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STABILIZING EFFECT OF VERTICAL PERIODIC MOTION ON ATHLETES – PADDLE SYSTEMS

A. Zelei

Department of Applied Mechanics,
Budapest University of Technology and Economics
H-1111. Budapest, Műegyetem rkp. 5., Hungary
zelei@mm.bme.hu

G. Stépán

Department of Applied Mechanics,
Budapest University of Technology and Economics
H-1111. Budapest, Műegyetem rkp. 5., Hungary
stepan@mm.bme.hu

Abstract

Paddling sports usually demand very sophisticated and well synchronized body motion, furthermore, balancing is a difficult task in itself for a beginner athlete. Although, it is usually the boat's lengthwise acceleration that is in the centre of interest of coaches, the boat oscillates in vertical direction, too, due to the rowing motion of the athlete. In the present study, the balancing effect of the vertical periodic motion of the athlete's body is examined. The so-called parametric excitation arises in these systems due to the vertical periodic motion of the mass centre. This unusual class of excitation is studied with respect to the stability of the system composed by the athlete and the row-vessel.

We construct a one-degree-of-freedom dynamical model by prescribing a non-stationary geometric constraint, that is, by moving the centre of gravity periodically in the vertical direction. This periodic movement causes parametric excitation, and the equation of motion is obtained in the form of the so-called Mathieu equation. By using the so-called Incze-Strutt stability chart of the Mathieu equation, we can find sets of parameters where the stabilization of a normally unstable floating rigid body is possible. This has an effect, for example, that can help in stabilization in case of a canoeist. Experimental and numerical investigations were also accomplished to support the idea

Keywords

Parametric excitation, Mathieu-equation, kayak, row-vessel, balancing

1 Introduction

Flat-water paddling sports especially kayaking and canoeing are fairly popular water-sports. Canoes and kayaks are pretty labile, using mechanical terminology, we call them statically unstable. In this work we focus on the balancing of these row-vessels. The boat basically moves in lengthwise direction and also oscillates in vertical direction due to the rowing motion of the athlete. We examine the stabilizing effect of the vertical periodic motion of the mass centre of the athlete-boat system. The main objective of this work is to investigate the possibility of the stabilization of a normally unstable floating body by parametric excitation. First, we briefly summarize the theory, since parametric excitation cannot be investigated analytically in closed form even in linear systems.

1.1 Floquet Theory

Floquet Theory was developed to investigate the stability of parametrically excited systems. Parametric excitation means that some of the parameters of the system change as a periodic/quasi-periodic/stochastic function of time. The most general periodic form of a parametrically excited system is given by the Floquet-equation [1]:

$$\mathbf{y}'(\tau) + \mathbf{A}(\tau)\mathbf{y}(\tau) = \mathbf{0}, \tag{1}$$

where matrix $\mathbf{A}$ is 2π -periodic:

$$\mathbf{A}(\tau + 2\pi) = \mathbf{A}(\tau). \tag{2}$$

This system can be reduced to the Hill equation, which is only one dimensional but still typical in mechanical systems due to the presence of the acceleration term:

$$x''(\tau) + p(\tau)x(\tau) = 0, \tag{3}$$

where the time dependent parameter p (that could be the stiffness of the systems) is periodic:

$$p(\tau + 2\pi) = p(\tau). \tag{4}$$

If the coefficient of x is harmonic, [2] we obtain the so-called Mathieu equation:

$$x''(\tau) + \left(\delta + \varepsilon\cos(\tau)\right) = 0. \tag{5}$$

The Mathieu-equation can be transformed to Floquet-equation form, when the periodic coefficient matrix is written as:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\delta - \varepsilon\cos(\tau) & 0 \end{bmatrix}. \tag{6}$$

1.2 Stabilization

Parametric excitation usually considered as an unexpected cause of instability problems, but under certain conditions, it can also be used for stabilizing unstable processes or equilibria. The main idea is to eliminate an oscillation with the help of another oscillation. The oldest known example is the inverted pendulum which can be stabilized by the harmonic vibration of the suspension pivot point [3, 4]. If the amplitude of the vibration is large enough, the upper equilibrium can be balanced by parametric excitation. In this work, we examine whether it is possible to stabilize a normally unstable floating rigid body in a similar way, and if so, for what ranges of the parameters.

