DIGITAL CONTROL AS SOURCE OF CHAOTIC BEHAVIOR

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In the present paper, we introduce and analyze a mechanical system, in which the digital implementation of a linear control loop may lead to chaotic behavior. The amplitude of such oscillations is usually very small, this is why these are called micro-chaotic vibrations. As a consequence of the digital effects, i.e. the sampling, the processing delay and the round-off error, the behavior of the system can be described by a piecewise linear map, the micro-chaos map. We examine a 2D version of the micro-chaos map and prove that the map is chaotic.

Keywords: Digital control; micro-chaos; 2D map.

1. Introduction

The stability properties of digital and analogue control systems are different [Kuo, 1977], moreover, the application of digital control often leads to chaotic behavior [Chen & Dong, 1998]. Even a linear control law may lead to chaos if there is some nonlinearity in the implementation of the control system or if the state-dependent delayed control is applied [Campbell et al., 2005; Sieber & Krauskopf, 2004].

In the present paper, we analyze a simple model of a digitally controlled mechanical system, which may perform chaotic vibrations. As a consequence of the digital effects, i.e. the sampling, the delay and the round-off error, the behavior of this system can be described by a piecewise linear map, the micro-chaos map [Enikov & Stépán, 1998]. It was proved in the mid-nineties [Haller & Stépán, 1996] that the 1D version of the micro-chaos map is chaotic. A couple of years later, we found that if dry friction is present in the system, the resulting behavior is transient chaotic [Tel, 1990]. We developed a method for the exact calculation of the mean lifetime of chaotic transients in the case of the 1D piecewise linear micro-chaos map [Csernák & Stépán, 2005]. This lifetime estimation method is already extended to the 2D case [Csernák & Stépán, 2007]. The goal of our present contribution is to prove that the 2D version of the map is chaotic. There are a few examples for proof of chaos in higher dimensional systems [Mischaikow, 1995; Zgliczinski, 1996, 1997; Gidea & Zgliczinski, 2004; Tucker, 1999; Bánhelyi et al., 2007], but these proofs are typically performed by computers, using interval-arithmetic. We will apply a more classical approach.

2. Mechanical Model

The mechanical model of a digitally controlled polishing or burring tool is shown in Fig. 1. It consists of a revolving cylinder of mass \( m \) sliding on the rough surface of a fixed block, under the action of an electric motor.

The velocity of the shaft is denoted by \( \dot{x} = v \), while the circumferential velocity of the polishing tool equals \( R\omega \). The characteristic of the damping force between the revolving polishing tool and the fixed workpiece is a mixture of the dry and
viscous friction characteristics and depends on the relative velocity \( v_{rel} = R \omega_0 + v \) between the surfaces. At low relative speeds, the combined dry and viscous friction force acting on the cylinder is locally decreasing as \( v_{rel} \) increases [Conti, 1875; Galton, 1878]. This effect may lead to the so-called stick-slip vibrations in several uncontrolled mechanical systems [Pupp & Stelter, 1990].

Linearizing this force about \( v_{rel}(t) \equiv R \omega_0 \), i.e. at \( v(t) \equiv 0 \), one obtains

\[
mg \mu(R \omega_0) + mg \mu'(R \omega_0)(v_{rel} - R \omega_0) = mg \mu(R \omega_0) + mg \mu'(R \omega_0) v_{rel},
\]

where the prime refers to the differentiation with respect to the relative velocity, thus, \( \mu'(R \omega_0) < 0 \). The states \( v = 0 \) are unstable due to this negative damping. To stabilize the tool in a certain position, a control force must be applied. Note, that in real technological situations the polishing pressure — and consequently, the friction force between the tool and workpiece — may vary in time. This effect may provide an additional irregular excitation to the system. Thus, the evolving motion can be chaotic in large scales as well. In engineering practice, special control systems are applied to hold the polishing pressure at a constant value [Yi et al., 2004]. This is why we will not take its variation into account.

The shaft of the polishing tool is driven by a DC motor, which exerts a so-called PD control force \( Q \), governed by a digital control system. \( P \) and \( D \) denote the proportional and differential gain, respectively. If we do not take into account the digital effects, the equation of motion of the system assumes the following form:

\[
\dot{m}v = -mg \mu'(v_{rel}) + mg \mu(v_{rel}) - Dv - P_x,
\]

where — after linearization — leads to

\[
\dot{m}v + (D + mg \mu'(R \omega_0))v + Px = 0. \tag{3}
\]

For the sake of simplicity, we introduce the notation \( f \equiv \mu'(v_0) = -\mu'(v_0) \), where \( f > 0 \) for low relative velocities. Using this notation the equation of motion can be rewritten as

\[
\dot{m}x + (D - fmg)\dot{x} + Px = 0. \tag{4}
\]

According to the Routh–Hurwitz criterion, the solution \( x = 0, v = 0 \) is stable if \( P > 0 \) and \( D > fmg \). Thus, if we use an analogue control system, the appropriate choice of \( P \) and \( D \) results in the asymptotic stability of the equilibrium state.

However, in the case of digital control, the computer samples the position \( x \) and velocity \( v \) at discrete time instances, \( t_j \equiv x(j \tau) \) and \( v_j \equiv v(j \tau) \), \( j \in \{1, 2, \ldots\} \). Let us introduce a nondimensional time \( T = t/\tau \). The differentiation with respect to this new variable \( T \) will be denoted by the prime.

Since some time is needed to process the measured signal, the force is exerted by the motor a bit after the sampling instant. This processing delay is often equal to the sampling time. In these cases, the control force exerted at \( T = j \) depends on the data sampled at \( T = j - 1 \).

