Regenerative effect is a widely accepted cause of self-excited vibrations on machine tools. Their prediction has always been one of the most difficult research areas. For conventional turning, the subcritical nature of Hopf bifurcations at the stability limits of stationary cutting has been proved recently. For thread cutting, subcritical co-dimension 2 Hopf bifurcations were observed also experimentally. These investigations were based on the analysis of autonomous delay-differential equations. The system behaviour ‘outside’ the unstable periodic motion is strongly affected by the loss of contact between the work-piece and the tool. Milling is a kind of cutting where loss of contact between teeth and work-piece occurs typically in a periodic way. This leads to non-autonomous governing equations similar to the damped, delayed Mathieu equation. The parametric excitation in the delay-differential equation leads to secondary Hopf bifurcations, and also to period doubling bifurcations if the speed of cutting is high enough. Low number of milling teeth and high speed together lead to highly interrupted cutting when the Poincaré mapping of the governing equation can be constructed in closed approximate form. The subcritical nature of both period doubling and secondary Hopf bifurcations are shown, and the development of chaotic oscillations is also explained. The global nonlinear dynamics of turning and high-speed milling are compared as self-interrupted and parametrically interrupted cutting processes.

1. Introduction

Machine tool vibrations have a negative effect on the quality of the machined surface of the workpiece. One of the most important causes of instability in the cutting process is the so-called regenerative effect. The physical basis of this phenomenon is well-known in the literature (see [1, 2]), still, it is one of the most difficult dynamical problems of engineering. Referring to its infinite dimensional nature, it is often compared to the problem of turbulence in fluid mechanics. Because of some external perturbations, the tool starts a damped oscillation relative to the workpiece, and the surface of the workpiece becomes wavy (see the simplest planar mechanical model of orthogonal cutting in Fig. 1.1). After a round of the workpiece (or tool) the chip thickness will vary at the tool because of this wavy surface. As a consequence, the cutting force depends on the actual and delayed values of the relative displacement of the tool and the workpiece. The length of the delay is equal to the time-period \( \tau \) of the revolution of the workpiece (or tool). This delay is the central idea of the regenerative effect (see [3]).
In case of conventional turning, the linear stability analysis of stationary cutting provides a complicated, sometimes fractal-like stability chart in the plane of the technological parameters (cutting speed, chip width, chip thickness). The existence of unstable periodic motions around the stable stationary cutting was already predicted in the early 80’s by the experiments in [4]. The corresponding subcritical Hopf bifurcation at the stability limit of turning has been proven analytically only recently [5-9]. The experiments and introductory analytical study in [10] prove the existence of saddle-like unstable quasi-periodic motions, too, in certain finite regions of the system parameters. The description of the global dynamics of the cutting process ‘outside’ the unstable limit cycle, or unstable torus, requires the analysis of the so-called self-interrupted cutting. This means that a stable quasi-periodic or even chaotic motion may exist when the tool leaves the work-piece for certain short time-periods during cutting. The time-intervals of these no-contact cases are regulated by the system itself. This is referred to as self-interrupted cutting. The next Section summarizes the corresponding results for turning in order to characterise the major differences between the nonlinear dynamics of turning and high-speed milling.

High-speed milling is one of the most preferred and efficient cutting processes nowadays. It is a challenging task for researchers to explore its special dynamical properties, including the stability conditions of the cutting process and the nonlinear vibrations that may occur near to the stability boundaries. These dynamical properties are mainly related to the underlying regenerative effect in the same way as it is in case of the classical turning process. Still, some new phenomena appear for low-immersion milling as predicted in [11, 12]. These phenomena were also reported in [13-15] in case of milling, independently from the immersion or speed characteristics of the milling processes.

High-speed milling usually means also low immersion (very small chip thickness) with relatively small number (2 to 4) of cutting edges on the tool. There is either no contact between the tool and the work-piece, or there is only one edge in contact for a (relatively short) time. In these cases, highly interrupted machining can well approximate the whole machining process. In the simplest models of highly interrupted machining, the ratio of time spent cutting to not cutting is a small parameter. This leads to a mechanical model where the free vibration of the tool is perturbed periodically by an impact to the work-piece. This impact results a sudden change in the tool oscillation velocity and it depends on the actual chip thickness, that is, on the difference of the present and the previous tool position. In this respect, the model includes the classical regenerative effect described for turning. There is one major difference, though: the periods of no-contact are regulated by the cutting speed parameter. Accordingly, high-speed milling is a kind of parametrically interrupted cutting as opposed to the self-interrupted cutting arising in unstable turning processes.

The simplest possible, but still nonlinear highly interrupted cutting model leads to a two-dimensional discrete mathematical model. Bifurcation analysis can be carried out along stability limits related to period doubling bifurcations and (secondary) Hopf bifurcations. These require center manifold reduction and normal form transformation. The tedious algebraic work can be carried out in closed form and leads to nonlinear phenomena similar to the one experienced in the case of the Hopf bifurcation in the turning process. The resulting subcritical bifurcations are presented in analytical form in the subsequent Sections. The existence of stable period two vibrations is also shown ‘outside’ the unstable period two vibrations. The stable ones can, however, quickly bifurcate to chaotic oscillations with increasing chip width. This is also shown by numerical investigation.


