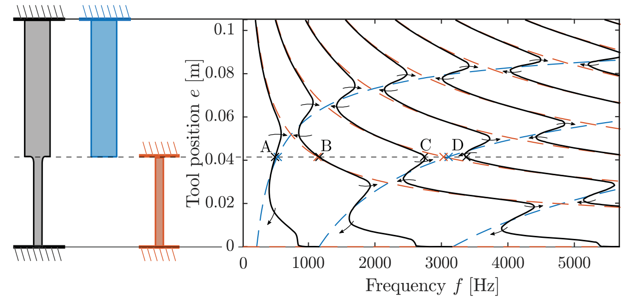

Fig. 6 Decomposition of the beam model; black curves represent natural frequencies in case of 85% cross section reduction, while blue- and red-dashed curves represent the natural frequencies of cross section reduction tending to 100%. (Color version online.)

replaced by different boundary conditions (see the schematic representation in Fig. 6). If we consider a very high stiffness ratio of the parts, then the stiffer (thick blue) beam is not influenced by the more flexible (thin red) one and the stiffer beam can be modeled as a cantilever beam. Meanwhile, the upper end of the more flexible beam is almost fixed by the stiffer one, so it can be modeled as a clamped-clamped beam. The corresponding natural frequencies can be determined from the following frequency equations [40], for the cantilever beam as

$$\cos\left((L-e)\sqrt[4]{\omega_{\mathrm{n}}^2 \frac{A_2\rho}{I_{z2}E}}\right)\cosh\left((L-e)\sqrt[4]{\omega_{\mathrm{n}}^2 \frac{A_2\rho}{I_{z2}E}}\right) = -1 \tag{27}$$

and for clamped-clamped beam as

$$\cos\left(e\sqrt[4]{\omega_{\mathrm{n}}^2 \frac{A_1\rho}{I_{z1}E}}\right)\cosh\left(e\sqrt[4]{\omega_{\mathrm{n}}^2 \frac{A_1\rho}{I_{z1}E}}\right) = 1 \tag{28}$$

The natural frequencies of the decomposed models (solutions of Eqs. (27) and (28)) are plotted by blue- and red-dashed curves, while natural frequencies relating to the original model are plotted by black curves for 85% cross-section reduction in Fig. 6.

Intersections of two backbones curves can be considered as attractive points, to which the curves of the natural frequencies tend (see points A and B in Fig. 6). Between these points, the natural frequencies of the original model follow one of the backbone curves. In the meantime, close to the attractive points, they swap to another backbone curve, where two adjacent frequencies are close to each other (see, for example, points C and D in Fig. 6). With the proposed decomposition method, the pattern in the fluctuation of the natural frequencies can be identified and visualized.

Recall that the presented workpiece models can be used for milling and turning operations, as well. In Sec. 4, the cutting force is derived for the milling process, and for the turning process, it can be obtained as its special case.

4 Milling Process

To achieve the stability calculation and to anticipate the quality of the machined surface, the cutting operation of the milling process is described.

4.1 Cutting Force. The widespread linear cutting force characteristic is applied, where the cutting force is linearly proportional to the instantaneous chip thickness [41]. It can provide a good estimation for the forced vibration [42] and can suitably describe the chatter phenomena [4], as well. Note that the description of models for general milling tool can be found in Ref. [43], but in our analysis, we focus on the description of traditional helical end-mills only. The radial and the tangential components of the elementary cutting forces acting on an elementary segment dz of the jth cutting edge (see Fig. 7) are given as

$$\begin{aligned} dF_{r,j}(t) &= K_r g\big(\varphi_j(t,z)\big) h_j\big(\varphi_j(t,z)\big)\, dz \cos\eta \\ dF_{t,j}(t) &= K_t g\big(\varphi_j(t,z)\big) h_j\big(\varphi_j(t,z)\big)\, dz \cos\eta \end{aligned} \tag{29}$$

where K_r and K_t are the radial and the tangential cutting coefficients, respectively. These parameters are usually identified by several cutting experiments for given cutting parameters and tool geometry [44]. The screen function is

$$g(\varphi) = \begin{cases} 1 & \text{if } \varphi_{\text{enter}} < \varphi < \varphi_{\text{exit}} \\ 0 & \text{otherwise} \end{cases} \tag{30}$$