Finally, since the output signal has a finite resolution \( h \) caused by the round-off errors at the digital processor, the equation of motion of the digitally controlled system can be written as

\[
v'(T) - f\tau v(T) = -\frac{h}{m} \text{Int} \frac{P x_{j-1} + D v_{j-1}}{h} T \in [j, j+1).
\]

This equation is valid between two successive sampling instants, and can be solved in this time period:

\[
v(T) = v_j e^{f\tau(T-j)} + (1 - e^{f\tau(T-j)}) \frac{h}{f mg} \times \text{Int} \frac{P x_{j-1} + D v_{j-1}}{h}. \tag{6}
\]

Thus,

\[
v_{j+1} = v_j e^{f\tau} + (1 - e^{f\tau}) u_j, \tag{7}
\]

and

\[
x_{j+1} = x_j + \int_{j}^{j+1} v(T) dT = x_j + \frac{e^{f\tau} - 1}{f \tau} v_j + \left(1 - \frac{e^{f\tau} - 1}{f \tau}\right) u_j. \tag{8}
\]

Fig. 1. The mechanical model of the polishing tool.
where \( u_i = (h/fmg) \text{Int}(P_{x,i} + Dv_{y,i})/h \) corresponds to the control force exerted at \( T = j \).

Thus, the behavior of this system can be described by the following map (micro-chaos map [Haller & Stépán, 1996]):

\[
\begin{bmatrix}
  x_{j+1} \\
  y_{j+1} \\
  v_{j+1}
\end{bmatrix}
= 
\begin{bmatrix}
  e^{f\sigma} - 1 & 1 - e^{f\sigma} & 1 \\
  0 & e^{f\sigma} & 1 \\
  0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
  x_j \\
  y_j \\
  v_j
\end{bmatrix}
+ 
\begin{bmatrix}
  0 \\
  0 \\
  h/fmg
\end{bmatrix}
\begin{bmatrix}
  P_{x,j} + Dv_{y,j}
\end{bmatrix}.
\]

(9)

Technically, \( \tau \) is in the range of 1–10 milliseconds, \( h \) is in the range of 10–100 milliNewtons, and \( f'(r_0) = -f \) is “small” negative.

3. Fundamental Properties of the 2D Micro-Chaos Map

3.1. Derivation of the 2D micro-chaos map

If we give up the idea of stabilizing the shaft of the cylinder at a certain position (choose \( P = 0 \)), the map (9) reduces to a 2D map:

\[ v_{j+1} = v_j e^{f\sigma} + (1 - e^{f\sigma}) \left( \frac{h}{fmg} \text{Int} \frac{Dv_{y,j}}{h} \right). \]

(10)

By introducing the new variable \( y_j = Dv_{y,j}/h \), and notations

\[ a = e^{f\sigma} > 1 \]
\[ b = \frac{D}{fmg} \text{Int} \frac{1}{h} > 0, \]

expression (10) assumes a more tractable form:

\[
\begin{bmatrix}
  y_{j+1} \\
  y_{j+1}
\end{bmatrix}
= 
\begin{bmatrix}
  a & b \text{Int}(y_j-1) \\
  1 & 1
\end{bmatrix}
\begin{bmatrix}
  y_j \\
  y_j
\end{bmatrix}.
\]

(13)

Our goal is to prove that this 2D version of the micro-chaos map is chaotic in large (nonzero measure) sets of the parameters.

Taking into account the effect of sampling but disregarding the round-off error \( h \to 0 \), the corresponding map assumes the following scalar form:

\[ y_{j+1} = ay_j - b \text{Int}(y_j). \]

(14)

In this case, the origin is the single fixed point, which — according to Jury’s criterion [Kuo, 1977], i.e. the Moebius-transformed Routh–Hurwitz criterion — is stable if

\[ (a, b) \in G = \{ (a, b) \in \mathbb{R}^2 \mid a > 1, b < 1, b > a - 1 \}. \]

(15)

Thus, the sampling time, which, in our case is equal to the processing delay, cannot be chosen arbitrarily: \( 0 < \tau < \log(2)/(fg) \). During the analysis of (13), we will restrict ourselves to the parameter domain \( G \) (see Fig. 2), and examine cases when the solution is positive \( (y_j > 0, y_{j+1} > 0) \). Our results can be naturally extended to the case of negative solutions, too.

Let us note that an even simpler, 1D version of the micro-chaos map can be obtained by neglecting the processing delay:

\[ y_{j+1} = ay_j - b \text{Int}(y_j). \]

(16)

The map (16) is chaotic as proven in [Haller & Stépán, 1996]. A chaotic solution is shown in Fig. 3.

3.2. Fixed points and basic branches of the 2D map

Divide the plane \( (y_{j-1}, y_j) \) into parallel bands, according to Fig. 4:

\[ M_m = \{ (y_{j-1}, y_j) \mid m \leq y_{j-1} < m + 1 \} \]

if \( m = 1, 2, 3, \ldots \).
\( M_m = \{(y_{j-1}, y_j) \mid m-1 < y_{j-1} \leq m\} \) if \( m = -1, -2, -3, \ldots \). (18)

and

\( M_0 = \{(y_{j-1}, y_j) \mid -1 < y_{j-1} < 1\} \). (19)

There can be at most one fixed point in each band: \( \bar{p}_m = \text{col}[(\bar{y}_m, \bar{y}_m)] \), where \( m \leq \bar{y}_m = mb/(a-1) < m+1 \) if \( m \geq 0 \), and \( m-1 < \bar{y}_m = mb/(a-1) \leq m \) if \( m \leq 0 \).

