THE MAGNUS EXPANSION FOR PERIODIC DELAY DIFFERENTIAL EQUATIONS

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ABSTRACT

In this paper, the application of the Magnus expansion on periodic time-delayed differential equations is proposed, where an approximation technique of Chebyshev Spectral Continuous Time Approximation (CSCTA) is first used to convert a system of delayed differential equations (DDEs) into a system of ordinary differential equations (ODEs), whose solution are then obtained via the Magnus expansion. The stability and time response of this approach are investigated on two examples and compared with known results in the literature.

1. INTRODUCTION

Time delayed systems are often used to describe several phenomena in different fields of science. For instance, the vibrations in machining processes such as milling or turning are often modeled by delayed differential equations (DDEs). Therefore, the investigation of the stability and the time response of these systems is a crucial problem in engineering. Two of the most successful methods for these studies are based on the idea of converting systems of DDEs either into finite dimensional discrete maps or into a set of ordinary differential equations (ODEs). These methods include the Chebyshev polynomial approach [1], the semi-discretization method [2], the spectral element method [3], the continuous time approximation [4][5], method of characteristic matrices [6][7], the Hill’s method for DDEs [8] and the pseudospectral collocation method [9], just to mention a few. The Chebyshev Spectral Continuous Time Approximation (CSCTA) was introduced by Butcher and Bobrenkov [10]. They showed that approximating DDEs by a large set of ODEs with the method of Chebyshev Spectral Continuous Time Approximation is possible with a considerably higher accuracy compared to the other, widely used methods such as finite differences. They also
proposed a technique to apply this theory on systems with multiple time delays.

Magnus’ expansion provides a solution for the nonlinear Hausdorff equation, which is associated with a linear time-varying system of ODEs. This technique allows one to symbolically approximate the logarithm of the monodromy matrix in terms of chosen parameters. Magnus' original version of the expansion was reordered and reorganized by Iserles, who successfully regrouped the terms using a graph-theoretic approach. The convergence and possible benefits of using Magnus’ expansion on non-delayed time-periodic systems is well-known in the literature.

In this paper DDEs with a single time delay are investigated, in particular the delayed Mathieu equation [11] and the delayed version of the system similar to the one described in [12] as a near-commutative case.

By using CSCTA, an alternative set of ODEs are investigated. Coupled problems of these sets of linear ODEs in the presence of time delay arise frequently in engineering applications. In most cases, one is interested in the dependence of stability on the system parameters, but the time response to certain inputs is also often in the focus of interest. There are various methods to obtain an approximate analytical fundamental solution matrix for time-periodic ODEs. One of these methods was developed by Magnus [13] by solving the nonlinear Hausdorff’s equation using Picard iteration. Several decades later Iserles [14] ‘rediscovered’ Magnus’ expansion by regrouping the terms of the expansion. Butcher et al. [12] studied the convergence of time-dependent ODEs based on Iserles’ reordering, and showed a technique using Chebyshev polynomials for the approximation of the terms of the expansion, which decreased the dependency of the accuracy of the approximation for parameters and initial conditions.

In this paper, we present an application of the expansion of periodic DDEs using the Chebyshev Spectral Continuous Time Approximation for converting a system of DDEs into a system of ODEs. It is shown that the analysis of the stability and the system and its response to different inputs can be performed using this method. The steps of the analysis is described and demonstrated on case studies.

2. CONTINUOUS TIME APPROXIMATION VIA CHEBYSHEV COLLOCATION FORMULATION

Consider a system of linear time-periodic DDEs with a single discrete time-delay as

\[
\begin{align*}
\dot{x}(t) &= A(t)x(t) + B(t)x(t-\tau) \\
x(t) &= \varphi(t), \quad -\tau \leq t \leq 0
\end{align*}
\] (1)

where \( x \in \mathbb{R}^n \) is a \( q \)-dimensional state vector, and \( A(t+T) = A(t) \), \( B(t+T) = B(t) \) are time periodic system matrices and \( T \) is the principal period. \( \varphi(t), t \in [-\tau, 0] \) is the initial vector function. Equation (1) can be considered as an abstract ODE by written in the form