which tracks whether the jth edge is in contact with the material or not in case of radial depth of cut a_e (see Fig. 7). The angular position of the jth cutting edge for constant helix angle and equally distributed teeth is

$$\varphi_j(t,z) = \Omega t + \frac{2\pi(j-1)}{Z} - \frac{2\pi z}{Z l_p} \tag{31}$$

where Z is the number of the cutting edges, Ω stands for the spindle speed in (rad/s), l_p is the helix pitch and $\eta = \arctan D\pi/(Zl_p)$ denotes the helix angle. Also, $h(\varphi_j(t,z))$ denotes the actual chip thickness, which is the sum of the stationary $h_{\text{stat}}(\varphi_j(t,z))$ and the dynamic $h_{\text{dyn}}(\varphi_j(t,z))$ chip thicknesses [4]. The stationary one can be given as the projection of the feed per tooth f_z into the tool tip velocity direction, which translates into

$$h_{\text{stat}}\big(\varphi_j(t,z)\big) = f_z \sin\varphi_j(t,z) \tag{32}$$

in the case of circular tooth path approximation [2]. It should be noted that the stationary chip thickness affects only the forced stationary vibration, and it has no influence on the stability of the cutting process in case of linear cutting force function [15]. The dynamic chip thickness relates to the surface regenerative effect of the cutting operation [9], which is relevant in the stability analysis of the machining process [4]. It reads as

$$h_{\text{dyn}}\big(\varphi_j(t,z)\big) = \big(y(e,t) - y(e,t-\tau)\big)\cos\varphi_j(t,z) \tag{33}$$

where $y(e,t)$ denotes the general coordinate that describes the motion of the workpiece relative to the tool at the contact point e (see in Fig. 7). Note that the delayed position $y(e, t-\tau)$ represents the vibration copied onto the surface caused by the preceding cutting edge, where $\tau = 2\pi/(\Omega Z)$ denotes the tooth passing period.

In order to obtain the resultant cutting force, we project the force components from the local tangential-radial coordinate system into the global $x-y$ coordinate system. Also note that the *SLE* computation requires the forced vibration perpendicular to the surface. This beam model is flexible only in this direction; so, the $dF_y(t)$ component is presented in the next steps. This elementary force components are integrated along the z-coordinate for the axial depth of cut a_p to obtain

$$F_y(t) = \sum_{j=1}^{N} \int_{a_p} \big[K_t\, K_r\big] h\big(\varphi_j(t,z)\big) g\Big(\big(\varphi_j(t,z)\big)\Big) \big[-\sin\varphi_j(t,z) \quad \cos\varphi_j(t,z)\big]^T dz \tag{34}$$

Note that for certain axial depth of cut values like $a_p = l_p$, $2l_p$, $3l_p$, …, also called trivial appropriate axial immersions [42,45], the stationary component of the cutting force is constant in time (see illustration in Fig. 8).

By means of the above-described force model, both the surface quality prediction and the stability calculation can be carried out.

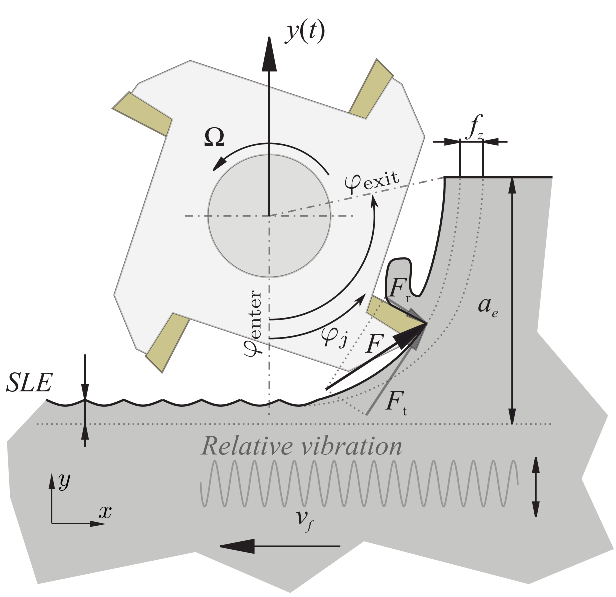

Fig. 7 Model of the milling process

4.2 Surface Location Error. In what follows, the calculation steps of *SLE* induced by the forced vibration are presented [45]. The machined surface profile is a result of the relative motion of the workpiece and the cutting edges. The forced stationary vibration of the contact point can be determined by means of the direct FRF $H(\omega, e)$ (Eqs. (4) or (24)) according to

