Control Barrier Functionals: Safety-critical Control with Time Delay

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Abstract: The scope of this work is to present the theoretical background for the safety-critical control of time delay systems with state delay. We extend the theory of control barrier functions by introducing control barrier functionals.

Keywords: Control of nonlinear systems, Safety-critical control, Delay systems, Infinite dimensional systems

1. INTRODUCTION

Safety considerations in modern control systems are increasingly important and extensively researched. A wide range of application areas can be listed, from self-driving autonomous vehicles, through robotic systems and human-robot collaboration, to biological applications and epidemiological models. In these applications, safety plays a key role for reliable autonomy or sustainable operation. Safety requires keeping the state of these systems within prescribed bounds for all time. Specifically, one can define a safe set over the state space, and define safety by means of the forward invariance of that set. An elegant way to achieve set invariance is to apply the theory of *control barrier functions* (CBFs) that allows safety-critical controller synthesis (Ames et al., 2017).

While most works in safety-critical control are applied to delay-free systems, time delay arises in many engineering applications. Time delays may compromise safety, thus their effects need to be taken into account to safely control delayed systems. Thereby, the goal of this research is to extend the CBF framework to control systems with state delay. Building on the concept of *safety functionals* that has been proposed to certify the safety of autonomous time delay systems (Orosz and Ames, 2019; Kiss et al., 2021), we extend CBFs to *control barrier functionals* to enforce safety in control systems with state delays.

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2. CONTROL BARRIER FUNCTIONALS

Let us consider the following control-affine system with state delay

$$\dot{x}(t) = \mathcal{F}(x_t) + \mathcal{G}(x_t)u(t), \tag{1}$$

where $x \in \mathbb{R}^n$ is the state and x_t represents the history of the state over a delay interval $[-\tau, 0]$ with $\tau > 0$. It is defined by the shift $x_t(\vartheta) = x(t+\vartheta)$, $\vartheta \in [-\tau, 0]$, that is an element of the Banach space $\mathcal{B} = \mathcal{C}^1([-\tau, 0], \mathbb{R}^n)$ of continuously differentiable functions over $[-\tau, 0]$. The control input is $u \in \mathbb{R}^m$, while $\mathcal{F} : \mathcal{B} \to \mathbb{R}^n$ and $\mathcal{G} : \mathcal{B} \to \mathbb{R}^{n\times m}$ are vector-valued and matrix-valued locally Lipschitz continuous functionals that act on the elements of the Banach space, respectively.

We consider the system safe if its state is contained within a *safe set* $\mathcal{S} \subset \mathcal{B}$ for all time. Accordingly, we frame safety-critical control as rendering set $\mathcal{S}$ forward invariant under dynamics (1): the controller needs to ensure for all initial conditions $x_0 \in \mathcal{S}$ that $x_t \in \mathcal{S}$, $\forall t \geq 0$ for the solutions of the corresponding closed loop system (assuming they exist $\forall t \geq 0$). Specifically, we define $\mathcal{S}$ as the 0-superlevel set of a continuously Fréchet differentiable functional $\mathcal{H} : \mathcal{B} \to \mathbb{R}$:

$$\mathcal{S} = \{x_t \in \mathcal{B} : \mathcal{H}(x_t) \geq 0\}, \tag{2}$$

where the selection of $\mathcal{H}$ is application-driven.

Designing a control input that guarantees the system to be safe motivates the following definition. Given the set $\mathcal{S}$, the corresponding $\mathcal{H}$ is called a **control barrier functional** (CBFal), if there exists an extended class $\mathcal{K}$ function α such that $\forall x_t \in \mathcal{B}$

$$\sup_{u\in\mathbb{R}^m} \dot{\mathcal{H}}(x_t, \dot{x}_t, u) > -\alpha\big(\mathcal{H}(x_t)\big), \tag{3}$$

where $\dot{\mathcal{H}}$ is the derivative of $\mathcal{H}$ along (1) that depends on x_t, $\dot{x}_t$ and u (see further discussion about it below).