1.2 Nonlinear Dynamics of Turning

In the following subsections, the nonlinear mechanical model of turning is introduced, and the corresponding time-delayed mathematical model is derived. Then the results of the local bifurcation analyses are summarised, and the global dynamics are explained in order to compare them to the nonlinear dynamics of high-speed milling.

1.2.1 Modelling of Turning

The simplest mechanical model of orthogonal cutting is presented in Fig. 1.1. There is one cutting edge only, and it is continuously in contact with the work-piece material. The elastic tool is characterized by modal parameters like the angular natural frequency \( \omega_n = \sqrt{k / m} \), the relative damping factor \( \zeta = b / (2m\omega_n) \), and the angular frequency \( \omega_d = \omega_n \sqrt{1 - \zeta^2} \) of the damped free tool oscillation. The theoretical chip thickness is \( h_0 \), the constant chip width is \( w \). The actual chip thickness is

\[
h(t) = h_0 + x(t - \tau) - x(t). \tag{1.1}
\]

The \( x \) component of the nonlinear cutting force \( F_x \) can be calculated in accordance with the experimentally verified three-quarter rule [23]:

\[
F_x = Kw h^{3/4}_0 = Kw(h_0 + x(t - \tau) - x(t))^{3/4} \tag{1.2}
\]

where \( K \) is an experimentally identified constant parameter. This formula is valid in a certain region around the theoretical chip thickness, but clearly, \( F_x = 0 \) for negative chip thickness in case of possible large \( x \) oscillations (see the cutting force characteristics in Fig. 1.1). The 3rd degree Taylor series approximation of the cutting force variation

\[
\Delta F_x = F_x - F_{x\theta}
\]

with respect to the chip thickness variation

\[
\Delta h = h(t) - h_0 = x(t - \tau) - x(t)
\]

assumes the form

\[
\Delta F_x \approx k_1 (\Delta h) - \frac{1}{8h_0} k_1 (\Delta h)^2 + \frac{5}{96h_0^2} k_1 (\Delta h)^3 \tag{1.5}
\]

The stationary cutting force \( F_{x\theta} \) and the so-called cutting coefficient \( k_1 \) assume the form

\[
F_{x\theta} = Kw h_0^{3/4}, \quad k_1 = \left. \frac{\partial F_x}{\partial h} \right|_{h_0} = \frac{3}{4} \frac{Kw}{h_0}, \tag{1.6}
\]

respectively.
The equation of motion has the simple form
\[ x(t) + 2\zeta\omega_n x(t) + \omega_n^2 x(t) = \frac{1}{m} \Delta F_x. \] (1.7)

Fig. 1.1: Mechanical model and cutting force characteristic

If the cutting force variation is substituted here using (1.5) and (1.4), and the dimensionless time \( \tau = \omega_n t \) is introduced, then the nonlinear delay-differential equation
\[ x'(\tau) + 2\zeta x'(\tau) + (1 + \tilde{w})x(\tau) - \tilde{w}(\tau - \tau) = \]
\[ -\frac{1}{8h_0} \tilde{w}(x(\tau) - x(\tau - \tau))^2 + \frac{5}{96h_0} \tilde{w}(x(\tau) - x(\tau - \tau))^3 \] (1.8)
is obtained with its linear part in the left-hand-side. The dimensionless chip width \( \tilde{w} \) can be calculated as the ratio of the cutting coefficient and the modal stiffness of the machine tool:
\[ \tilde{w} = \frac{k_1}{m\omega_n^2} = \frac{k_1}{k} = \frac{3}{4} \frac{K}{k^4h_0} w, \] (1.9)
while the dimensionless time delay \( \tau = \omega_n \tau \) and the dimensionless angular velocity \( \tilde{\Omega} = \Omega/\omega_n \) are related to the angular velocity \( \Omega \) of the cylindrical workpiece in the following way:
\[ \tilde{\tau} = \frac{2\pi}{\tilde{\Omega}} = \frac{2\pi}{\Omega} \omega_n = \tau \omega_n. \] (1.10)

If \( v \) denotes the cutting speed, \( rv \) is the circumference of the cylindrical workpiece.
1.2.2 Bifurcation Analysis of Turning

The stability analysis of the stationary turning means the investigation of the trivial solution of the delay-differential equation (1.8) that can be based on the characteristic function

$$\lambda^2 + 2\zeta\lambda + (1 + \tilde{w}) - \tilde{w}\exp(-2\pi\lambda/\tilde{\Omega}) = 0. \quad (1.11)$$

The locations of the infinitely many characteristic roots \(\lambda\) depend on 3 dimensionless parameters only. The so-called stability chart is presented in the plane of the parameters \(\tilde{\Omega}\) and \(\tilde{w}\) related to the chosen technology, while the modal damping \(\zeta\) of the machine tool is a fixed parameter chosen to be 0.05 here. The shaded region in the chart of Fig. 1.2 shows the parameter domain where the characteristic roots are in the left half of the complex plane, that is, where the stationary cutting is stable.