\[
\mathcal{S}'(t) = \mathcal{S}(t) \mathcal{S}(t),
\] (2)

where \( \mathcal{S}(t) \) is an infinite-dimensional vector, while \( \mathcal{S}(t) \) is an infinite-dimensional operator. In this paper we use the CSCTA method for the approximation of these infinite dimensional quantities by finite dimensional ones. It has been shown in [10], that this method gives considerably more accurate results for a certain level of discretization than other conventional finite difference formulations of semi-discretization, which are using an equispaced grid for discretizing infinite dimensional quantities. The comparison of semi-discretization, spectral element and collocation methods was discussed in [15], where the Legendre-Gauss-Lobatto grid determined by the set of Legendre polynomials was used. In this paper we chose the set of Chebyshev polynomials for our discretization, although Legendre polynomials could also be used instead.

The main idea of the CSCTA method is, that instead of discretizing \( \mathcal{S}(t) \) by a uniform grid on a \( \tau \)-periodic interval \([-\tau, t]\). \( N \) number of uneven \( \Delta \tau \) intervals are formed, such that \( \Delta \tau_1 + \Delta \tau_2 + \ldots + \Delta \tau_N = \tau \). These are constructed by using the so-called Chebyshev collocation points, which correspond to the extremum points of the Chebyshev polynomials of the first kind of degree \( N \). By graphical interpretation, these points are the projections of equispaced points on the upper half of a circle with unit radius i.e. \( t_j = \cos\left(j\frac{\pi}{N}\right), \quad j = 0, 1, \ldots, N \), and are defined in the domain \([-1, 1]\) shown in Fig. 1.
Figure 1. CHEBYSHEV COLLOCATION POINTS DEFINED AS (a) THE PROJECTION OF EQUISPACED POINTS ALONG THE UPPER HALF OF THE UNIT CIRCLE AND (b) THE SOLUTION $x(t)$ DISCRETIZED ON THE TIME INTERVAL $\tau$.

We denote the number of collocation points by $m = N + 1$, and we approximate $\mathcal{R}(t)$ by the following finite dimensional state vector:

$$y(t) = \begin{bmatrix} x(t_0) & x'(t_0) & \cdots & x'(t_{N-1}) & x(t_N) \end{bmatrix}^T,$$

$$y(t) = \begin{bmatrix} y_1(t) & y_2(t) & \cdots & y_m(t) \end{bmatrix}^T.$$

$\mathcal{R}(t)$ is approximated by $\hat{A}(t)$, a finite dimensional, $\tau$-periodic matrix $\hat{A}(t+\tau) = \hat{A}(t)$, and Eqn. (2) is discretized as

$$\begin{bmatrix} \dot{y}_1(t) \\ \dot{y}_2(t) \\ \vdots \\ \dot{y}_m(t) \end{bmatrix} = \begin{bmatrix} \hat{A}(t) & 0 & \cdots & 0 \\ 0 & \hat{A}(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \hat{A}(t) \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_m(t) \end{bmatrix} + \frac{2\tau}{\tau} \begin{bmatrix} D_d(\tau) \\ \vdots \\ D_d(\tau) \end{bmatrix},$$

where $0_q$ is a $q \times q$ dimensional zero matrix, and $D_d(\tau)$ denotes the $mq \times mq$ differential operator $D_d$ for which only the rows starting with $q+1$ are used. This differential operator of the order of $m$ is formed by using the Kronecker product with the $q \times q$ identity matrix on the differentiation matrix $D:

$$D = D \otimes I_q.$$

The elements of $D$ are:

$$D_{ij} = \begin{cases} \frac{2N^2 + 1}{6}, & i = j, \\ -\frac{f_i}{2(1-t^2)}, & 1 \leq i \leq N-1, j = i+1, \\ \frac{c_i (-1)^j}{c_j (1-t^2)}, & i \neq j, j = 0, \ldots, N, \quad c_0 = 2, i = 0, N. \end{cases}$$

Since the Chebyshev collocation points are defined over the interval $[-1,1]$, a rescaling of $2/\tau$ is applied such that the collocation interval is defined on $[0, \tau]$, which appears as the coefficient of the differential operator.