The stability chart of the Mathieu equation was derived in 1928 (see [5, 6]). It is known as Incze-Strutt diagram (see Fig. 1.).

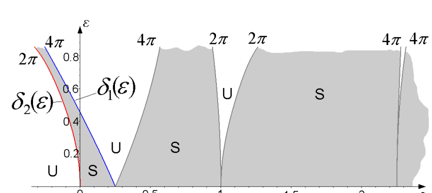


Fig. 1. Stability chart of Mathieu-equation

The basic idea was to find the stability boundaries in a double series expansion with respect to the parameter δ and the solution x as a function of the "small" parameter ε that is the amplitude of the parametric excitation:

$$\delta(\varepsilon) = \delta_0 + \delta_1\varepsilon + \delta_2\varepsilon^2 + \dots, \qquad (7)$$

$$x(\tau,\varepsilon) = u_0(\tau) + u_1(\tau)\varepsilon + u_2(\tau)\varepsilon^2 + \dots \qquad (8)$$

In this work, we are interested in the $\delta < 0$ region of the Incze-Strutt diagram, were the equilibrium is obviously unstable for $\varepsilon = 0$ but it can become stable for a narrow $\varepsilon > 0$ region. After the application of the Floquet Theory, the truncated stability boundaries in question appear in the following form:

$$\delta_1(\varepsilon) = -\frac{1}{2}\varepsilon^2, \qquad (9)$$

$$\delta_2(\varepsilon) = \frac{1}{4} - \frac{1}{2}\varepsilon - \frac{1}{8}\varepsilon^2. \qquad (10)$$

1.3 Phase-space geometry

The geometric background of the Floquet Theory can be demonstrated in a phase-space of a two dimensional dynamical system extended by the time axis. Due to the 2π-periodic coefficient matrix, the motion having the same initial conditions 2π later (see Fig. 2. dashed line) is exactly the same (Fig. 2. continual curve).

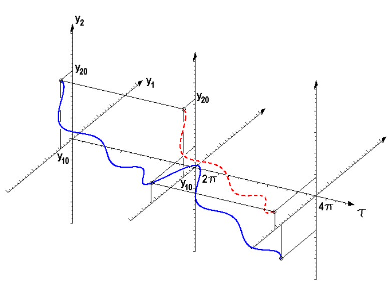


Fig. 2. Phase-space of a 2-dimensional dynamical system

Therefore we can fold back the phase space, where the $\tau = 0$ plane is its so-called Poincaré-section. The 2π-periodic folded phase space can be seen in Fig. 3. presenting a cylindrical structure

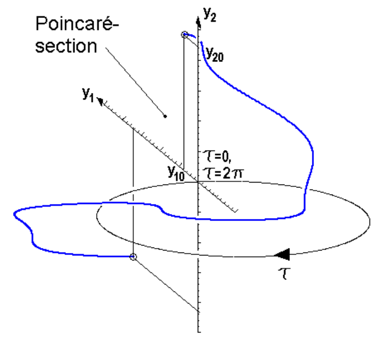


Fig. 3. Folded cylindrical phase-space

The stability boundaries are defined by the 2π and the 4π periodic solutions of the Mathieu equation. The cross sections of the trajectories at the $\tau = 0$ plane (Fig. 3., 4.) represent the periodicity of the solutions.

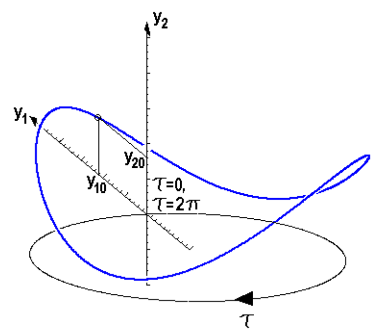


Fig. 4. 2π periodic solution

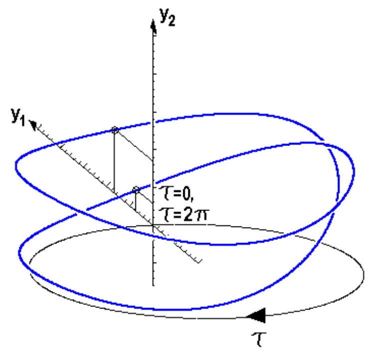


Fig. 5. 4π periodic solution

1.4 Goal of study

In the subsequent sections, we investigate the application of the Floquet Theory for floating bodies. The fundamental question is whether the parametric excitation helps the paddling athletes to balance themselves.