There are 2 sets of points where closed form bifurcation analysis can be carried out. The details of these long algebraic calculations are presented in [5] and [10]. We briefly summarise these results here to make them comparable to the special nonlinear phenomena of high-speed milling.

At the minimum points

$$\tilde{\Omega}_{cr,j} = \frac{\sqrt{1 + 2\zeta}}{j - \frac{1}{\pi} \arctan \frac{1}{\sqrt{1 + 2\zeta}}} , \quad j = 1, 2, 3, \ldots , \quad \tilde{w}_{cr} = 2\zeta(1 + \zeta) \quad (1.12)$$

![Fig. 1.2: Stability chart of turning](image-url)
of the stability lobes, Hopf bifurcation occurs referring to self-excited vibrations with the dimensionless vibration frequency \( \omega = \sqrt{1 + 2\zeta} \) also shown above the chart of Fig. 1.2. The infinite dimensional centre manifold reduction and the normal form calculation proves the subcritical sense of the bifurcation, that is, unstable periodic vibration exists around the stable stationary cutting approximated in the form

\[
x(t) = \frac{8}{\sqrt{5}} \frac{h_0(1 + \frac{11}{30} \zeta)}{1 - \frac{2\zeta}{2(1 + \zeta)}} \cos(\sqrt{1 + 2\zeta} \omega_a t),
\]

(1.13)

At the peak points \( T_j, j = 1, 2, 3, \ldots \) of the stability lobes, co-dimension 2 Hopf bifurcations occur. This means, that a sector can be identified in the stable parameter domain (see the grey-shaded region below \( T_j \) in Fig. 1.2) where 2 unstable periodic motions exist together with an unstable quasi-periodic oscillation ‘around’ the stable stationary cutting. The corresponding 2 dimensionless vibration frequencies \( \omega_{1,2} \) of these oscillations are also presented above the stability chart in Fig. 1.2. The structure of the corresponding phase space is explained when the global dynamics of the system is discussed in the next subsection.

### 1.2.3 Global Dynamics of Self-Interrupted Cutting

Since there are unstable periodic motions, the existence of a global attractor is not explained by the local bifurcation analysis presented in the previous section. Experiments and numerical simulations show that there is no attractor in the delayed system – the large amplitude stable motion exists only when the tool leaves the workpiece for certain time periods. As Fig. 1.3. shows, the vibration amplitudes of the tool may become so large that the tool is not in contact with the workpiece between points \( Q_1 \) and \( Q_2 \) then between \( Q_3 \) and \( Q_4 \).

\[\text{Fig. 1.3: Self-interrupted cutting process}\]

In these intervals, there is zero cutting force, no regenerative effect, and the system is described by the equation of the damped oscillator:

\[
x''(\tau) + 2\zeta x'(\tau) + (1 + \varsigma) x(\tau) = 0.
\]

(1.14)
Since the trivial solution in this system is globally asymptotically stable, the tool quickly enters the material of the workpiece again, and the de-stabilising delay effect is switched on again. The system will have switches between these two dynamics: the damped free-flight of the tool described by (1.14) and the regenerative cutting described by (1.8). This may be either periodic, quasi-periodic, chaotic or transient chaotic motion. The bifurcation diagram of Fig. 1.4 represents the global dynamics of turning in the vicinity of the stability limit at the minimum point (1.12) of the stability lobes. Transient chaotic motion may occur when the tool entering the workpiece arrives back to the infinite dimensional phase space of regenerative chatter inside the unstable limit cycle, and the chaotic switches between the two different dynamics disappear suddenly, and stable stationary cutting is approached.

\[
\begin{align*}
\text{vibration amplitudes} & \\
n_0 & \\
\text{stable} & \\
\text{transient chaotic} & \\
\text{unstable periodic} & \\
\text{unstable stationary} & \\
& \approx 0.92 \tilde{w}_{cr} \\
& = 2\zeta(1+\zeta)
\end{align*}
\]

**Fig. 1.4:** Bifurcation diagram and global dynamics of turning

It is still unresolved, however, how the unstable periodic motion is connected to the branch of the (possibly chaotic) attractor ‘outside’. The location of this turning point can be estimated where the vibration amplitude of the unstable limit cycle reaches the theoretical chip thickness \( h_0 \). Then for small damping values \( \zeta < 0.05 \), formulae (1.12) and (1.13) yield

\[
h_0 = \frac{8}{\sqrt{5}} h_0 (1 + \frac{1130}{30} \frac{\tilde{w}}{w_{cr}}) \Rightarrow \tilde{w}^* = \left( 1 - \frac{5}{64} - \frac{22}{384} \zeta \right) w_{cr} \approx 0.92 w_{cr} . \quad (1.15)
\]

This means, that globally stable stationary cutting is likely to be guaranteed only at about 8% below that critical chip width calculated from linear stability theory. The experimental results in [4] give somewhat even smaller values for the limit of global stability. Near a notch of the stability chart, for example, they measured \( \tilde{w}^* = 2.7 \) [mm], \( w_{cr} = 3.1 \) [mm], that is, the global limit is about 1-2.7/3.1=13% below the linear one.