In the neighborhood of the fixed point \( \bar{p}_m \), Eq. (13) can be rewritten as

\[
F_m \begin{bmatrix} y_{j-1} \\ y_j \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & a \end{bmatrix} \begin{bmatrix} y_{j-1} \\ y_j \end{bmatrix} - b \begin{bmatrix} 0 \\ m \end{bmatrix}.
\] (20)

The fixed points of (20) are hyperbolic with the eigenvalues \( \lambda_1 = 0 \) and \( \lambda_2 = a > 1 \).
The unstable manifold (unstable line) of the fixed point $p^\infty$ can be given as $U_m, y_j = ay_j - bm$. A direct consequence of the definition of map (13) is that any point $y_j = col[y_j, y^\infty_j]$ in band $m$ is im($y_j$) is mapped to $p_{j+1} = col[y_j, ay_j - b_m]$, i.e. to the unstable line $U_m$ of the fixed point $p^\infty$. Thus, the solutions immediately arrive at the family of unstable lines and stay on these lines. Consequently, a possible strange attractor must consist of segments of the unstable lines.

The intersection point of $U_m$ and the line $y_{j-1} = m$ is $p^m_m = col[m, (a - b)m]$, while the intersection point of $U_m$ and the line $y_{j+1} = m+1$ is $p^{m+1}_m = col[m+1, a(m+1) - bm]$. The upper index refers to the serial number of the unstable line while the lower index shows the first coordinate of the point.

Similar results can be obtained if $m < 0$. The intersection point of $U_m$ and the line $y_{j+1} = m$ is $p^m_m = col[m, (a - b)m]$, while the intersection point of $U_m$ and the line $y_{j-1} = m-1$ is $p^{m-1}_m = col[m-1, a(m-1) - bm]$ in this case.

Since points in the right hand are mapped onto the unstable manifold of $p^\infty$. For $p^\infty_m \equiv p^{m\infty}_m \equiv col[x^\infty_m, y^\infty_m] = col[(a - b)m, (a^2 - ab - b)m]$, the leftmost point that can be reached from $U_m$. Moreover, since $p^m_m \in M_{m+1}$, its "upper" image $p^{m\infty}_m = col[p^{m\infty}_m, p^{m\infty}_m] = col[(a - b)m + a, (a^2 - ab - b)m + a^2]$ is the rightmost accumulation point of those that can be reached from $U_m$. According to these results, the basic branch [Tel & Grus, 2006] of the fixed point $p^\infty$, i.e. the longest piece of the manifold emanating from the fixed point, is defined as

$$y_j = ay_j - bm, \quad (a - b)m \leq y_j < (a - b)m + a \quad \text{if } m \geq 0,$$

$$y_j = ay_j - bm, \quad (a - b)m > y_j \geq (a - b)m - a \quad \text{if } m < 0,$$

and

$$y_j = ay_j, \quad -a \leq y_j \leq a \quad \text{if } m = 0.$$

As computer experiments have shown, there are finite domains in the parameter plane $(a, b)$, where the solution stays on the basic branches of certain fixed points. These branches form the attractor of the system, which is equivalent to a multivalued version of the 1D map — see Fig. 4.

The successive iteration steps can be followed easily on a modified col-web diagram: Starting at a certain initial point, the next point is found as follows: one projects the point "horizontally" to the diagonal, then "vertically" to the line of the attractor. In the multivalued domains, the appropriate branch is selected according to the previous value of the coordinate $y_j$.

4. Proof of Chaos

In this section, we will prove that map (13) has chaotic solutions in certain parameter domains. For the sake of simplicity, we restrict ourselves to the cases when $(a, b) \in G$, the basic branches are not dashed, solutions cannot assume negative values, and the endpoints of the basic branches are $p^m_m$ and $p^{m+1}_m$, respectively. Certain parameter domains, where these conditions are fulfilled, are shown in Fig. 5.

The greatest domain, denoted by "01", corresponds to the parameter values where the basic branches lie on the unstable lines of the fixed points $p^0$ and $p^1$. According to the detailed analysis of the aforementioned conditions — which can be found in the Appendix — the boundary curves of this domain are $n^{\text{sup}}_{m+1} = a^2(a - 1), n^{\text{inf}}_{m+1} \equiv n^{\text{inf}}_{m+1} = a^2(a - 1)/(a^2 - 1), d^1 = a(a^2 - 1)$, and $d^0 = a(a^2 - 1)/(a^2 + a - 1)$.

Similarly, if the parameters are found in the domain "012", the attractor consists of three branches: the third branch is on the unstable line $U_2$. The boundary curves are $d^1 = a(a^2 - 1)/(a^2 + a - 1) d^2 = 2a(a^2 - 1)/(a^2 + a + 1), n^{\text{sup}}_{m+2} = 2a^2(a - 1)/(a^2 + 1), g^{\text{sup}} = 2a - a^2, g^{\text{sup}} = (2a^2 - a)/(2a + 1)$.

4.1. Sensitive dependence on initial conditions

We know that solutions arrive at the family of unstable lines $y_j = ay_j - bm, m = 0, 1, 2, \ldots$, and stay on these lines. The distance of two neighboring lines is $b/\sqrt{1 + a^2}$.