$$y(e,t) = \mathcal{F}^{-1}\big(H(\omega,e)\phi_y(\omega)\big) \tag{35}$$

where $\phi_y(\omega)$ denotes the Fourier transformation of the cutting force (34). The motion of the workpiece relative to the jth cutting edge is described by means of the superposition of the rotating edges and the forced stationary vibration $y(e,t)$, that is

$$r_j(e,t,z) = y(e,t) - \frac{D}{2}\cos\varphi_j(t,z) \tag{36}$$

The *SLE* is determined as the extremum of these in the form

$$\begin{aligned} SLE(e,z) &= \max_{t,j}\big(r_j(e,t,z)\big) - \frac{D}{2} \quad (\text{up}-\text{milling}) \\ SLE(e,z) &= \min_{t,j}\big(r_j(e,t,z)\big) + \frac{D}{2} \quad (\text{down}-\text{milling}) \end{aligned} \tag{37}$$

As a result, the $SLE(e,z)$ depends not only on the tool position e only but also on the axial coordinate z of the tool. Consequently, the so-called *maximum surface location error* $MSLE(e)$ parameter

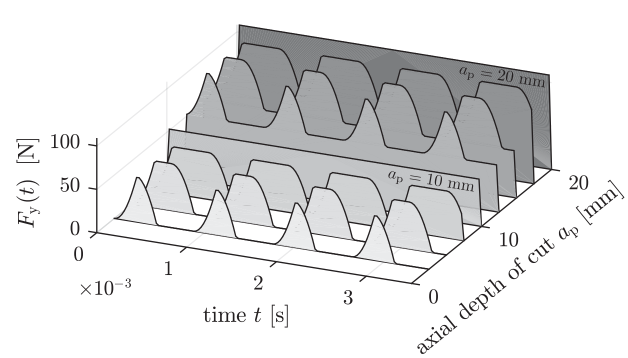

Fig. 8 Cutting force in different axial depth of cuts during one tool revolution for spindle speed $N = 17{,}000$ rpm and helix pitch $l_p = 10$ (mm). Parameters are given in Table 2.

is introduced [45], which can be obtained by

$$MSLE(e) = \max_{z}\,(SLE(e, z)) \tag{38}$$

The described method is capable to forecast the *MSLE* values along the tool position for a given set of technological parameters. Note that this calculation is based on stable forced vibration, so it is valid only if the milling operation is in the chatter-free (stable) domain.

4.3 Stability Analysis. In this section, the calculation of stability boundaries is briefly discussed according to the surface regenerative effect, which is based on the method of the so-called extended multi-frequency solution (EMFS) [11]. The EMFS is an effective computational algorithm in order to calculate the stability boundaries. It is based on the real and the imaginary parts of the truncated Hill's infinite determinant. Then, the stability chart is determined in the plane of the spindle speed Ω and the tool location e by means of the so-called multi-dimensional bisection method [46]. The combination of MDBM and EMFS is capable to determine the stability boundaries in a computationally effective way, even if closed stable or unstable islands appear. Figure 9 shows the stability chart for parameters presented in Tables 1 and 2 considering that the entire width of the workpiece is machined ($a_p = w$). It should be noted that this stability diagram does not show the traditional lobe structure since the vertical axis is not the axial immersion a_p, but the tool position coordinate e. Still, the typical phenomenon, that one side of the lobes tend to the natural frequencies or their integer quotients (ω_n/k, $k = 1, 2, \ldots$) can be observed in Fig. 9.

5 Case Study

In this section, the previously presented calculation methods are applied in two different case studies and compared with laboratory tests, for which, peripheral milling tests of beam-type workpieces are carried out with $w = a_p$. The technological parameters are shown in Tables 1 and 2, and the direct FRF is presented in Fig. 4.

5.1 Numerical Results. The so-called superchart (see in Figs. 10 and 11) [16,45] visualizes the stability chart together with the *MSLE* values in the plane of the spindle speed N and the tool position coordinate e.