The bifurcation scenario is even more complicated at the peak point \( T_j \) of the stability lobes in Fig. 1.2. Related experiments with thread cutting around a co-dimension 2 bifurcation point are described in details in [10]. When the system parameters are taken form the dark-shaded sector below the point \( T_j \) in Fig. 1.2, the surface plot of the machined workpiece in Fig. 1.5 clearly presents the shadow of an unstable quasi-periodic oscillation with the two nearby frequencies \( \omega_{1,2} \) also shown in Fig. 1.2. This explains the clear beating effect in the first part of the signal.

As Fig. 1.5 also shows, the tool starts leaving the workpiece during its \( 4^{th} \) round between the points \( Q_1-Q_2 \) and \( Q_3-Q_4 \) in the same way as explained in Fig. 1.3. From that point, the
possibly chaotic oscillation leaves a fractal-like material surface behind in a similar way as explained for the co-dimension 1 Hopf bifurcation case at the notches of the stability lobes.

The measured surface of a workpiece in Fig. 1.5 hides an intricate infinite dimensional phase space structure for the co-dimension 2 Hopf bifurcation case at a certain peak $T_j$ of the stability lobes. The governing equations (1.8) and (1.14) with the switching condition $h(t)=0$ in (1.1) contain the time delay both in (1.8) and in (1.1). This time delay is responsible for the infinite dimensional nature of the phase space (see details in [16, 17]).

In Fig. 1.6, the plane $(x,s)$, together with the axis $x_\omega$ imitating all the other coordinates of infinite number, represent this infinite dimensional phase space. It shows two saddle-like unstable periodic solutions $P_1$ and $P_2$. Their stable manifolds – embedded in the infinite dimensional space – cross each other at the origin $O$ only. It also shows a saddle-like unstable torus $T$ that combines the frequencies of the two periodic solutions. The torus is covered by a quasi-periodic solution.

![Fig. 1.5: Measured surface of a machined workpiece](image1)

![Fig. 1.6: Self-interrupted cutting – global dynamics in infinite dimensional phase space](image2)
Solutions starting ‘inside’ the periodic solutions, like the ones with initial data \( x_0 \) or \( x_1 \), converge the \( O \) representing the stable stationary cutting. However, a trajectory starting ‘outside’ the periodic solutions and the torus, like the one starting at \( x_t \) close to \( P_1 \), gets involved in the dynamics of the limit cycle, spiralling ‘outwards’, then approaches the torus, gets involved in the quasi-periodic motion, then it leaves this torus, too, and spirals further ‘outwards’ while it hits the (non-stationary!) hyper-surface \( S \). This surface represents the wavy workpiece surface here, and crossing this surface means that the tool leaves the workpiece.

The loss of contact results a switch to the damped free-flight of the tool starting at \( Q_1 \). Then the trajectory will continue to spiral towards the origin \( O \) in the phase plane \((x,x)\) in accordance with the damped, one degree-of-freedom oscillator (1.14). Before the trajectory could reach the origin, it will hit the (moving!) hyper-surface \( S \) at the point \( Q_2 \) – the tool enters the workpiece again. The trajectory is switched back to the infinite dimensional phase space of regenerative chatter (1.8). Then the process is continued close to the unstable periodic motion \( P_1 \), then around the unstable torus \( T \), then free-flight between \( Q_3 \) and \( Q_4 \), and so on. The resulting motion is likely to be chaotic, with a chance of jumping back to the regenerative chatter ‘inside’ the unstable periodic and quasi-periodic motion. This could happen after a long chaotic transient, if it can happen at all.

The above-described phase space structure represents a cutting process where the free-flights of the tool are regulated by the cutting process itself. This is called self-interrupted cutting. High-speed milling has a qualitatively different global dynamics in this respect.