2 Mechanical model

2.1 Planar model of a floating body

A floating body, like a ship, has 6 degrees of freedom (DoF), which are the 3 rotations, namely roll, pitch and yaw and the 3 transversal motions, namely the surge, sway and heave. In order to use as simple model as possible for the analytic calculations we investigate planar motion only (see the vertical plane in Fig. 6.).

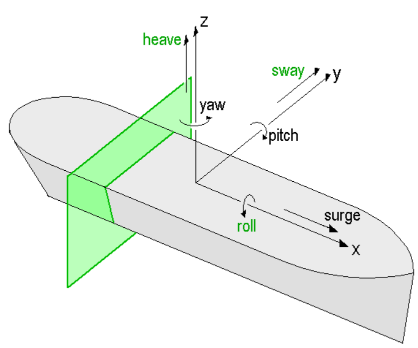


Fig. 6. Spatial ship motions

Roll, sway and heave are the general coordinates of the planar mechanical model. The generalized coordinates are chosen to be the roll angle φ and the position coordinates x and y of the centre of gravity. The parameters also shown in Fig. 7. are:

l : length of the body in direction z

a : width of the body

p : height of the mass centre

m : mass;

J_C : moment of inertia

We introduce the modified density parameter:

$$\mu = \rho a l \,. \tag{11}$$

The shallow dive h can be calculated as:

$$h = \frac{m}{\mu} \,. \tag{12}$$

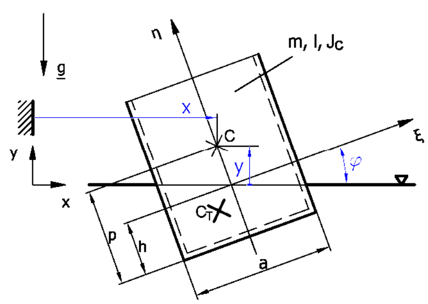


Fig. 7. Planar mechanical model of the floating body

The water surface is assumed to be ideally flat and steady. The potential function of the system can be seen in Fig. 8. and Fig. 9. Rectangular and triangular regions (denoted by R and T in Fig. 8., 9.) are distinguished. If the roll angle is large, only one corner of the body is in the water, thus the wetted part of the body is triangular shaped. The critical value φ_{CRIT} separates the two regions.

It is well known that the vertical position of a symmetric floating body is an equilibrium position. First, we identify other possible equilibria of the square shaped floating body. The stability of each equilibrium can be examined by the analysis of the potential function of these conservative systems. The vertical position is stable in Fig. 8. and unstable in Fig. 9. The stability loss of the vertical position leads to two tilted equilibrium positions of the body.

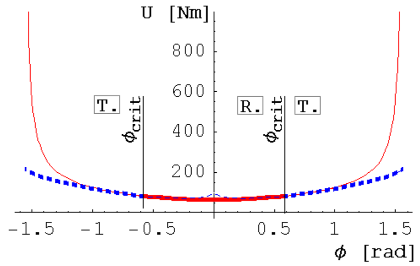


Fig. 8. Potencial function of a vertically stable case

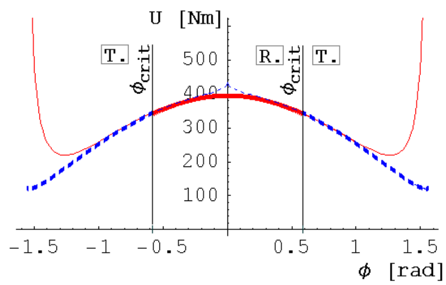


Fig. 9. Potencial function of a vertically unstable case. Two stable equilibria arise.

The Lagrangian is given by the kinetic and the potential energy. The damping effect of the water is completely neglected. Then the Lagrangian can be written as:

$$L = \frac{1}{2} m\left(\dot{x}^2 + \dot{y}^2\right) + \frac{1}{2} J_C \dot{\varphi}^2 - mgy - \mu g\left(\frac{p^2}{2}\cos\varphi - py + \frac{y^2}{2\cos\varphi} + \frac{a^2}{24} tg\varphi \sin\varphi\right). \tag{13}$$

The general coordinate x does not appear in the Lagrangian, so x is a so-called cyclic coordinate. Therefore, we can fix the horizontal motion of the body.