The convergence of the CSCTA method of the collocational polynomial of degree $N$ was discussed in [16]. In this paper we only present the results obtained for the maximum error of a simple eigenvalue $\lambda'$ of $\hat{A}(t)$:

$$\max |\lambda' - \hat{\lambda}| \leq \frac{C_2}{\sqrt{N}} \left( \frac{C_1}{N} \right)^N,$$

where $\hat{\lambda}$ corresponds to the eigenvalue of $\mathcal{R}(t)$, $C_1$ and $C_2$ are constants independent of $N$.

3. APPLICATION TO THE DELAYED MATHIEU EQUATION

Consider the delayed Mathieu equation:

$$x(t) + (a + b \cos(\omega t)) x(t) = c x(t-2\pi/\omega),$$

for which the stability boundaries are known [11]. The number of the sufficient Chebyshev collocation points is determined by a comparison of the value of the total integral square error for different values of $N$. Rewriting Eqn. (8) into the form of Eqn. (1) gives

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}.$$ 

The integral square error

$$x_{\text{err}} = \int_0^\tau (x(t) - x(t))^2 \, dt$$

is used to determine an appropriate value of $N$ to use in the CSCTA discretization, where $x_m$ denotes the solution obtained by using $m$ collocation points and $x$ is considered to be the ‘exact’ solution obtained by a conventional numerical solver of high resolution.
Figure 2. INTEGRAL SQUARE ERROR OF SOLUTIONS TO THE DELAYED MATHIEU EQUATION USING DIFFERENT NUMBER OF COLLOCATION POINTS

It can be seen in Fig. 2 that the solution is sufficiently accurate already at 5 collocation points, since the error over the first period is $\varepsilon < 10^{-6}$. For the sake of accuracy, in the following calculations we use 10 collocation points. The relative difference between the integral error for 10 and 20 collocation points is 0.12%.

4. INTRODUCTION TO MAGNUS’ EXPANSION

We are interested in the approximation of the analytical fundamental solution matrix for time-periodic delay systems, which are approximated using CSCTA in Eqn. (4). By setting $t_0 = 0$, the fundamental solution of Eqn. (4) can be given as:

$$\Phi (t) = \hat{A} (t) \Phi (t), \quad \Phi (0) = I,$$  \hspace{1cm} (11)

$$y (t) = \Phi (t) y_0.$$  \hspace{1cm} (12)

Magnus’ expansion approximates the logarithm of the fundamental matrix, $\Omega (t)$ analytically, such that

$$\Phi (t) = e^{\Omega (t)}.$$  \hspace{1cm} (13)

The approximation of $\Omega (t)$ by Magnus is based on the Picard iteration of the solution to Hausdorff’s equation. The general form of the Hausdorff’s equation is

$$\hat{\Omega} (t) = \sum_k \frac{B_k}{k!} ad_{\Omega (0)} \hat{A} (t).$$  \hspace{1cm} (14)

where $\hat{A} (t)$ is the system matrix, $B_k$ is the $k^{th}$ Bernoulli number satisfying

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{B_k}{k!} x^k$$  \hspace{1cm} (15)

and $ad_{\Omega} (Y)$ denotes the matrix commutation operator where

$$ad_{\Omega} (Y) = XY - YX = [X,Y].$$  \hspace{1cm} (16)

$$ad_{\Omega} (Y) = \sum_{q=0}^{p} \binom{p}{q} X^{p-q} Y X^q.$$  \hspace{1cm} (17)

While Magnus’ original form of the expansion is well-known, its pattern was not obvious, for which Iserles [14] gave an explanation and reordered the terms of the expansion into indexing sets, so called ‘Trees’ and ‘Forests’ as:

$$\Omega (t) = \sum_{k=0}^{\infty} \sum_{\mathcal{J}_k} \alpha_k [H_k].$$  \hspace{1cm} (18)

where $\alpha_k$ are the coefficients determined by the Bernoulli numbers for the matrix functions $H_k$, while $\mathcal{J}_k$ denotes the indexing set of trees used for the reordering of the terms by Iserles. The hats over the $\hat{A}$ are omitted in the following for brevity.