### 1.3 Nonlinear Vibrations of High-Speed Milling

In case of milling, the number \( z \) of cutting edges is more than 1, and they are not in contact with the workpiece continuously, even when the stationary cutting is stable. The dimensionless governing equation of the mechanical model in Fig. 1.7 assumes the form

\[
x''(\tilde{t}) + 2\zeta x'(\tilde{t}) + \left[1 + \bar{\nu}(\tilde{t})\right]x(\tilde{t}) - \bar{\nu}(\tilde{t})x(\tilde{t} - \tau) = 0 \quad \text{(h.o.t.).}
\]  

(1.16)

The dimensionless chip width varies periodically with a period equal to the dimensionless time delay \( \tau = \tau_0 \), that is

\[
\bar{\nu}(\tilde{t} + \tau) = \bar{\nu}(\tilde{t}) ,
\]  

(1.17)

due to the periodically varying number (1 or 2 in Fig. 1.7) of cutting edges being in contact with the workpiece. The time delay \( \tau \) is the tooth pass period here, that is,

\[
\tau = 2\pi / (z\Omega) .
\]  

(1.18)

The derivation of (1.16) can be found in [18]. The stability analysis of this delayed Mathieu-type equation is difficult due to the combination of the delay and the parametric excitation [19]. The model and analysis of high-speed milling is somewhat easier since the number of cutting edges is small \((z=2)\), the immersion is low, and consequently the time spent cutting to not cutting is very small for each cutting edge. The corresponding mechanical model and its analysis are presented in the subsequent subsections.
1.3.1 Modelling of High-Speed Milling

The mechanical model of low-immersion high-speed milling is presented in Fig. 1.8. The modal parameters of the machine tool are the same here as they are for turning (see Section 1.2.1, Fig. 1.1):

\[
\omega_n = \sqrt{\frac{k}{m}}, \quad \zeta = \frac{b}{2m\omega_n}, \quad \omega_d = \omega_n \sqrt{1 - \zeta^2}.
\]

(1.19)

The cutting force characteristics are the same as in Fig. 1.1. The only difference between the models of turning and high-speed milling is the contact time between the tool and workpiece. In case of high-speed milling, the cutting is highly interrupted, that is, the contact periods are very short compared to the tooth pass period \( \tau \) in (1.18). Consequently, the tool edge spends most of the time in free-flight.
The theoretical chip thickness is denoted by \( h_0 \), again. The actual chip thickness is either zero for no contact, or

\[
h(t_j) = h_0 + x(t_j - \tau) - x(t_j),
\]

where \( t_j - \rho \tau \) is the instant of the \( j \)th contact period between the tool and the work-piecex. We consider that the contact time \( \rho \tau \) is so short (\( \rho << 1 \)) that the position of the tool, and also the chip thickness, do not change during this period of time. This approximation is deeply analyzed and justified in [11], and confirmed also experimentally in [12].

Thus, the equations of motion can be constructed for the two parts of the tool motion in the following way. For the free-flight of the tool, we have

\[
x(t) + 2\zeta \omega_n x(t) + \omega_n^2 x(t) = 0, \quad t \in [t_j - \tau, t_j - \rho \tau)
\]

with initial conditions

\[
x_{j-1} = x(t_{j-1}) \approx x(t_j - \rho \tau), \quad v_{j-1} = \dot{x}(t_{j-1}),
\]

where, again, we consider that the position of the tool does not change much during the short contact period. For the contact period, we have

\[
x(t) = -c x(t) - k x(t) + F_x \approx F_x, \quad t \in (t_j - \rho \tau, t_j)
\]

where the usual condition of the classical impact theory is applied: all the forces except the contact ones (actually, except the cutting force) are negligible. The initial conditions are as follows

\[
x_j = x(t_j - \rho \tau) \approx x(t_j), \quad v_j = x(t_j - \rho \tau).
\]

The nonlinear cutting force can be calculated in accordance with (1.2):

\[
F_x = K w h^{3/4} = K w (h_0 + x(t - \tau) - x(t))^{3/4} \approx K w (h_0 + x_{j-1} - x_j)^{3/4}
\]

where \( w \) denotes the chip width, again. The chip thickness variation (1.4) simplifies to

\[
\Delta h = h(t_j) - h_0 = x_{j-1} - x_j,
\]

and the Taylor series (1.5) of the cutting force with respect to the chip thickness assumes the form

\[
F_x \approx F_{x0} + k_1(x_{j-1} - x_j) - \frac{1}{8h_0}k_1(x_{j-1} - x_j)^2 + \frac{5}{96h_0^2}k_1(x_{j-1} - x_j)^3,
\]

where the stationary cutting force \( F_{x0} \) and the cutting coefficient \( k_1 \) are defined in (1.6).

The solution of the impact equation (1.23) for the time interval \( \rho \tau \) assumes the form

\[
m(v_j - v_j^\tau) = \rho \tau F_x,
\]

that is
\[ v_j = \frac{\rho \tau}{m} F_0 + \frac{\rho \tau}{m} k_1 (x_{j-1} - x_j) + v_j^- - \frac{\rho \tau}{8 m h_0} k_1 (x_{j-1} - x_j)^2 + \frac{5 \rho \tau}{96 m h_0} k_1 (x_{j-1} - x_j)^3 \] (1.29)

where \( x_j \) and \( v_j^- \) can be calculated as a linear function of \( x_{j-1} \) and \( v_{j-1} \) by means of the well-known solution of the equation of motion (1.21) of the free damped oscillation of the tool with the initial conditions (1.22).

