Balancing with Reflex Delay

G. STÉPÁN AND L. KOLLÁR
Department of Applied Mechanics
Technical University of Budapest
H-1521 Budapest, Hungary
stepan@mm.bme.hu

Abstract—we consider the simplest possible model describing the “man-machine” system when somebody places the end of a stick on his fingertip, and tries to move the lowest point of the stick in a way that its upper position should be stable. The stability analysis of the mechanical model provides surprisingly simple and clear results in spite of the fact that modelling the time delay makes the corresponding differential equation infinite dimensional in mathematical sense. The calculated critical delay of reflexes and the frequency of the nonlinear vibrations arising as a result of a Hopf bifurcation at the limit of linear stability shows good agreement with experimental observations. By means of the above results, we also get some in-view into the work of the organ called “labyrinthus” in the inner ear which helps in self-balancing of the human body even when our eyes are closed.

© 2000 Elsevier Science Ltd. All rights reserved.

Keywords—Delay, Hopf bifurcation, Balancing, Inner ear.

1. INTRODUCTION

Unstable equilibria of mechanical systems often have to be stabilized by the human operators, e.g., by the sportsmen. A typical basic example for this is the self-balancing of the human body in target shooting.

The analysis of the problem of balancing an inverted pendulum proves that the human operator has to apply a quite complicated control strategy if he wants to achieve his goal in the presence of the time delay of his/her reflexes. It is a rule of thumb, that increasing time delay tends to destabilize any dynamical system. To avoid the instability naturally occurring in the mechanical system and also caused by the time delay, the human operator has to choose the control parameters from a narrow region which can be found only after more or less practicing. Above a critical value of the reflex delay, the balancing is impossible.

Stabilization of the inverted pendulum is a challenging basic example, hence, a long series of publications has appeared in this line for the last 40 years (see, e.g., [1-5] as not a complete list) either about its experimental or theoretical aspects. This problem is interesting not only in biology, but also in robotics, to construct biped robots (see, e.g., [6]).

In the subsequent sections, the stability chart in the space of the control parameters is constructed and the above-mentioned critical reflex delay is calculated. The surprisingly simple analytical results have interesting physical meaning and they show a good correlation with simple experimental observations. These results also provide some in-view into the work of the organ
called "labyrinthus" in the inner ear which helps in the self-balancing of the human body when our eyes are closed.

2. MECHANICAL MODEL OF BALANCING

Consider the simplest planar mechanical model of the inverted pendulum shown in Figure 1. Its lowest point slides smoothly along the horizontal line. This mechanical model is the simplest possible model describing the "man-machine" system when somebody places the end of the stick on his fingertip, and tries to move this lowest point of the stick in a way that the stick is balanced at its upper position. The system has two degrees of freedom described by the general coordinates \( \varphi \) and \( z \). The angle \( \varphi \) is detected together with its derivatives and the horizontal control force \( Q \) is determined by them in a way that the upper \( \varphi \equiv 0 \) position should be asymptotically stable.

![Figure 1. Mechanical model of stick balancing.](image)

The nonlinear equations of motion are derived by means of the Lagrangian equations of the second kind. They assume the form

\[
\begin{pmatrix}
m & \frac{1}{2} m l \cos \varphi \\
\frac{1}{2} m l \cos \varphi & \frac{1}{3} m l^2
\end{pmatrix}
\begin{pmatrix}
\ddot{\varphi} \\
\ddot{z}
\end{pmatrix}
- \begin{pmatrix}
\frac{1}{2} m l \dot{\varphi}^2 \sin \varphi \\
\frac{1}{2} m l g \sin \varphi
\end{pmatrix} = \begin{pmatrix}
Q \\
0
\end{pmatrix},
\]

from which the "cyclic" coordinate \( z \) can easily be eliminated to be left with the single second-order equation

\[
(4 - 3 \cos^2 \varphi) \ddot{\varphi} + \frac{3}{2} \dot{\varphi}^2 \sin(2\varphi) - \frac{6g}{l} \sin \varphi = -\frac{6}{ml} Q \cos \varphi. \tag{1}
\]

In these equations, \( g \) stands for the gravitational acceleration.