By differentiating and reordering Eqn. (18) into forests $\mathcal{F}_k$, the first seven terms for $H_k$ can be obtained in the form

$$k = 0, \mathcal{F}_0 \rightarrow H_0 \in \{ I \},$$

$$k = 2, \mathcal{F}_2 \rightarrow H_2 \in \left\{ \int \int A A \right\},$$

$$k = 3, \mathcal{F}_3 \rightarrow H_3 \in \left\{ \int \int \int A A \right\},$$

$$k = 4, \mathcal{F}_4 \rightarrow H_4 \in \left\{ \int \int \int \int A A A \right\}.$$  \hspace{1cm} (19)

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For better visualization, let us introduce the following graphical representation for describing integration and commutation in the trees of the expansion:

\[ P \sim \int P, \quad Q \sim [P, Q] \quad (20) \]

If matrix \( A \) is represented by the symbol \( \bullet \), the first 7 trees described in (19) can be drawn as it is seen on Fig. 3. The first 7 coefficients \( \alpha_\tau \), determined by the combination of Bernoulli numbers are:

\[
\alpha_\tau = \begin{cases}
  1, & \frac{1}{2}, \frac{1}{12}, \frac{1}{24}, \frac{1}{72}, \frac{1}{720}, \frac{1}{5040}, \ldots
\end{cases}
\quad (21)
\]

With the help of Eqn. (21) and using the trees listed in Eqn. (19), Eqn. (18) gives the approximation for \( \Omega(t) \) in the form

\[
\Omega(t) = 1 \int_0^t A(s) ds + \frac{1}{2} \int_0^t \left[ A(s) \int_0^s A(s') ds'' \right] ds' + \frac{1}{4} \int_0^t \left[ A(s) \int_0^s \left[ A(s') \int_0^{s''} A(s''') ds''' \right] ds'' \right] ds' + \ldots \quad (22)
\]

For example, applying the expansion on the non-delayed Mathieu equation, with the coefficient matrix

\[
A(t) = \begin{bmatrix}
  0 & 1 \\
  -a - b \cos t & 0
\end{bmatrix}
\]

the solution can be given as \( x(t) = e^{\Omega(t)} x(0) \). The corresponding integral square error of the approximate solution compared to the exact solution obtained by a numerical solver of high resolution can be seen on Fig. 4. The difference between the error values for the expansions of different order is negligible, but it was found that the magnitude of the error strongly depends on the initial conditions and the forcing frequency \( \omega \). Results obtained at different \( \omega \) values showed that by increasing the forcing frequency, the global error will decrease significantly. This phenomenon was explained in details by Iserles [17], where also a modification of the Magnus expansion was presented for achieving a better performance in the presence of high oscillations.

\[ H_\tau = A \]

\[ H_\tau = \int_{t_0} \int_{t_0} A \cdot A \]

\[ H_\tau = \int_{t_0} \left[ \int_{t_0} A \right] A \cdot A \]

\[ H_\tau = \left[ \int_{t_0} A \right] \left[ \int_{t_0} A \right] A \]

\[ H_\tau = \left[ \int_{t_0} A \right] \left[ \int_{t_0} A \right] \left[ \int_{t_0} A \right] A \]

Figure 3. GRAPHICAL, TREE REPRESENTATION OF THE FIRST 7 TERMS OF MAGNUS’ EXPANSION BY ISERLES [14]