With the above-determined coefficients we can construct the nonlinear discrete model

\[
\begin{bmatrix}
  x_j \\
  v_j
\end{bmatrix} = A \begin{bmatrix}
  x_{j-1} \\
  v_{j-1}
\end{bmatrix} + \sum_{b=h, k=0} b_h k x_{j-1} v_{j-1}^k \begin{bmatrix}
  0 \\
  0
\end{bmatrix} \left[ \frac{\rho \tau}{m} F_0 \right] + \sum_{b=h, k=2,3,4,5} b_{hk} x_{j-1} v_{j-1}^k \begin{bmatrix}
  0 \\
  0
\end{bmatrix}
\] (1.30)

where the coefficients in \( A \), and \( b_{hd} \) in the nonlinear part, are determined from the corresponding solutions of the series of free-flights and impacts of the tool, that is, from Eqs. (1.21), (1.22) and (1.29). The linear part assumes the actual form

\[
A = \begin{bmatrix}
  e^{-\omega_0 \tau \zeta} \cos(\omega_0 \tau - \varepsilon) \\
  e^{-\omega_0 \tau \zeta} \sin(\omega_0 \tau)
\end{bmatrix} + \frac{\rho \tau}{m} \left( 1 - e^{-\omega_0 \tau \zeta} \cos(\omega_0 \tau - \varepsilon) \right) \begin{bmatrix}
  \frac{e^{-\omega_0 \tau \zeta}}{\sqrt{1-\zeta^2}} \\
  -\frac{\zeta e^{-\omega_0 \tau \zeta}}{\sqrt{1-\zeta^2}}
\end{bmatrix} \left( \cos(\omega_0 \tau + \varepsilon) - \frac{\rho k_1}{m} \sin(\omega_0 \tau) \right)
\]

In these formulae, the phase angle \( \varepsilon \) satisfies \( \tan \varepsilon = \frac{\zeta}{\sqrt{1-\zeta^2}} \).

### 1.3.2 Bifurcation Analysis of High-Speed Milling

In case of high-speed milling, the linear stability analysis of stationary cutting is based on the characteristic equation of the linear part of the difference equation (1.30): \[ \det(\lambda I - A) = 0 \] (1.32)

In stable cases, the characteristic multipliers \( \lambda_{1,2} \) are located in the open unit disc of the complex plane. The stability boundaries in the stability chart of Fig. 1.9 are calculated from the condition

\[ |\lambda_{1,2}| = 1 \] (1.33)

The analysis of (1.32) with (1.31) shows that there are two kinds of loss of stability: period doubling (or flip) bifurcation occurs when \( \lambda_1 = -1 \) at

\[
\tilde{\Omega}_{cr} = \frac{\rho k_1}{m \omega_0} = \frac{1-\zeta^2}{\omega_0 \tau} \frac{\sin(\omega_0 \tau)}{\chi(\omega_0 \tau) + \cos(\omega_0 \tau)} , \quad \omega_0 \tau = \frac{2\pi}{4\tilde{\Omega}} ,
\] (1.34)

and Neimark-Sacker (or secondary Hopf) bifurcation occurs when \( \lambda_{1,2} = \exp(\pm \phi) \) at
The bifurcations along the stability limits can be distinguished with the help of the vibration frequencies of the self-excited vibrations above the stability chart of Fig. 1.9 (the full structure of these frequencies is presented in [20] experimentally, too). The chart is constructed in the plane of the same parameters as in Fig. 1.2 with somewhat modified definitions related to the new parameters $\rho$ and $z$:

$$\omega = \frac{2\pi}{z\Omega}, \quad \varpi = \frac{\rho k_1}{k m \Omega^2} = \frac{3}{4} \frac{K_p}{k^4 h_0} w.$$ \hspace{1cm} (1.36)

These parameters coincide with the ones in (1.9) and (1.10) defined for turning with $z=1$ and $\rho=1$.

**Period doubling bifurcation**

At the stability boundary (1.34) of period doubling vibrations, the eigenvalues of the coefficient matrix $A$ are

$$\lambda_1 = -1, \quad \lambda_2 = e^{-\omega_0 \tau} \left( \text{sh}(\zeta \omega_0 \tau) + \cos(\omega_0 \tau) \right),$$ \hspace{1cm} (1.37)

where $|k_2| < 1$. With the help of the corresponding critical eigenvectors $s_1$ and $s_2$, the Jordan transformation matrix can be constructed in the form $T = [s_1 \ s_2]$ and the new variables $\xi$ and $\eta$ are introduced in accordance with $\begin{bmatrix} x \\ y \end{bmatrix} = T \begin{bmatrix} \xi \\ \eta \end{bmatrix}$.
This way, and also by using a constant shift transformation, system (1.30) can be transformed into

\[
\begin{bmatrix}
\xi_j \\
\eta_j
\end{bmatrix} = \begin{bmatrix}
-1 & 0 \\
0 & \lambda_2
\end{bmatrix}
\begin{bmatrix}
\xi_{j-1} \\
\eta_{j-1}
\end{bmatrix} + \sum_{h+k=2, h,k \geq 0} c_{hk} \xi_j^h \eta_j^{k-1} + \sum_{h+k=2, h,k \geq 0} d_{hk} \xi_j^h \eta_j^{k-1},
\]