The control force \( Q \) is considered in the simplest form of a PD controller with constant gains \( P \) and \( D \) chosen by the operator appropriately

\[
Q(t) = D\dot{\varphi}(t - \tau) + P\varphi(t - \tau). \tag{2}
\]

In this formula, the human reflex delay is also modelled by a constant time lag (or dead time) \( \tau \). Clearly, the trivial solution \( \varphi \equiv 0 \) in (1),(2) describes the equilibrium to be stabilized.

3. STABILITY ANALYSIS

A stability analysis of the \( \varphi \equiv 0 \) position is required to find suitable control parameters \( P, D \) in (2). The variational system of the motion equation (1) with (2) at the trivial solution assumes the form of a linear retarded differential difference equation (RDDE)

\[
\dot{\varphi}(t) - \frac{6g}{l} \varphi(t) + \frac{6}{ml} (D\dot{\varphi}(t - \tau) + P\varphi(t - \tau)) = 0. \tag{3}
\]
THEOREM 3.1. If there is no delay in the system, i.e., \( \tau = 0 \), then the trivial solution of (3) is asymptotically stable in Lyapunov sense if and only if
\[
P > mg \quad \text{and} \quad D > 0.
\]
This statement can easily be proved by means of the well-known Routh-Hurwitz criterion since (3) becomes a simple ordinary differential equation in this case.

THEOREM 3.2. Let the time delay be positive in (3), i.e., \( \tau > 0 \). Approximate the delay effect with its first-order Taylor polynomial, that is
\[
\varphi(t - \tau) \approx \varphi(t) - \dot{\varphi}(t)\tau \quad \text{and} \quad \dot{\varphi}(t - \tau) \approx \dot{\varphi}(t) - \ddot{\varphi}(t)\tau.
\]
The trivial solution of the resulting ordinary differential equation is asymptotically stable if and only if
\[
P > mg, \quad D > \tau P, \quad \text{and} \quad D < \frac{ml}{6\tau}.
\]
Furthermore, there exists a “critical delay” \( \tau_{cr} \) with the following properties:

(i) for every \( \tau < \tau_{cr} \) there exist parameters \( P \) and \( D \) such that the zero solution of the resulting ordinary differential equation is asymptotically stable,
(ii) for every \( \tau > \tau_{cr} \) and for any \( P \) and \( D \) the zero solution of the resulting ordinary differential equation is unstable.

The value \( \tau_{cr} \) is given by
\[
\tau_{cr} = \sqrt{\frac{l}{6g}}.
\]
The proof of the first statement of this theorem can be based on the analysis of the corresponding transcendental characteristic function
\[
\lambda^2 - \frac{6g}{l} \tau^2 + \frac{6}{ml} \tau D e^{-\lambda} + \frac{6}{ml} \tau^2 Pe^{-\lambda},
\]
whether all its infinitely many characteristic roots satisfy \( \Re \lambda < 0 \). This analysis is supported by the method presented in [7]. The simple results in the last statement (7) of the above theorem

THEOREM 3.3. Let the time delay be positive in (3), i.e., \( \tau > 0 \). The trivial solution of (3) is asymptotically stable if and only if
\[
mg = P_{\min} < P < P_{\max}(\omega) = \left( mg + \frac{ml}{6\tau^2} \omega^2 \right) \cos \omega,
\]
where \( \omega \) is the only value satisfying \( D\omega = P\tau \tan \omega \) in the interval \((0, \pi/2)\).
Furthermore, there exists a “critical delay” \( \tau_{cr} \) with the following properties:

(i) for every \( \tau < \tau_{cr} \) there exist parameters \( P \) and \( D \) such that the zero solution of (3) is asymptotically stable,
(ii) for every \( \tau > \tau_{cr} \) and for any \( P \) and \( D \) the zero solution of (3) is unstable.

The value \( \tau_{cr} \) is given by
\[
\tau_{cr} = \sqrt{\frac{l}{3g}}.
\]
can be proved by checking the limit condition \( P_{\text{min}} = P_{\text{max}}(\omega) \) for the existence of any stability domain for \( P \) in (6).