2.2 Parametrically excited model – effect of paddling

The aim is to derive a 1 DoF model of the parametrically excited ship model. We prescribe the height of the centre of gravity by a harmonic function $y(t)$:

$$y(t) = y_0 + y_1 \cos(\omega t)\,. \tag{14}$$

Because both coordinates of the centre of gravity are constrained, the only generalized coordinate is the roll angle.

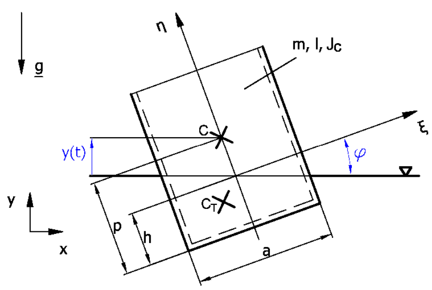


Fig. 10. 1 DoF planar model

The constant term of the excitation y_0 is also determined by the roll angle.

$$y_0 = (p - h)\frac{1}{\cos\varphi}\,. \tag{15}$$

The two parameters of the parametric excitation are the amplitude y_1 and the angular frequency ω.

We use the Lagrangian equation to derive the equation of motion of the shaken body:

$$J_C\ddot{\varphi} + mg\left(\frac{1}{2}\frac{m}{\mu} - p\right)\sin\varphi$$
$$+ \mu g\left[\frac{{y_1}^2\cos^2(\omega t)tg\varphi}{2\cos\varphi}\right. \quad . \tag{16}$$
$$\left. + \frac{a^2}{24}\left(\sin\varphi + \frac{tg\varphi}{\cos\varphi}\right)\right] = 0$$

Since we only investigate the stability of the vertical position, where $\varphi = 0$, the equation of motion can be linearized. We use the dimensionless time $\tau = \omega t$. Finally the linearized equation of motion (17) is in complete correspondence with the Mathieu-equation:

$$\varphi'' + \{\delta + \varepsilon\cos(\tau)\}\varphi = 0, \tag{17}$$

where the parameters are:

$$\delta = \frac{\mu g}{4\omega^2 J_C}\left[\frac{m}{\mu}\left(\frac{1}{2}\frac{m}{\mu} - p\right) + \frac{y_1^2}{4} + \frac{a^2}{12}\right], \tag{18}$$

$$\varepsilon = \frac{\mu g}{4\omega^2 J_C}\frac{y_1^2}{4}. \tag{19}$$

If $\varepsilon = 0$ $\delta < 0$ then the vertical position is unstable and the boat capsizes, But it can be stable if the amplitude y_1 increases, that is, when $\varepsilon \neq 0$.

3 Stability of the parametrically excited floating body

We can apply the Incze-Strutt diagram for the equation (17) of motion of the floating body with the physical parameters (18) and (19). The transformed stability chart shows the physical parameters of the excitation only, namely the angular frequency and the amplitude. The stability boundaries are obtained by substituting expressions (18) and (19) into the equations of the boundaries $\delta_1(\varepsilon)$ (9) and $\delta_2(\varepsilon)$ (10).

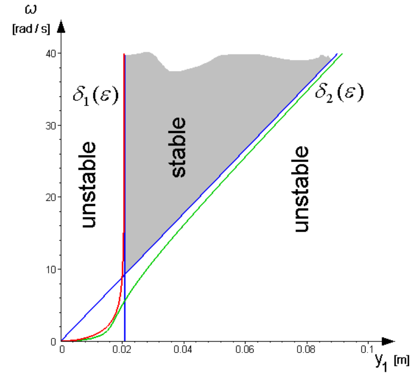


Fig. 11. Stability chart of the parametrically excited floating body

Appropriate finite values of the amplitude and the frequency of the harmonic oscillation are needed for the stabilization of the normally unstable floating body. The stability boundaries are highly sensitive to the changes in the geometrical parameters and the mass.

4 Verification

We investigate the possibility of stabilizing a row-vessel that is unstable without parametric excitation. The parametric excitation is generated by the vertical periodic motion of the sportsman's body. Due to the harmonic motion of the mass centre, the shallow dive also changes periodically. This effect helps the athletes to balance the boat.