For the test case presented in Fig. 10, the chatter-free parameter regions (pockets) can be found among the stability lobes near to the resonant spindle speeds ($N \approx 15{,}000$, 21,000, 41,000 rpm). On the one hand, chatter can be avoided for the total length of the

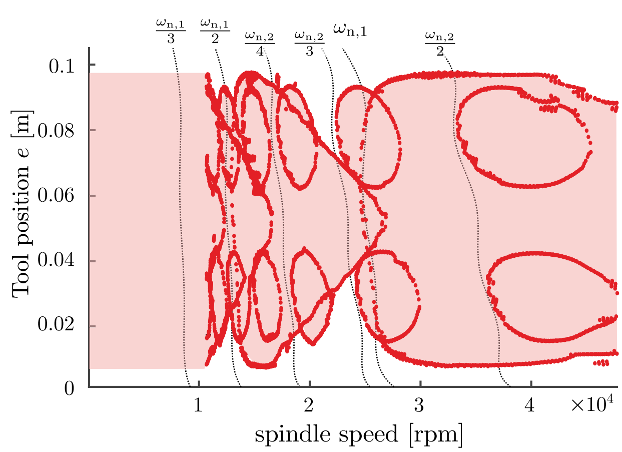

Fig. 9 Stability diagram in the plane of the tool path e and the spindle speed Ω. The red curves show the stability boundaries corresponding different vibration modes provided by the EMFS and the light red shaded area presents the corresponding unstable domains. (Color version online.)

workpiece. On the other hand, these regions are usually not recommended from the viewpoint of the surface errors since significant resonant vibrations take place (see Fig. 10). However, it can be observed that negligible *MSLE* values are resulted along the spindle speed and tool position curves belonging to the angular frequencies $\omega_{n,1}/2$ and $\omega_{n,2}/2$.

Figure 11 shows the results of the second test with a wider workpiece where $w = a_p = 2l_p$. Accordingly, the cutting force does not vary in time, and no resonance occurs that would generate large *MSLE*. In the meantime, however, the unstable region became larger due to the increased axial immersion.

It is well-known that large resonant stationary vibrations can occur if one of the natural frequencies is excited by one of the Fourier components of the cutting force variation. As a result, considerable *MSLE* can be generated at resonant spindle speeds

$$\Omega = \frac{\omega_n}{iZ}, \quad i = 1, 2, \ldots, \tag{39}$$

which can be observed at the vertical dotted lines in Fig. 12. However, for helical cutting tools, there exist so-called trivial and non-trivial appropriate axial immersions [42,45], where no resonant vibrations take place and negligible *MSLE* can be realized even along these "resonant" vertical lines of the stability chart. The trivial appropriate axial immersion can be given by

$$w = jl_p, \quad j \in \mathbb{N} \tag{40}$$

represented by horizontal dashed lines in Fig. 12. Note that in this case, the cutting edges cover full periods $[0, k2\pi]$; thus, they result in constant cutting force independently from the spindle speed Ω. The non-trivial appropriate axial immersion (see the slanting black lines in Fig. 12) can be given as

$$w(\Omega) = k\frac{\Omega}{\omega_n} l_p Z, \quad k \in \mathbb{N} \tag{41}$$

for which, no resonant vibrations emerge at the "resonant" spindle speeds. This way, we can find low *MSLE* values for certain large axial immersions in the stable pockets along the vertical resonant spindle speed lines.

Table 1 Parameters of the milled workpieces (Workpiece I. and II.)

Parameter	Symbol	Workp. I.	Workp. II.
Width	w	15 (mm)	20 (mm)
Thickness	v_2	3 (mm)	5 (mm)
Length	L	105 (mm)	
Material		AlMgSi05	
Density	ρ	$2935\left(\frac{\text{kg}}{\text{m}^3}\right)$	
Young's modulus	E	50 (GPa)	
Stiffness prop. damping coeff.	α_K	$1.43 \cdot 10^{-6}$ (s)	
Mass prop. damping coeff.	α_M	$45\left(\frac{1}{\text{s}}\right)$	

Table 2 Parameters of the case study [44]