With this definition we state our main result that ensures safety for systems with state delay. If $\mathcal{H}$ is a CBFal for (1), then any locally Lipschitz continuous controller $\mathcal{K}: \mathcal{B} \times \mathcal{B} \to \mathbb{R}^m$, $u = \mathcal{K}(x_t, \dot{x}_t)$ satisfying

$$\dot{\mathcal{H}}(x_t, \dot{x}_t, \mathcal{K}(x_t, \dot{x}_t)) \geq -\alpha\big(\mathcal{H}(x_t)\big), \tag{4}$$

$\forall x_t \in \mathcal{S}$ renders $\mathcal{S}$ forward invariant. Similar to CBFs, this result allows safety-critical controller synthesis using CBFals, for example, by solving optimization problems to find the nearest safe action to a nominal but potentially unsafe control input (Ames et al., 2017).

In (3) and (4), the left hand side is the time derivative of $\mathcal{H}$ that characterizes how its value changes over time along the solution x_t of (1). In finite dimensional delay-free systems, this time derivative is usually referred to as the directional derivative (or *Lie derivative*) of the CBF along the solution. In the presence of time delay and infinite dimensional dynamics, this derivative has an intricate representation which we break down below.

2.1 *Time Derivative of the Control Barrier Functional $\mathcal{H}$*

Consider the system (1) and let $\mathcal{H}\colon \mathcal{B} \to \mathbb{R}$ be a continuously Fréchet differentiable functional. Then there exists a unique $\eta\colon \mathcal{B} \times \mathbb{R} \to \mathbb{R}^n$ that is of bounded variation in its second argument such that the time derivative of $\mathcal{H}$ along the system can be expressed as

$$\dot{\mathcal{H}}(x_t, \dot{x}_t, u) = \int_{-\tau}^{0} \mathrm{d}_\vartheta \eta(x_t, \vartheta) \dot{x}_t(\vartheta), \tag{5}$$

with

$$\dot{x}_t(\vartheta) = \begin{cases} \mathcal{F}(x_t) + \mathcal{G}(x_t)u & \text{if } \vartheta = 0, \\ \dot{x}(t+\vartheta) & \text{if } \vartheta \in [-\tau, 0). \end{cases} \tag{6}$$

The integral in (5) is a so-called Stieltjes type.

For example, when the CBFal $\mathcal{H}$ involves multiple discrete (point) delays $\tau_k \in [-\tau, 0]$, $k \in \{0, \dots, l\}$ (including $\tau_0 = 0$) and a continuous (distributed) delay described by a bounded kernel over $[-\sigma_1, -\sigma_2] \subseteq [-\tau, 0]$, the bounded variation η is illustrated in Fig. 1 for $l = 1$ discrete delay and has the form

$$\eta(x_t, \vartheta) = w_0(x_t)\theta(\vartheta) + \sum_{k=1}^{l} w_k(x_t)\hat{\theta}(\vartheta + \tau_k) + \eta_{\mathrm{d}}(x_t, \vartheta), \tag{7}$$

where

$$\eta_{\mathrm{d}}(x_t, \vartheta) = \begin{cases} 0 & \text{if} \quad \vartheta < -\sigma_1, \\ \int_{-\sigma_1}^{\vartheta} w_{\mathrm{d}}(x_t, s)\,\mathrm{d}s & \text{if} \quad -\sigma_1 \leq \vartheta \leq -\sigma_2, \\ \int_{-\sigma_1}^{-\sigma_2} w_{\mathrm{d}}(x_t, s)\,\mathrm{d}s & \text{if} \quad -\sigma_2 < \vartheta. \end{cases} \tag{8}$$

Here the weights $w_k\colon \mathcal{B} \to \mathbb{R}^n$ and $w_{\mathrm{d}}\colon \mathcal{B} \times \mathbb{R} \to \mathbb{R}^n$ are (potentially complicated nonlinear) functionals of x_t that depend on the specific form of $\mathcal{H}$ (see an example below). Furthermore, θ and $\hat{\theta}$ denote the right and left continuous Heaviside step functions, respectively.