Higher order terms can be constructed in a similar way, but the complexity of the expressions increases quickly. In subsequent analysis, we use an approximation of these terms by the shifted Chebyshev polynomials. It is shown that this technique provides a simpler algorithm for constructing new higher order terms by utilizing its recursive property, although it still requires the construction of the trees.
\[ \mathbf{T} = \begin{bmatrix} 1 & 2 & -1 & 8 & 7 \end{bmatrix} \mathbf{T}_n \mathbf{T}_n^T \]

and \( \hat{\mathbf{G}} = \mathbf{I} \otimes \mathbf{G} \) is defined in a similar way as \( \hat{\mathbf{T}}(\tau) \). \( \mathbf{B}_{k,l} \) contains the investigated system, more precisely the \( l^{th} \) tree of forest \( k \). Since the forest \( \mathcal{F}_0 \) possesses one tree, this single term can be described as

\[ \mathbf{B}_{0,1} = \sum_i \hat{\mathbf{A}}_i \otimes \mathbf{d}_i. \] (28)

Here, \( \hat{\mathbf{A}}_i \) denotes the constant coefficient matrix corresponding to the normalized system

\[ \hat{\mathbf{A}} = \mathcal{T} \mathbf{A} = \sum_i \hat{\mathbf{A}}_i f_i(\tau). \] (29)

where \( \tau = t/T \), \( \mathbf{d}_i \) contains the coefficients of the Chebyshev expansion of \( f_i(\tau) \). \( f_i(\tau) = \mathbf{d}_i \mathbf{T}^T(\tau) \). The terms of the vector \( \mathbf{d}_i \) can be approximated either by the integral formula

\[ \mathbf{d}_i^{(n)} = \begin{cases} \int \frac{f_i(\tau) \mathbf{T}_n(\tau)}{\sqrt{T - \tau^2}} d\tau & \text{if } k = 0 \\ 2 \int \frac{f_i(\tau) \mathbf{T}_n(\tau)}{\sqrt{T - \tau^2}} d\tau & \text{otherwise} \end{cases} \] (30)

or by using Bessel functions, as explained later.

The next tree listed in Eqn. (19), the element of \( \mathcal{F}_1 \) can be written as

\[ \hat{\mathbf{T}}^T(\tau) \hat{\mathbf{G}}^T \mathbf{B}_{2,1} = \hat{\mathbf{T}}^T(\tau) \hat{\mathbf{G}}^T \left( \mathbf{Q}_{0,1} \mathbf{b}_{0,1} - \mathbf{Q}_{0,1} \hat{\mathbf{G}} \mathbf{b}_{0,1} \right). \] (31)

Introducing \( \mathbf{b}_{2,j} \), the notation

\[ \mathbf{b}_{2,j} = \mathbf{Q}_{0,1} \mathbf{d}_j - \mathbf{Q}_{0,1} \hat{\mathbf{G}}^T \mathbf{d}_j \] (32)

provides a recursive formula for which \( \mathbf{B}_{2,1} \) and all the successive terms of the expansion can be calculated as

\[ \mathbf{B}_{2,1} = \sum_0 \mathbf{\hat{A}} \mathbf{\hat{A}} \otimes \mathbf{b}_{2,j}. \] (33)

Thus, the first tree of forest \( \mathcal{F}_1 \) can be expressed as

\[ \hat{\mathbf{T}}^T(\tau) \hat{\mathbf{G}}^T \mathbf{B}_{2,1} = \sum_{0,k} \mathbf{\hat{A}} \mathbf{\hat{A}} \otimes \mathbf{b}_{1,j}. \] (34)

\[ \mathbf{b}_{1,j} = \mathbf{Q}_{0,1} \mathbf{b}_{1,j} - \mathbf{Q}_{0,1} \hat{\mathbf{G}}^T \mathbf{b}_{1,j}. \] (35)

Recursively the second tree of forest \( \mathcal{F}_2 \) can be written as

\[ \hat{\mathbf{T}}^T(\tau) \hat{\mathbf{G}}^T \mathbf{B}_{2,1} = \sum_{0,k} \mathbf{\hat{A}} \mathbf{\hat{A}} \otimes \mathbf{b}_{2,j}. \] (36)
\[ \tilde{T}^{\tau}(\tau) \hat{G}^{\tau} b_{2,1} = \sum_{\nu} \tilde{A} \tilde{A} \tilde{A} \otimes b_{2,\nu}, \]
\[ b_{2,\nu} = Q_{\nu,2} d_{\nu} - Q_{\nu,1} \hat{G}^{\tau} b_{1,\nu}, \]
\[ \tilde{T}^{\tau}(\tau) \hat{G}^{\tau} b_{4,1} = \sum_{\mu} \tilde{A} \tilde{A} \tilde{A} \tilde{A} \otimes b_{4,\mu}, \]
\[ b_{4,\mu} = Q_{\mu,4} d_{\mu} - Q_{\mu,3} \hat{G}^{\tau} b_{3,\mu}, \]