(1.39)

where \( \lambda_2 \) is given in (1.37), and the coefficients \( c_{hk}, d_{hk} \) of the nonlinear terms are obtained from the coefficients \( b_{hk} \) in (1.30) with the same linear transformation (1.38).

In (1.39), one can calculate the 2\textsuperscript{nd} degree approximation of the center manifold in a much simpler way as it is in case of the infinite dimensional delayed model of turning. In this case, the center manifold is a curve in the \((x,y)\) plane tangent to the subspace spanned by the eigenvector \( s_1 \) corresponding to the critical eigenvalue \( \lambda_1 = -1 \), and tangent to the \( \zeta \) axis in the \((\xi,\eta)\) plane. The solutions of Eq. (1.39) converge to this manifold, since \( \lambda_2 < 1 \) is true always for the other eigenvalue of the linear coefficient matrix \( A \) (see [21]). The 2\textsuperscript{nd} order approximation of the center manifold simply reads

\[
\eta = \frac{d_{20}}{1-\lambda_2} \xi^2 + \ldots
\]

(1.40)

where the parameter

\[
d_{20} = -\frac{\omega_n}{2h_0} \sin(\omega_n \tau) \frac{\cos(\omega_n \tau) + \cosh(\omega_n \tau) \left(\xi^3 + 3 \xi \eta \right)}{\cos(\omega_n \tau) + \cosh(\omega_n \tau) + 2 \sinh(\omega_n \tau)}
\]

(1.41)

is the result of a lengthy algebraic calculation. The dynamics restricted to this center manifold is described by the scalar one-dimensional discrete dynamical system

\[
\xi_j = -\xi_{j-1} + c_{20} \xi_{j-1}^2 + \left(c_{30} + c_{11} \frac{d_{20}}{1-\lambda_2}\right) \xi_{j-1}^3
\]

(1.42)

The 2\textsuperscript{nd} degree term of (1.42) can be eliminated by means of a near-identity nonlinear transformation

\[
\xi = u + \frac{c_{20}}{2} u^2 + \ldots
\]

(1.44)

to obtain the normal form

\[
u_{j+1} = (-1 + \mu) u_j + \delta u_j^3
\]

(1.45)

with \( \mu = 0 \) for the unperturbed case. In the perturbed normal form (1.45), \( \mu \) is a small positive perturbation of the critical characteristic multiplier \( \lambda_1 = -1 \). Then the sub- or supercritical
nature of the period two solution is determined by the parameter $\delta$, since the amplitude of the emerging period two vibration is estimated by

$$r = \sqrt{-\mu / \delta}.$$  \hfill (1.46)

If $\delta > 0$ then the flip bifurcation is supercritical, if $\delta < 0$ then it is subcritical. The parameter $\delta$ (similar to the Poincare-Ljapunov constant of the Hopf bifurcation) is the result of a long algebraic calculation that was also checked by a computer algebraic software:

$$\delta = c_{20}^2 + c_{30} + c_{11}d_{20} = \frac{-5}{12b_0^2} \sin^2(\omega_n \tau) \left( \cos(\omega_n \tau) + \cosh(\omega_n \tau) \right) + \frac{1}{2} \left( \sin(\omega_n \tau) \right) < 0. \hfill (1.47)$$

Since the sign of $\delta$ is always negative, the flip bifurcation is subcritical, just as it is for turning. This means that an unstable period two motion (limit cycle) exists around the stable period one motion (stable fix point of the iteration). The corresponding bifurcation diagram is shown in Fig. 1.10.

**Neimark-Sacker bifurcation**

At the stability boundary (1.35) of quasi-periodic self-excited vibrations, the eigenvalues of the coefficient matrix $A$ are complex conjugate:

$$\lambda_{1,2} = \exp(\pm i\varphi).$$  \hfill (1.48)

where $|\lambda_{1,2}| = 1$. With the help of the corresponding critical eigenvectors and the new variables $\xi_j$ and $\eta_j$, system (1.30) can be transformed into

$$\begin{bmatrix}
\xi_j \\
\eta_j
\end{bmatrix} = \begin{bmatrix}
\cos \varphi & \sin \varphi \\
-\sin \varphi & \cos \varphi
\end{bmatrix} \begin{bmatrix}
\xi_{j-1} \\
\eta_{j-1}
\end{bmatrix} + \sum_{h+k=2,\, h,k \geq 0} c_{hk} \xi_{j-1}^h \eta_{j-1}^k + \sum_{h+k=2,\, h,k \geq 0} d_{hk} \xi_{j-1}^h \eta_{j-1}^k.$$  \hfill (1.49)

The bifurcation calculation has the same steps as in the case of the flip bifurcation. There is no need for center manifold reduction in this case, the normal form calculation is even more complicated, though. However, the final result is the same: the bifurcation is subcritical, again. The details of the lengthy algebraic work can be found in [22]. This means that unstable quasi-periodic motion exists around the stable stationary cutting in the neighborhood of the Neimark-Sacker type stability limit.