Let us fix a constant $\delta = b/(1 + a^2) < b/\sqrt{1 + a^2}$. We will show that for any two points $p_j \neq q_j$, with $|p_j - q_j| < \delta$, there exists $N \geq 1$ such that the distance of the $N$th iterates of these points is greater than $\delta$.

$$|p_N - q_N| \geq \delta.$$  

According to the conditions stated above, $p_N$ and $q_N$ must lie on the same line and if they arrive at
different lines, we are done. Since the map exponentially expands the distances of points taken from the same band and the same line, without loss of generality we can assume that after \( n \) iterations the two points arrive at different bands. If \(|p_n - q_n| \geq \delta\), we are done. If not, then we have \(|p_n - q_n| < \delta = b/(1 + a^2) < 1\), thus, we can assume that \( p_n = \text{col}[u, av - mb] \in M_i \) and \( q_n = \text{col}[v, av - mb] \in M_{i+1} \). The next iterates of these points are \( p_{n+1} = \text{col}[au - mb, a(au - mb) - ib] \) and \( q_{n+1} = \text{col}[av - mb, a(av - mb) - (i + 1)b] \). Consequently,

\[
|p_{n+1} - q_{n+1}| = |\text{col}(u - v, a^2(u - v) - b)| > |\text{col}(0, a^2(u - v) - b)| = b - a^2|u - v| > b - a^2\delta = \delta. \tag{25}
\]

Hence the choice \( N = n + 1 \) completes the proof. \( \blacksquare \)

### 4.2. Existence of attractor

Expression (13) can be rewritten as

\[
P_j = \begin{bmatrix} y_j \\ y_{j+1} \end{bmatrix} = \begin{bmatrix} 0 & \ -1 \\ \ -b & \ a \end{bmatrix} \begin{bmatrix} y_{j-1} \\ y_j \end{bmatrix} = \begin{bmatrix} 0 \\ y_j \end{bmatrix} \chi_j^{-1} = \begin{bmatrix} 0 \\ y_j \end{bmatrix} \chi_j^{-1} = B^j p_0 + \sum_{k=0}^{j-1} B^k b \chi_k, \tag{26}
\]

where \(-1 < \chi_k < 1\).

\[
B = \begin{bmatrix} 0 & \ 1 \\ \ -b & \ a \end{bmatrix}, \ \ b = \begin{bmatrix} 0 \\ b \end{bmatrix}, \ \ p_0 = \begin{bmatrix} 0 \\ y_1 \end{bmatrix}, \ \ \text{and} \ \ p_0 = \begin{bmatrix} 0 \\ y_1 \end{bmatrix}. \tag{27}
\]

According to numerical evidence, the numbers \( \chi_k \) vary irregularly and the sequence (26) is neither convergent, nor divergent for almost all initial vectors \( p_0 \). However, for sufficiently large index \( j \) the vectors \( p_j \) stay in the neighborhood of the origin. In the following, we prove this property and give an estimation for the maximal possible norm \( |p|_\infty \) of the vectors \( p_j \), in the limit \( j \to \infty \). The norm of this maximal vector provides an estimation for the size of the attracting domain at the origin.

The eigenvalues of the matrix \( B \) are

\[
\lambda_\pm = \frac{a \pm \sqrt{a^2 - 4b}}{2}. \tag{28}
\]

(A) If \( a^2 \geq 4b \), the eigenvalues are real numbers. Moreover, if \((a, b) \in G\), both eigenvalues are less than one: \( 0 < \lambda_- \leq \lambda_+ < 1 \). Since \( B \) is not a normal matrix, the Euclidean norm of the matrix is given by the greatest singular value \( \sigma_1 \), which is not equal to the spectral radius \( \rho = \lambda_+ \). Thus, the assumption \( |B| = \rho \) is not valid and cannot be exploited to approximate the maximal norm \( |p|_\infty \), as it was done in Enikov & Stepin, 1998.

The singular values \( \sigma_{1,2} \) of \( B \) can be obtained as the positive roots of the equation

\[
(\sigma^2)^2 - \sigma^2(\sigma^2 + 1) + \sigma^2 = 0. \tag{29}
\]
Although the greatest singular value — the norm of $\mathbf{B}$ — can be greater than one, 
\[ \lim_{j\to\infty} \|\mathbf{B}^j\| = \lim_{j\to\infty} \rho^j = 0 \quad (30) \]

is fulfilled. Thus, 
\[ |\mathbf{p}_\infty| = \max_k \left| \sum_{k=0}^\infty \mathbf{B}^k \mathbf{v}_k \right|. \quad (31) \]

The following transformation matrix can be constructed using the eigenvectors of $\mathbf{B}$: 
\[ \mathbf{T} = [s_1 \ s_2] = \begin{bmatrix} 1 & \lambda_+ & 1 \\ 0 & \lambda_+ & 0 \end{bmatrix}. \quad (32) \]

The matrix $\mathbf{B}$ and the vector $\mathbf{b}$ expressed in the basis of the eigenvectors:
\[ \tilde{\mathbf{B}} = \mathbf{T}^{-1} \mathbf{B} \mathbf{T} = \begin{bmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{bmatrix}, \quad \tilde{\mathbf{b}} = \mathbf{T}^{-1} \mathbf{b} = [b_1 \ b_2] = \begin{bmatrix} b \\ \sqrt{\alpha^2 - 4b} \end{bmatrix}. \quad (33) \]

Thus, 
\[ \sum_{k=0}^\infty \tilde{\mathbf{B}}^k \mathbf{v}_k = \begin{bmatrix} \sum_{k=0}^\infty \lambda_+^k \mathbf{v}_k \\ \sum_{k=0}^\infty \lambda_-^k \mathbf{v}_k \end{bmatrix}. \quad (34) \]