The corresponding stability chart in the plane of the gain parameters \( P, D \) for constant delays \( \tau_1 < \tau_2 < \cdots \) is presented qualitatively in Figure 2. The encircled numbers show the number of the characteristic roots \( \lambda \) having positive real parts. For example, if too great proportional gain \( P \) is applied by an untrained operator, two complex conjugate characteristic roots turn up in the right half of the complex plane. This refers to a Hopf bifurcation resulting periodic motion around the desired equilibrium. The recent experiments of [5] also show a strong periodic component in the angle signals produced by untrained operators when balancing an inverted pendulum. The stability chart in Figure 2 also shows that the shaded stability domain shrinks as the time delay \( \tau \) increases, and at the above mentioned critical value, it disappears.

4. EXPERIMENTAL OBSERVATIONS

In spite of the fact, that the model is strongly simplified, and formula (2) of the control force describes only the basic components of the human operator’s behaviour, the above results are quite reliable even quantitatively. The delay of our reflexes is in the range of 0.1 seconds through our eyes and arms. Formula (7) means that after a short practice, everybody is able to balance a stick of length \( l = 1.2 \) meters, when the critical delay is \( \tau_{\text{crit}} \approx 0.2 \) seconds. Anybody can experience that the longer the stick is, the easier it is to equilibrate it, since \( \tau_{\text{crit}} \) becomes greater. It is impossible to balance short sticks like pencils, etc. The critical length of the stick is at about 0.3 meters. Finally, if one is a bit tipsy, the long stick cannot be equilibrated either, because the delay of the reflexes becomes too great. This may cause problems even in the self-balancing of the human body.

The self-balancing of human beings is, of course, a very complicated phenomenon. The body is controlled by us to stabilize it in a position which is physically unstable with a lot of degrees of freedom. However, even a simple inverted pendulum cannot be balanced by means of a single position signal or a single velocity signal. As Figure 2 shows, there is no stability if either \( D = 0 \) or \( P = 0 \) in (3). The human brain also has to use both signals, and the ear does provide them. Roughly speaking, the semicircular canals sense the angular velocity, while the attitude is sensed by means of the otolith organs as shown in Figure 3 (see also [8]).

5. FUNCTIONING OF THE HUMAN BALANCING ORGAN

The organ can be found in the inner ear. It consists of four general parts: utricle, saccule, cochlea, and three sets of semicircular canals oriented in nearly mutually orthogonal planes. The dynamic receptors found in the semicircular canals are the receptor hairs covered by a gelatinous dividing partition, the cupula. These canals are filled with a fluid called endolymph. It forms
closed fluid-circles. An angular acceleration of the head causes a flow of the endolymph relative to the duct wall, via the inertia of the endolymph. This flow deflects the cupula, initiating a signal to the brain. A constant angular velocity of the head does not send signal to the receptors, because the friction between the duct wall and the endolymph carries the fluid away. When the rotation stops, the endolymph flows further and the cupula is deflected in the opposite direction. Thus, these receptors sense the angular acceleration. In the subsequent section, this kind of receptor is described mathematically in the inverted pendulum model.

6. IMPROVED MODEL OF RECEPTOR

Figure 4 shows an improved model of the receptor installed in the stick balancing model. This system has four degrees of freedom described by the general coordinates $x, \varphi, \psi, \beta$. The receptor hair is connected to the pendulum (to the human body) through the duct wall. The torsional spring stiffness $s_t$ describes the elasticity of this hair. The disc models the endolymph with its inertia $\Theta$, the torsional damping $k_t$ presents its viscosity.

The angle $\varphi$ of the pendulum and the angle variation $\varphi - \psi$ of the torsional spring are detected,
and the control force $Q$ is determined by them in a way that the $\varphi \equiv 0$ position should be asymptotically stable

$$Q(t) = P\varphi(t - \tau) + D(\varphi(t - \tau) - \psi(t - \tau)), \quad (8)$$

where $\tau$ is the reflex delay as before.