**1.3.3 Global Dynamics of Parametrically Interrupted Cutting**

Since the bifurcations are subcritical either for the period doubling bifurcations or for the secondary Hopf bifurcations, unstable period two, or unstable quasi-periodic oscillations exist around the otherwise stable stationary cutting in case of high-speed milling in a similar way as it is in the case of simple turning. The study of the self-interrupted cutting ‘outside’ the unstable motions in subsection 1.2.3 helps to understand the global behavior of high-speed milling, too. A scenario, similar to the bifurcation diagram in Fig. 1.4 is described below in the neighborhood of a period doubling bifurcation denoted by the thick arrow in Fig. 1.9.
It is likely, that ‘outside’ the unstable period two oscillation, there is a region where the tool ‘leaves’ the workpiece, more exactly, it does not enter the workpiece at each revolution. For example, the existence of a stable period two oscillation is expected, where the tool enters the workpiece only at every second round, and it flies over the possible contact region in between. This means, for example, that a milling tool with two cutting edges will operate in a peculiar way: only one of the edges will enter the workpiece after each revolution of the tool, while the other edge will always fly without any contact with the workpiece.

Analytical calculations confirm this hypothesis at certain parameter regions. If a discrete mapping is constructed in a way that two free-flight periods described by (1.21) and (1.22) are connected to one impact described by (1.28) and (1.29), another nonlinear discrete mapping can be constructed, similar to (1.30). The analysis of this even more complicated mapping proves the existence of stable period two oscillations outside the unstable period two oscillations (see details in [22]). Further experimental study and observations related to these special vibrations are presented in [24].

Numerical simulations clearly confirm the analytical predictions. The simulation results also show, that the tool does not reach the workpiece in each tooth pass period when it operates in an unstable parameter regime. In Fig. 1.10, the bifurcation diagram is calculated numerically in the neighborhood of the flip bifurcation at $\Omega_{cr} \approx 2.2$, $\bar{w}_{cr} \approx 0.05$. The unstable period two vibration is presented by a broken line as a result of the analytical predictions. They are also identified numerically by a simple kind of shooting method. The connection (the existence of a likely turning point) of the stable and unstable period two oscillations has not been explored yet.

![Bifurcation diagram and global dynamics of high-speed milling](image)

**Fig. 1.10:** Bifurcation diagram and global dynamics of high-speed milling

The path-following of the stable period two oscillation that can also be considered as a kind of parametrically interrupted oscillation shows that this motion can quickly bifurcate to chaotic oscillations at increasing values of the dimensionless chip thickness $\bar{w}$ (see Figure 1.10).
1.4 Conclusions

High-speed milling is usually combined with low immersion and low number of cutting edges. Since the dynamics of milling contains parametric excitation due to the periodically varying number of active teeth, high-speed milling is very sensitive for this due to the small ratio of time spent cutting to not cutting. This parametric excitation is combined with the regenerative effect related to a time delay equal to the time-period of the parametric excitation itself. While the dynamics of conventional turning contains regenerative effect only, the combination with parametric excitation results new ways of loss of stability for milling processes.

In case of high-speed milling, the mathematical model describes the new ways of loss of stability via period two vibrations, and the model also describes the (secondary) Hopf bifurcations similar to the ones of turning. The local nonlinear analysis proves that all these bifurcations are subcritical due to the nonlinear cutting force characteristics either for turning or milling, either for the flip or for the Hopf case. The subcritical nature of the bifurcations explains for the engineers why the local stability analysis provides limited result in practice, why the otherwise linearly stable stationary cutting may be sensitive to slight perturbations.

The study of the global dynamics of cutting ‘outside’ the unstable motions shows that chaos can easily appear either for turning or high-speed milling. These chaotic oscillations are related to the loss of contact between the tool and the workpiece. While the loss of contact is self-regulated in case of turning, it is determined parametrically in case of high-speed milling. Both global vibrations might be chaotic, but there is a major difference that can be expressed with an extended terminology used for self-excited vibrations and parametrically excited vibrations. In case of unstable stationary cutting, or in case of large perturbations of slightly stable cutting processes, the global attractor of turning describes a kind of self-interrupted cutting process, while the global attractor of high-speed milling is either a period 2 or a chaotic oscillation representing a parametrically interrupted cutting process.

References

Nonlinear Dynamics of High-Speed Milling