The supremum of the norm of this vector can be obtained with $\chi_k = 1, k = 0 \ldots \infty$, or $\chi_k = (-1)^k, k = 0 \ldots \infty$. We determined the sums of the series and transformed back the obtained vectors to the original $(y_{j-1}, y_j)$ coordinate system. The resulting vectors are 
\[ \mathbf{p}_{\infty,1} = \mathbf{T} \tilde{\mathbf{p}}_\infty = \begin{bmatrix} b \\ 1 + b - a \end{bmatrix} \quad (35) \]

and 
\[ \mathbf{p}_{\infty,-1} = \begin{bmatrix} b \\ 1 + b - a \end{bmatrix}. \quad (36) \]

respectively. Consequently, there exists a so-called absorbing ball [Robinson, 1995], i.e. a circle of radius $R = |\mathbf{p}_{\infty,1}| = |\mathbf{p}_{\infty,-1}| = \sqrt{2b}/(1 + b - a)$, which attracts every solution and can be considered as a global attractor. For example, at $a = 1.2$ and $b = 0.22$, the radius is $R \approx 15.556$. We detected twelve separated local attractors in this case, all of them are inside the global attractor, as seen in Fig. 6.

(B) If $\alpha^2 < 4b$, the eigenvalues of $\mathbf{B}$ form a complex conjugate pair and the spectral radius is $\rho = \sqrt{\lambda_+ \lambda_-} = \sqrt{\alpha^2} < 1$. In this case 
\[ \sum_{k=0}^\infty \tilde{\mathbf{B}}^k \mathbf{v}_k = \begin{bmatrix} b \\ \sqrt{4b - \alpha^2} \end{bmatrix} \quad (37) \]

where $i = \sqrt{-1}$ and $\phi = \arctan \sqrt{(4b - \alpha^2)/\alpha}$. Due to the phase factors in expression (37), the domain of possible vectors $\mathbf{p}_\infty$ cannot be determined as in the previous case. It is easy to check that the norm

![Fig. 6. Twelve local attractors within the absorbing ball at $a = 1.2$ and $b = 0.22.$](image)
obtained with $\chi_k = 1, k = 0 \ldots \infty$ is not maximal. However, an upper estimate can be obtained for the norm of $\mathbf{p}_\infty$:

$$|\mathbf{p}_\infty| \equiv \max \left| \sum_{k=0}^{\infty} b_k^* \mathbf{b}_k \right|$$

$$= \max \left| T \sum_{k=0}^{\infty} b_k^* \mathbf{b}_k \right|$$

$$= \max \left| T \left( \sum_{k=0}^{\infty} \lambda_k^i b_k \right) \right|$$

$$\leq |T| \left( \sum_{k=0}^{\infty} |\lambda_k^i|^k |b_k| \right)$$

(38)

Since $|\lambda| = |\lambda| = |b| = |b| = b/\sqrt{b^2 - a^2}$, and the Euclidean norm of $T$ is $||T|| = \sqrt{1 + b^2 + \sqrt{b^2 - a^2} + b^2 + 1 - 2b}$, we obtain

$$|\mathbf{p}_\infty| \leq \frac{\sqrt{2(1 + b^2 + \sqrt{b^2 - a^2} + b^2 + 1 - 2b)b}}{\sqrt{b^2 - a^2}(\sqrt{b^2 - a^2})}$$

(39)

In Fig. 7, we show the solution as $a = 1.4$, $b = 0.8$ and the circle of radius $R = 17.2536$. As it can be seen, this circle is not invariant, since the solution may leave it temporarily, but finally arrives back inside the circle.

We proved so far that every solution tends to the neighborhood of the origin. It follows from the definition of the points introduced in Sec. 3.2 and the construction of the parameter domains shown in Fig. 5, that in these domains the basic branches between col$[x_l, y_l]$ and col$[x_h, y_h]$ form an invariant and attractive set $\mathcal{A}$. More precisely, the attractor consists of segments of lines $y_j = a\tilde{y}_{j-1} - b$, where

$$y_{j-1} \in (x_l, x_{\text{col}}^{m_{\text{min}}-1})$$

$$y_{j-1} \in (x_{\text{inf}}, x_{\text{col}}^{m_{\text{min}}})$$

... (40)

$$y_{j-1} \in (x_{\text{sup}}, x_{\text{col}}^{m_{\text{max}}})$$

$$y_{j-1} \in (x_{\text{inf}}, x_{\text{col}}^{m_{\text{max}}+1})$$

... (41)

$$y_{j-1} \in (x_{\text{inf}}, x_{\text{col}}^{m_{\text{max}}+1})$$

... (42)

As an example, see Fig. 4, where $m_{\text{min}} = 2$ and $m_{\text{max}} = 3$. The set

$$y_{j-1} \in (m_{\text{min}} - 1, m_{\text{min}})$$

$$y_{j-1} \in (y_{m_{\text{min}}-1}, y_{m_{\text{min}}})$$

... (43)

$$y_{j-1} \in (m_{\text{min}} - 1, m_{\text{min}})$$

$$y_{j-1} \in (y_{m_{\text{min}}}, y_{m_{\text{min}}+1})$$

... (44)

$$y_{j-1} \in (m_{\text{min}} - 1, m_{\text{min}})$$

$$y_{j-1} \in (y_{m_{\text{min}}}, y_{m_{\text{min}}+1})$$

... (45)

$$y_{j-1} \in (m_{\text{min}} - 1, m_{\text{min}})$$

$$y_{j-1} \in (y_{m_{\text{min}}}, y_{m_{\text{min}}+1})$$

... (46)

![Fig. 7. Solutions tend to the attractor ($a = 1.4$, $b = 0.8$).](image)
\( y_{j-1} \in (m_{\text{max}} + 1, m_{\text{max}} + 2) \),
\( y_j \in ([y_{m_{\text{max}}+1}, y_{m_{\text{max}}+1}) \) \hspace{1cm} (47)

is the basin of attraction of \( A \).