Parameter	Symbol	Value
Feed per tooth	f_z	0.05 (mm)
Number of teeth	Z	4
Helix pitch	l_p	10 (mm)
Tool diameter	D	8 (mm)
Radial immersion	a_e	0.4 (mm)
Radial force coefficient	K_r	$300 \cdot 10^6\left(\frac{\text{N}}{\text{m}^2}\right)$
Tang. force coefficient	K_r	$800 \cdot 10^6\left(\frac{\text{N}}{\text{m}^2}\right)$

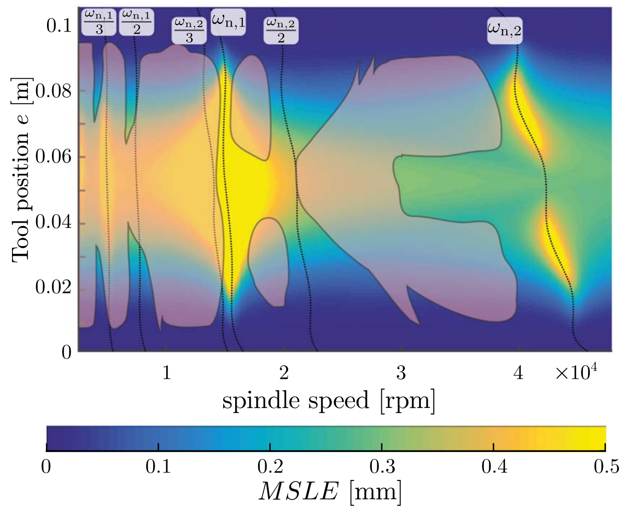

Fig. 10 The superchart with maximum surface location error and stability diagram in the plane of the spindle speed and the tool path for up-milling of Workpiece I; chatter appears within the pink domains. Parameters: $l_p = 10$ mm; $a_p = w = 15$ mm, $v = 3$ mm. (Color version online.)

From a practical point of view, the favorable cases are when the ratio of the helix pitch and width of the workpiece can be expressed as a ratio of two integers:

$$\frac{i}{k} = \frac{l_p}{w}, \quad i, k \in \mathbb{N} \tag{42}$$

in order to provide a condition for the number i of the higher harmonics of the resonant spindle speeds $\omega_n/(iZ)$ (see in Fig. 12).

In the case study of Workpiece I in Fig. 10, $l_p/w = 2/3$ and Eq. (42) holds for parameter pairs $[i, k] = [2, 3], [4, 6], [6, 9], \ldots$. This means that every second (even) resonant spindle speeds lead to negligible $MSLE$ values (e.g., $\omega_{n,1}/2$ relates to point A in Fig. 12), while the odd ones create resonant vibrations and large surface errors.

In the case study of Workpiece II in Fig. 11, $l_p/w = 1/2$ and Eq. (42) holds for parameter pairs $[i, k] = [1, 2], [2, 4], [3, 6], \ldots$, which means that none of the "resonant" spindle speeds lead to

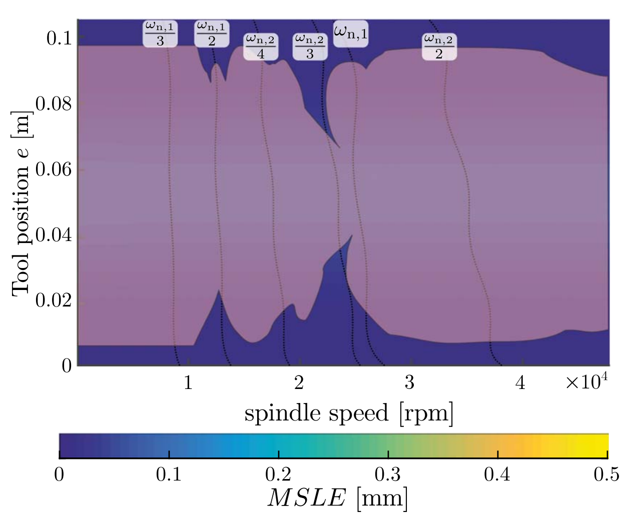

Fig. 11 The superchart with maximum surface location error and stability diagram in the plane of the spindle speed and the tool path for up-milling of Workpiece II; chatter appears within the pink domains. Parameters: $l_p = 10$ mm; $a_p = w = 20$ mm, $v = 5$ mm. (Color version online.)