The nonlinear equations of motion assume the form

$$\begin{pmatrix}
m & \frac{1}{2}ml\cos\varphi & 0 & 0 \\
\frac{1}{2}ml\cos\varphi & \frac{1}{3}ml^2 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & \Theta
\end{pmatrix}
\begin{pmatrix}
\ddot{x} \\
\ddot{\varphi} \\
\ddot{\psi} \\
\ddot{\beta}
\end{pmatrix}
+ \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & s_t & -s_t & 0 \\
0 & -s_t & s_t & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
x \\
\varphi \\
\psi \\
\beta
\end{pmatrix}
+ \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & -\frac{1}{2}ml\varphi^2\sin\varphi & 0 & 0 \\
0 & 0 & -\frac{1}{2}mg\sin\varphi & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
Q \\
0 \\
0 \\
0
\end{pmatrix}. \quad (9)$$

The trivial solution $\varphi \equiv 0$ in (8),(9) describes the equilibrium to be stabilized.

7. STABILITY ANALYSIS OF THE IMPROVED MODEL

The variational system of the equations (9) with (8) at the trivial solution assumes the form of a system of linear RDDEs. In the following theorems, however, small delays are considered, and the corresponding approximate linear models are simplified to ODEs.

**Theorem 7.1.** If there is no reflex delay in the system, i.e., $\tau = 0$, then the trivial solution of the variational system of (9) with (8) is asymptotically stable in Lyapunov sense if and only if $P > mg$ and $D > -2s_t$.

**Theorem 7.2.** Let the time delay be positive in (8),(9), i.e., $\tau > 0$. Approximate the delay effect with the first-order Taylor polynomial with respect to $\tau$ as in (4). The trivial solution of the resulting system of ordinary differential equations is asymptotically stable if and only if $P > mg$ and $H_3 > 0$, where

$$H_3 = a_1a_2a_3 - a_0a_2^2 - a_1^2a_4,$$

with the coefficients

$$a_0 = 6ls_tk_tP - 6mgls_tk_t,$$
$$a_1 = 6ls_t\Theta P - 6ls_tk_t\tau P - 6mgls_t\Theta,$$
$$a_2 = ml^2s_tk_t + 6lk_t\Theta D - 6mgkl_tk_t + 6lk_t\Theta P + 12s_tk_t\Theta - 6ls_t\Theta \tau P,$$
$$a_3 = ml^2s_t\Theta - 6lk_t\Theta D - 6lk_t\Theta \tau P,$$
$$a_4 = ml^2k_t\Theta.$$

Furthermore, there exists a "critical delay" $\tau_{cr}$ with the following properties:

(i) for every $\tau < \tau_{cr}$ there exist parameters $P$ and $D$ such that the zero solution of the resulting system of ordinary differential equations is asymptotically stable,

(ii) for every $\tau > \tau_{cr}$ and for any $P$ and $D$ the zero solution of the resulting ordinary differential equation is unstable.
Let us choose parameters to describe the balancing of the human body standing in the gravitational field, and let the muscles of legs display the control force. Approximate parameters of the balancing organ can be calculated from the research results of Van Buskirk and Grant (see [9]). Thus, the numerical values of the parameters are as follows: \( m = 70 \text{[kg]} \), \( g = 9.81 \text{[m/s}^2\text{]} \), \( I = 1.2 \text{[ml, st = 8.89 x 10^{-12}[Nm]} \), \( k_t = 5.175 \times 10^{-12} \text{[Nms]} \), \( \Theta = 1.36 \times 10^{-12} \text{[kgm}^2\text{]} \). Finally, choose a realistic, but short reflex delay: \( T = 0.05 \text{[s]} \).

The stability chart in Figure 5 presents the shaded stability domain in case of the above parameters. This domain shrinks as the delay \( \tau \) increases and, at a certain critical value, it disappears. When all the other parameters are fixed as above, this critical reflex delay is at \( \tau_{cr} = 0.113 \text{[s]} \).

8. CONCLUSION

The balancing of the human body can be modelled by a simple inverted pendulum and control force. This model can be improved by modelling the mechanical behaviour of the human balancing organ. Although the models and also their analysis can still be developed further, the numerical results with realistic parameter values provide reliable stability charts and critical reflex delays. One obvious application of the results is related to the analysis of balancing problem of elderly people, to find correlation between slower reflexes and increasing number of fall-overs.

REFERENCES