There are cases when the attractor consists of only two mangled basic branches. In such cases \( m_{\text{min}} = m_{\text{max}} + 1 \). Although these numbers cannot be interpreted as the indices of the leftmost and rightmost whole basic branches, the above expressions still hold.

**4.3. Topological transitivity**

In the following section we restrict ourselves to the parameter domain denoted by “01” in Fig. 5. Our goal is to show that there are certain subdomains in “01” where the attractor \( A \) can be partitioned in such a way that the partition is irreducible and primitive. In an irreducible and primitive partition, every region can be reached from any other partition and there is at least one region whose image fully covers at least two other regions.

As an example, consider the attractor of map (13) at parameters \( a = 1.3 \) and \( b = 0.66 \) (see Fig. 8).

The partition of the attractor is constructed as follows: the open intervals on the branches between the line \( y_{j-1} = 1 \) and lines \( A_0 \) or \( A_1 \) are the basic regions, denoted by \( B^L \) and \( B^U \) on the lower and upper branches, respectively, in Fig. 8 (the subscripts 4 and 9 are the serial numbers of these regions, according to the symbolic dynamics). One endpoint of both sections coincides with the endpoint of the corresponding basic branch, according to the conditions stated in Sec. 5. Thus, these basic regions are well defined if the parameters are chosen from the domain “01”. The preimages of these segments form the remaining regions. The inverse of \( F_m \) is defined such that the resulting point lies on the \( n \)th basic branch:

\[
F^{-1}_m \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} \frac{z_2 + bn}{a^2 + bn} \\ \frac{z_2 + bn}{a} \end{bmatrix}.
\hspace{1cm} (48)
\]

As it can be seen in Fig. 8, the fourth preimage of region \( B^L \equiv B^L_{14} \) on the lower branch — branch 1 — would stretch over \( \text{col}[x_r, y_r] \), this is why the rightmost region of this branch — region 1 — consists of region \( F^{-1}_1(B^L) \) and the remaining part of the branch 1. This construction ensures that the closure of the image of region 1 fully covers region \( F^{-1}_1(B^L)_{12} \), whose image also fully covers region \( F^{-1}_1(B^U)_{31} \), etc. It is also important that the image of region \( B^L \equiv B^L_{14} \) fully covers region 1 — this fact is not a consequence of the construction of the regions, thus, should be checked in the general case.

---

**Fig. 8.** Partition of the attractor \( (a = 1.3, b = 0.66) \).
The upper branch — branch 0 — can be partitioned in a similar way: certain preimages of section $B_0^\theta$ are marked in the figure: $F_0^{-1}(B_0^\theta)$, $F_0^{-1}(B_0^\theta)$, and $F_0^{-1}(B_0^\theta)$. As it can be observed, the image of region $B_4$ is fully covered by region $B_4$, i.e. $F_0(B^\theta)$, fully covering region $F_0^{-1}(B_3^\theta)$ and the remaining part of the upper basic branch. The condition for this case should also be checked in the general case. To construct a primitive partition, it is enough to form only a single additional region (number 5) of the preimages of regions $F_0^{-3}(B_0^\theta)$, and the remaining regions can be reached from any other region and the image of region 4 fully covers two other regions.

In order to find the parameter domain where similar primitive partitions can be constructed, introduce the following notation: $F_0[p^1_1] = col[a - b, a(a - b)]$ denotes the rightmost accumulation point of $F_0(B^\theta)$ — this is the image of point $p^1_1$. Point $F_0[p^1_2]$ is in a preimage of $B_0^\theta$, $F_0(B^\theta)$ fully covers at least two preimages of $B_0^\theta$ if $F_0^{-1}(F_0[p^1_2])$ is still on the upper branch. This condition can be formulated as follows:

$$\frac{a - b}{a} \geq y' \equiv a(a^2 - ab - b), \quad \text{thus (49)}$$

$$b \geq \frac{c^2_m}{a^2 + a^2 - 1} \equiv \frac{a(a - 1)}{a^2 + a^2 - 1}, \quad \text{(50)}$$

In the example, at $a = 1.3, b \geq 0.5953$.

$F_1[p^1_1] = col[a, a^2 - b]$ denotes the rightmost accumulation point of $F_1(B^\theta)$ — this is the image of point $p^1_2$. Point $F_1[p^1_2]$ is in a preimage of $B_0^\theta$. $F_1(B^\theta)$ fully covers one preimage of $B_0^\theta$ if $F_1^{-1}(F_1[p^1_2])$ is still on the lower branch. This condition means that

$$\frac{a + b}{a} \leq y'' \equiv a^3 - b, \quad \text{thus (51)}$$

$$b \leq c^2_i \equiv \frac{a^4 - a}{a + 1}. \quad \text{(52)}$$

In the example, at $a = 1.3, b \leq 0.6766$. If $b \geq c^3_m$, the image $F_0(B^\theta)$ covers more than one region on the upper branch. Moreover, if $b < c^4_i$, the image $F_1(B^\theta)$ covers at least one region on the lower branch. Thus, the partition is irreducible and primitive.