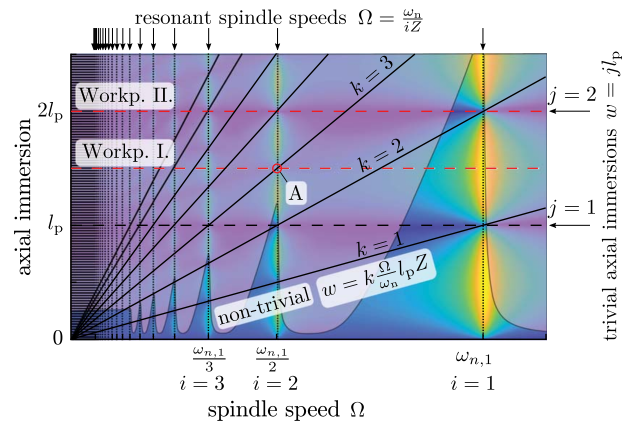

Fig. 12 Schematic representation of superchart with *MSLE* colormap and stability lobes in the plane of spindle speed and axial immersion together with the resonant spindle speeds (vertical dotted lines). The trivial appropriate axial immersions (horizontal dashed lines) and the non-trivial appropriate axial immersions (slanting black lines) are given for a single-degree-of-freedom milling model. (Color version online.)

resonance. This case demonstrates a special case, where the axial immersion a_p is the double of the helix pitch l_p; hence, this situation corresponds to the trivial appropriate axial immersion which relates to the horizontal red-dashed line at $w = 2l_p$ in Fig. 12. Thus, the cutting force is constant, there is no forced vibration and the small $MSLE$ is determined by the static deformation only.

5.2 Measurement Results. Measurements were performed both on Workpiece I and II fixed as a clamped-clamped beam in the NCT EmR-610Ms 3 axis CNC milling machine. The photo of the experimental setup is presented in Fig. 13. The corresponding parameters are presented in Tables 1 and 2, which were identified by modal analysis and cutting tests [41,44].

The FRF along the tool path for Workpiece I before the milling operation was measured by modal testing where the workpiece was excited at different positions. The first two natural frequencies and the corresponding mode shapes are visualized in a Pulse B&K data acquisition system, as shown in Fig. 14. Note that in this case, the natural frequencies do not change along the plate length, since there is no changing geometry. The predicted natural frequencies of the equivalent model can be seen from Fig. 4 at $e = 0$ m, and they are compared with the measured values in Table 3. It should be noted that there was a slight difference between the measured and the predicted natural frequencies (3.77% and 7.64%). The major effect of the discrepancies can be the ideal modeling of the clamped-clamped ends, which may be a too strict condition in practice. For accurate modeling of the clampings, it could be replaced by flexible spring-damper elements, which parameters can be tuned. There is a mild asymmetry in the mode shapes, which also indicates

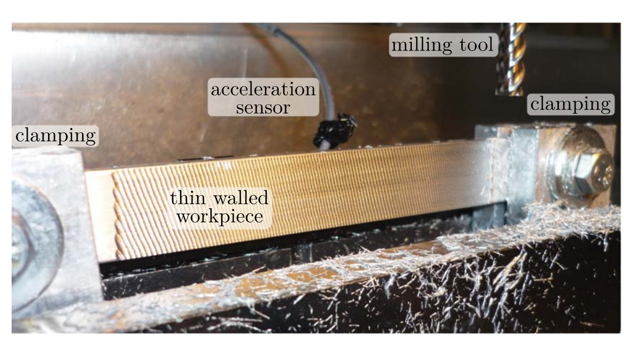

Fig. 13 Experimental setup

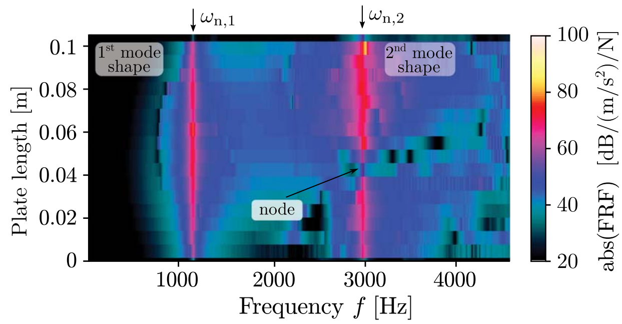

Fig. 14 FRF of Workpiece I along the tool position (waterfall diagram) for the first two natural frequencies and mode shapes