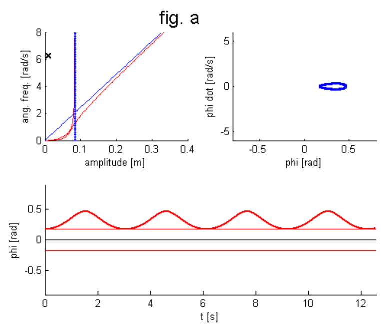
4.1 Numerical simulation

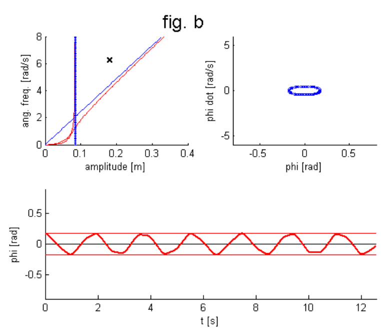
Athletes are told to keep a good rhythm of rowing and not to row faster but row more powerfully when they want to go faster. Frequency of 80-85 strokes per minute is typical. Usually, accelerations of 2m/s^2 and 150N force can be measured in horizontal direction [7]. The specific literature rarely refers to the vertical displacements, accelerations and forces, which, in our view, are important in the stabilization process. The forces acting between the shell and the rower and between the water and paddle blade change the resting waterline causing oscillations of 4–6 cm [8].

The chosen parameters of the numerical simulation are:

J_C=12[kg · m^2],
m=90[kg],
l=3.5[m],
a=0.55[m],
p=0.6[m],
ω=6.28[rad/s].

The nonlinear equation (16) of motion was used for the numerical simulation. A constant angular frequency has been set, and the stability boundaries were crossed by increasing the amplitude parameter y_1. We investigate the time history of the roll angle φ and the Poincaré section (see Fig. 12.).





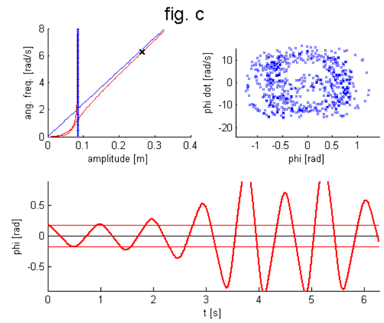


Fig. 12. Numerical results (a.: amplitude is too small to the stabilization, b.: stable, c.: amplitude is too large to the stabilization)

4.2 Experiment

An experiment verified the practical validity of the mechanical model and its analytical and numerical study. A small boat was constructed and 2 eccentric counter-rotating rotors provided the necessary parametric excitation in the vertical direction (see Fig. 13., 14.).

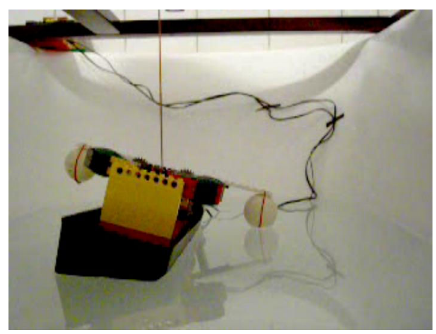


Fig. 13. Stopped rotors

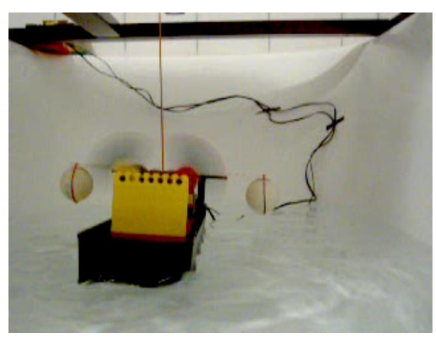


Fig. 14. Running rotors

The experiment clearly showed that the ship stabilization is possible via parametric excitation: the boat floated stably with the rotors running, while it capsized immediately when the rotors were stopped.

5 Conclusion

The possibility of the stabilization of an unstable floating body by parametric excitation has been proved by analytic calculations. There are realistic and finite parameter domains, which can be feasible for sportsmen. Consequently, the parametric excitation induced by the athlete assists in the stabilization of the boat, and the athlete's balancing effort may be less this way.

The investigation of true ship geometries and the consideration of water-boat interaction require further mechanical modelling and numerical computation.

In future work, a model should be investigated that is stabilized by a PD controller and parametric excitation together, which is likely to be the realistic scenario in case of paddle systems.

Acknowledgements

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