The conditions, implying that the image of the basic region $B^\theta$ covers at least one region on the upper branch and the image of the basic region $B^\theta$ covers more than one region on the lower branch, can be obtained in a similar way:

$$b \geq c^2_i \equiv \frac{a^4 - a}{a + 1}. \quad \text{(53)}$$

$$b \leq c^4_i \equiv \frac{a(a^3 - 1)}{1 + a + a^2}. \quad \text{(54)}$$

The parameter domains, where conditions (50) and (52) or (53) and (54) are fulfilled, are shown in Fig. 9.

$$a_{ij} = \begin{cases} 1 & \text{if } p_j \supset I_j, \\ 0 & \text{otherwise} \end{cases} \quad \text{(55)}$$

Fig. 9. Parameter domains, where primitive partitions can be constructed.
In the example, the transition matrix of the constructed partition can be written as

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}. \tag{56}
\]

In the marked parameter domain, shown in Fig. 9, the structure of the partition and the transition matrix is similar to the one shown in the example, i.e. there is one nonzero element in every row and column, not in the main diagonal, and there is an additional nonzero element \( a_{kj} \):

\[
a_{ni+1} = 1, \quad i = 1 \cdots m - 1 \tag{57}
\]

\[
a_{m1} = 1 \tag{58}
\]

\[
a_{kj} = 1, \quad \text{where} \ k \neq j, \ k + 1 \neq j. \tag{59}
\]

These matrices are irreducible and primitive, as well, i.e. there exists an integer number \( n > 0 \), such that every element of \( A^n \) is positive. This property can be proved as follows: in the case shown above (59), \( A^2_{ij} > 0 \) if \( k > 1 \), or \( A^2_{kk} > 0 \) if \( k = 1 \), and \( A^2_{i+j} > 0 \) if \( j < m \), or \( A^2_{k+1} > 0 \) if \( j = m \). Thus, the existence of the additional nonzero element \( a_{kj} \) implies the appearance of two additional nonzero elements. Moreover, \( A^2_{ni+2} > 0 \) if \( i < m - 1 \), or \( A^2_{ij} > 0 \) if \( i = m - 1 \), or \( A^2_{ij} > 0 \) if \( i = m \). Thus, the diagonal nonzero elements are simply shifted to the right. It can be proved by induction, that the number of nonzero elements increases at least by one during the multiplication by \( A \). Thus, \( A^n \) contains only positive elements, where \( N \leq m^2 - m \).

In the example, \( N = 65 < 9^2 - 9 = 72 \).

We must emphasize at this point that the constructed partition is not necessarily a generating partition. It is obvious, that to every possible semi-infinite symbol sequence corresponds at least one point on the attractor. However, there may exist points, whose itinerary cannot be described with the symbolic dynamics on the introduced partition, i.e. the partition is not Markovian. Let us introduce the set \( \mathcal{L} \subseteq \mathcal{A} \) as the set of points that correspond to a symbol sequence. The proved property of the transition matrix \( A \) implies that the micro-chaos map is topologically transitive on \( \mathcal{L} \) and there exists a countable infinity of periodic orbits, an uncountable infinity of nonperiodic orbits and a dense orbit (see the proof in [Haller & Stépán, 1996]). Thus, according to e.g. [Robinson, 1995; Wiggins, 1990], the 2D micro-chaos map is chaotic on the set \( \mathcal{L} \).

Note, that in [Haller & Stépán, 1996] the fractal dimension of a similar subset of the attractor was calculated in the case of the one dimensional micro-chaos map. However, it was meaningless to give an upper estimate for that non-Markovian symbolic dynamics: the result was less than one, while the fractal dimension of the attractor was exactly one in the considered case.

Micro-chaotic vibrations may occur in mechanically realistic systems, too. For example, at parameters \( m = 1 \text{kg}, \tau = 10 \text{ms}, g = 9.81 \text{m/s}^2, D = 3.5 \text{Ns/m}, \) and \( f = 0.1 \text{s/m}, \) one obtains \( a \approx 1.0 \) and \( b \approx 0.035 \). These parameters do not belong to the parameter domain “01”, but the proof can be extended to this case without significant changes. The resulting chaotic attractor can be seen in Fig. 11.

5. Conclusions
As a consequence of the digital effects — the sampling, the processing delay, and the round-off error — controlled systems may behave irregularly. We considered a simple mechanical system under linear control law, with differential gain \( D \), sampling time and processing delay \( \tau \) and resolution \( h \) of the digital-analogue converter. The solutions of our mathematical model can be described by a 2D map. We proved that the solutions are sensitive to the initial conditions, there is an attractive set in the neighborhood of the origin, and showed that the formulae describing the invariant attractor can be derived easily in certain parameter domains. We also showed an algorithm with which the attractor can be partitioned into an irreducible and primitive partition. These properties implied in the case of the 1D micro-chaos map [Haller & Stépán, 1996] that the map is topologically transitive on a certain set, there exists a countable infinity of periodic orbits, an uncountable infinity of nonperiodic orbits and a dense orbit, thus, the 1D micro-chaos map is chaotic. Exploiting that the graph of the 1D micro-chaos map and the attractor of the 2D map are similar, these last steps of the proof, published in [Haller & Stépán, 1996], can be applied to the 2D case without significant change. Consequently, the 2D micro-chaos map is chaotic, too.
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References


Appendix

Important Relations

There are several relations with which the important properties of the dynamics of the system can be characterized:

(A) If $y_{m+1} > y_{m} - b$ and $m ≥ 0$, the solution can jump from the $m$th basic branch to the next $(m + 1)$th basic branch, over the corresponding fixed point. Consequently, the solution may reach the $(m + 2)$nd basic branch, as well. This situation may occur if

$$b < n^m_{\text{sup}} = \frac{a^2(a - 1)(m + 1)}{n^{2a} + 1}, \quad (A.1)$$

For example, $n^2_{\text{sup}} \approx 0.614 > b$ in the case shown in Fig. 4, thus, $p^2_{\text{sup}}$ is over the fixed point $p^3$. Consequently, the solutions can reach band $M_4$ as they move away from $p^2$, and finally, they reach the unstable line of $p^3$. Since $n^4_{\text{sup}} \approx 0.58 < b$, the solutions cannot go further, to the unstable line of $p^5$.