Table 3 Natural frequencies of Workpiece I before machining (see Figs. 14 and 4 at $e = 0$ (m))

	$\omega_{n,1}$	$\omega_{n,2}$
Calculated	1142 (Hz)	3155 (Hz)
Measured	1099 (Hz)	2914 (Hz)
Relative error	3.77%	7.64%

not symmetrical clamping conditions. For our investigation, this difference (<8%) is acceptable, because, only to the change of the workpiece dynamics is in the focus of the paper. Thus, the effects and description of the optimized constraints are out of the scope of this paper, although there exist several models in the literature [27,31].

The first measurement was performed for Workpiece I at $N =$ 17,000 rpm, which is close to the first-resonant spindle speed belonging to the first natural frequency of the workpiece. Figure 15 shows the scaled photo of the surface contour of the workpiece and its laser scanned profile (colored curves) together with the theoretically predicted values (black curve). The detailed surface profile measurement method is presented in Ref. [47]. It can be seen that the pattern of the large *MSLE* along the length of the workpiece has a similar form to the respective first mode shape. In this test, no chatter marks were observed. Similar shapes were predicted in Fig. 10 based on the numerical analysis of the corresponding mechanical model, but the asymmetry resulted from the different clampings was not captured by the model. As Fig. 15 shows, there is a slight difference between the measured and predicted surface profile (black curve). Since the amplitude of the resulting vibration, consequently, the magnitude of surface errors are depending on the "distance" from the resonant spindle speed, a slight deviation in the natural frequencies may cause this discrepancy in the magnitude of the surface errors.

The spindle speed $N = 22{,}000$ rpm was also tested which is near to the second-resonant spindle speed belonging to the second-natural frequency. In this case, only negligible *MSLE* values were measured as predicted in Eq. (42) for [i, k] = [2, 3]. However, chatter marks can be observed in regions according to the second-mode shape shown in Fig. 16; this is also presented by the high fluctuations of the measured surface profile. The reasons for this could be that the domain of the two unstable islands is underestimated and the line of the selected spindle speed crosses them in the chart Fig. 10.

During the measurement performed for Workpiece II, we find chatter marks along the whole workpiece for all the tested spindle speeds as predicted by the numerical results in Fig. 11. The spectrogram of a typical vibration signal is shown in Fig. 17, which represents that the varying chatter frequency along the tool path relates to the fluctuated natural frequency. Note that, there is no one single-dominant chatter frequency, but multiple ones, which is typical in highly interrupted milling [48]. As an example, the machined surface at $N = 17{,}000$ rpm is presented in Fig. 18, which shows

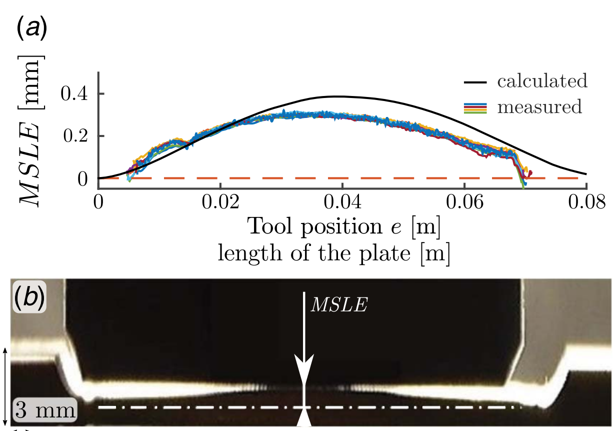

Fig. 15 (*a*) Measured surface profile at along the workpiece and (*b*) scaled photo of the surface contour in the function of the tool position for spindle speed $N = 17{,}000$ rpm; parameters from Tables 1 and 2 for Workpiece I