Then $\dot{\mathcal{H}}$ can be written in an affine form of u as

$$\dot{\mathcal{H}}(x_t, \dot{x}_t, u) = \mathcal{L}_{\mathcal{F}}\mathcal{H}(x_t, \dot{x}_t) + \mathcal{L}_{\mathcal{G}}\mathcal{H}(x_t)\, u, \tag{9}$$

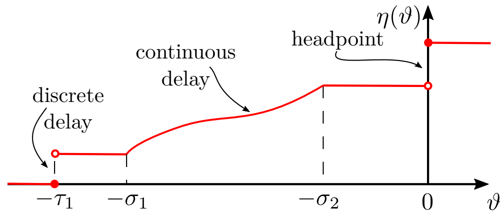

Fig. 1. Illustration of the bounded variation function η.

where the directional derivatives read

$$\begin{aligned} \mathcal{L}_{\mathcal{F}}\mathcal{H}(x_t, \dot{x}_t) =& w_0(x_t)\mathcal{F}(x_t) + \sum_{k=1}^{l} w_k(x_t)\dot{x}_t(-\tau_k) \\ &+ \int_{-\sigma_1}^{-\sigma_2} w_{\mathrm{d}}(x_t, \vartheta)\dot{x}_t(\vartheta)\,\mathrm{d}\vartheta, \\ \mathcal{L}_{\mathcal{G}}\mathcal{H}(x_t) =& w_0(x_t)\mathcal{G}(x_t). \end{aligned} \tag{10}$$

The calculation of these expressions is demonstrated for an example below. Substituting (9) into (4) yields an affine constraint for the control input. This constraint can be used to synthesize safe-critical controllers, for example, by incorporating it into optimization problems.

2.2 *Example*

Consider the system (1) with the CBFal $\mathcal{H}$ that contains a point delay τ and a distributed delay over $[-\sigma_1, -\sigma_2]$, defined as

$$\mathcal{H}(x_t) = h\left(x_t(0), x_t(-\tau), \int_{-\sigma_1}^{-\sigma_2} \rho(\vartheta)\,\kappa\big(x_t(\vartheta)\big)\,\mathrm{d}\vartheta\right). \tag{11}$$

Here $h\colon \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ and $\kappa\colon \mathbb{R}^n \to \mathbb{R}^n$ are continuously differentiable, while $\rho \in \mathcal{C}^1([-\sigma_1, -\sigma_2], \mathbb{R}^{n\times n})$ is the kernel function of the distributed delay that takes into account the states between time moments $t-\sigma_1$ and $t-\sigma_2$. One may directly take the time derivative of (11) as

$$\begin{aligned} \dot{\mathcal{H}}(x_t, \dot{x}_t, u) =& \underbrace{\nabla_0 h(.)}_{w_0(x_t)} \cdot \big(\mathcal{F}(x_t) + \mathcal{G}(x_t)u\big) + \underbrace{\nabla_1 h(.)}_{w_1(x_t)} \cdot \dot{x}_t(-\tau) \\ &+ \int_{-\sigma_1}^{-\sigma_2} \underbrace{\nabla_2 h(.)\rho(\vartheta)\nabla\kappa\big(x_t(\vartheta)\big)}_{w_{\mathrm{d}}(x_t,\vartheta)} \cdot \dot{x}_t(\vartheta)\,\mathrm{d}\vartheta \end{aligned} \tag{12}$$

where ∇ denotes gradient, ∇_k is the gradient with respect to the kth argument of a function, and $(.)$ is a shorthand notation for evaluation at the argument of h as in (11). This provides the necessary expressions to calculate (9) and (10) and find safe control inputs by (4).

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