Similarly, if condition $y_{m+1} < y_{m-1}$ is fulfilled at $m > 0$, which means that

$$b > n^m_{\text{inf}} = \frac{a^2(a - 1)m}{n^{2a} - 1}, \quad (A.2)$$

the solutions can jump from the $n$th basic branch over the fixed point of the $(m - 1)$st basic branch. Thus, the $(m - 2)$nd branch can be reached by them. At $a = 1.5$ and $b = 0.6$ (Fig. 4), $n^4_{\text{inf}} \approx 0.587 < b$, and $p^4_{\text{inf}}$ is under the fixed point $p^2$. Consequently, the solutions can reach band $M_4$ as they move away from $p^3$, and finally, they reach the basic branch of $p^5$. 


Note that if \( g_{1}^{l} < g^{0} \), the positive solutions may jump to the negative domain. This situation occurs if \( b > n_{1}^{m} \), where

\[
\begin{align*}
\text{if } y_{m}^{m} < g^{m-1} \text{ and } m \leq 0, \\
\quad b < n_{m}^{m} = \frac{a^2(a - 1)(m - 1)}{ma^2 - 1}, \quad \text{while} \quad (A.3)
\end{align*}
\]

\[
\begin{align*}
\text{if } y_{m}^{m} > g^{m+1} \text{ and } m \leq 0 \implies \\
\quad b > n_{m}^{m} = \frac{a^2(a - 1)m}{ma^2 + 1}. \quad (A.4)
\end{align*}
\]

The same problem occurs if \( g_{m}^{m} < y_{m-1}^{m-1} \) and \( m \geq 1 \):

\[
\begin{align*}
\text{if } y_{m}^{m} > y_{m+1}^{m+1} \text{ and } m \leq -1, \\
\quad b > g_{m}^{m} = \frac{a^2m + a(1 - m)}{am + 1}, \quad (A.7)
\end{align*}
\]

\[
\begin{align*}
\text{if } y_{m}^{m} < y_{m-2}^{m-1} \text{ and } m < 0, \\
\quad b < g_{m}^{m} = \frac{a^2(m - 1) - a(m - 2)}{am + 1}, \quad (A.8)
\end{align*}
\]

\[
\begin{align*}
\text{if } y_{m}^{m} < y_{m+1}^{m+1} \text{ and } m > 2, \\
\quad b < g_{m}^{m} = \frac{a^2(m - 1) - a(m - 2)}{am + 1}. \quad (A.9)
\end{align*}
\]

For example, if \( a = 1.55 \) and \( b = 0.8 \) (see Fig. 10), \( g_{m}^{m} = 0.6975 \), thus, the rightmost accumulation point on the branch of \( U \) is not \( p_{m}^{m} \), the basic branch stretches over this point.

![Fig. 10. “Overstretching” of a basic branch, \( a = 1.5, b = 0.8 \).](image)
For example, see Fig. 11, where \( d^3 \approx 0.0197 < b = 0.035 \), and \( F_1(p_0^n) \) is below \( p_0^n \). As a consequence, a gap appears on the basic branch belonging to \( U_1 \).

(D) Note, that there are cases when more than two basic branches overlap. The condition of \( k \)-fold overlap of branches is

\[
y_m > y_{m+k-1},
\]

which leads to

\[
b > \frac{a(k-2)}{k-1}.
\]

If conditions (A.1)–(A.4) are not fulfilled, the basic branches of \( \bar{p}_{m+1} \) or \( \bar{p}_{m-1} \) become mangled, since the solution cannot jump to the other side of the corresponding fixed point.

Thus, the maximal and minimal numbers \( m \), due to which whole basic branches occur, are

\[
m_{\text{min}} = \frac{b}{a^2(b+1-a)}
\]

\[
m_{\text{max}} = \frac{a^2(a-1)-b}{a^2(b+1-a)} + 1.
\]

The rightmost accumulation point of the attractor can be obtained as \( F_{m_{\text{max}}+1}(p_{m_{\text{max}}}) \):

\[
col[x_r, y_r] = \text{col}[a(a-b)-b]m_{\text{max}} + a^2,
\]

\[
ax_r - b(m_{\text{max}} + 1).
\]

The leftmost point of the attractor can be obtained as \( F_{m_{\text{min}}-1}(p_{m_{\text{min}}}) \):

\[
col[x_l, y_l] = \text{col}[a(a-b)-b]m_{\text{min}},
\]

\[
ax_l - b(m_{\text{min}} - 1).
\]

The introduced points are shown in Fig. 4, together with the auxiliary lines \( A_{0}: y_j = (a^2 - ab - b/a - b)y_{j-1} \) and \( A_{\pm}: y_j = (a^2 - ab - b/a - b)y_{j-1} \pm (ab/a - b) \), passing through the endpoints of the basic branches.