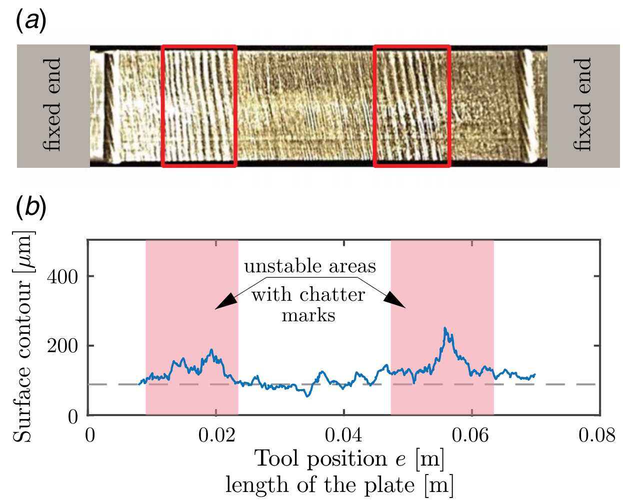

Fig. 16 Photo of the chatter marks (*a*) and the laser-scanned surface profile (*b*) for Workpiece I at spindle speed $N =$ 22,000 rpm; parameters from Tables 1 and 2

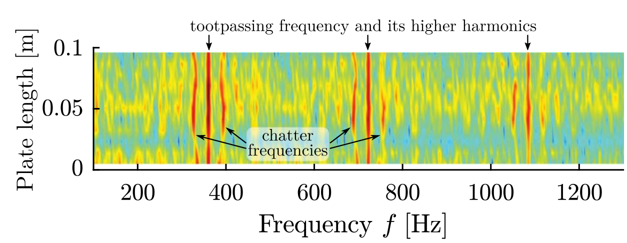

Fig. 17 Typical spectrum for a fully unstable milling case at spindle speed $N = 21{,}660$ rpm

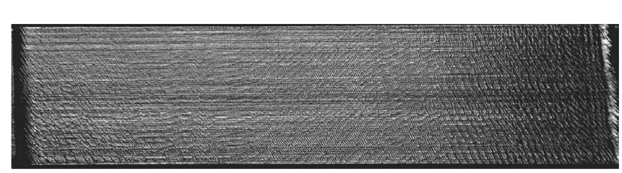

Fig. 18 Chatter marks on the milled surface for Workpiece II at $N = 17{,}000$ rpm; parameters from Tables 1 and 2

chatter marks along the full length of the tool path. Note that negligible shape deviation is detected, because trivial appropriate axial immersion is applied (see Eq. (40) for $a_p = 2l_p$).

All the results of the performed measurements are in correlation with the theoretical predictions. The measurements also show that the varying dynamical properties have relevant influence on the surface errors and the stability properties in case of flexible workpieces. The differences can be explained by the non-ideal and non-symmetric clampings, which may lead to change in natural frequencies, asymmetry in mode shapes, and consequently in surface errors.

All the above-presented steps of MSLE and stability computation can be performed in case of more complex geometry. However, in a general situation, an analytical solution may not exist, but from a detailed FE model, one can compute the FRF along the tool path. With this extended FRF, the stability chart and the surface quality can be predicted at each step along the tool path. Based on these methodologies, the cutting technology of the milling operations can be optimized in order to achieve efficient production and reach acceptable surface errors.

6 Conclusions

In this contribution, it is shown how the change of the frequency response function affects the stability and surface errors of the milling operations. To demonstrate this, the workpiece is considered as a flexible beam; moreover, the stiffness variation caused by the material removal and the change in the excitation point (tool position) are also taken into consideration. First, finite element formulation is derived for the direct frequency response function, then its parameters are validated through an analytical closed-form solution (called distributed transfer function method). The pattern of the natural frequency fluctuation caused by the material removal is explained through extreme cross-sectional reduction, for which analytical solution is provided in the limit case.

The stability boundaries together with the *MSLE* values are presented in the superchart as a function of spindle speed and tool position.

The case studies show that the chatter-free parameter domains are located at the resonant spindle speeds. It is also shown that the maximum surface location errors can be significant at these spindle speeds. However, if the trivial or the non-trivial appropriate axial immersions are applied, then good surface quality can be achieved even for resonant spindle speeds. In this sense, the results of this paper may help understand the connections between the mode shapes/natural frequencies and the location of the unstable areas and surface errors. The numerical and the experimental case studies show agreement, which validates the theoretical predictions.

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