The first tree in forest \( \mathcal{F}_1 \) can be expressed in the form:
\[ \tilde{T}^{\tau}(\tau) \hat{G}^{\tau} b_{4,1} = \sum_{\mu} \tilde{A} \tilde{A} \tilde{A} \tilde{A} \otimes b_{4,\mu}, \]
\[ b_{4,\mu} = Q_{\mu,4} d_{\mu} - Q_{\mu,3} \hat{G}^{\tau} b_{3,\mu}, \]

the second tree in forest \( \mathcal{F}_2 \) reads
\[ \tilde{T}^{\tau}(\tau) \hat{G}^{\tau} b_{4,1} = \sum_{\mu} \tilde{A} \tilde{A} \tilde{A} \tilde{A} \otimes b_{4,\mu}, \]
\[ b_{4,\mu} = Q_{\mu,4} d_{\mu} - Q_{\mu,3} \hat{G}^{\tau} b_{3,\mu}, \]

and finally the third tree of forest \( \mathcal{F}_3 \) can be written as
\[ \tilde{T}^{\tau}(\tau) \hat{G}^{\tau} b_{4,1} = \sum_{\mu} \tilde{A} \tilde{A} \tilde{A} \tilde{A} \otimes b_{4,\mu}, \]
\[ b_{4,\mu} = Q_{\mu,4} d_{\mu} - Q_{\mu,3} \hat{G}^{\tau} b_{3,\mu}. \]

Note that for the construction of \( b_{2,\nu} \) of the length \( m \), a vector of coefficients \( d_{\nu} \) of the length \( 2m \) is required, while for the vector \( b_{4,\mu} \) of the same length, \( 4m \) terms are needed from the vector of coefficients. Using the same parameters for the non-delayed Mathieu equation as in Fig. 4, error calculations on the first principal period for different number of Magnus terms for are shown on Fig. 5. Here, the order of error for the expansions with a single term is significantly larger than that of the higher order expansions. Comparison of the results of the direct integration method in Fig. 4 and the method of the Chebyshev polynomials in Fig. 5 shows that the second method provides significantly more accurate results.

6. EXAMPLES

The application of Magnus’ expansion to periodic DDEs is presented on the following two examples.

6.1. Delayed Mathieu equation

The Magnus’ expansion on time-delayed systems will be shown using the example of a delayed Mathieu equation, described in Eqn. (8). The terms of the expansion can be constructed using the finite-dimensional matrix \( \tilde{A} \) and by rescaling the system to unit period, the form of the terms reads
\[ \tilde{A}(\tau) = \sum_{i} \tilde{A}_i \tilde{f}_i(\tau), \]
where \( \tilde{A}_i = T \tilde{A}, \)
\[ \tilde{f}_i(\tau) = \cos(2\pi \tau) \]
and \( \tilde{f}_i(\tau) = 1. \)
Furthermore, \( \tilde{A}(\tau) = \tilde{A}(\tau + 1) \) and \( \tilde{f}_i(\tau) = \tilde{f}_i(\tau + 1). \) Note that the principal period in the case of Eqn. (8) is \( T = 2\pi. \)

Using this notation it is clear that
\[ \tilde{A}_1 = T \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ -b & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}, \]
\[ \tilde{A}_2 = T \begin{bmatrix} 0 & -1 & 0 & \cdots & 0 & 0 & 0 \\ -a & 0 & 0 & \cdots & 0 & c & 0 \end{bmatrix}, \]

Since the first term of the vector of shifted polynomials is \( T_1(\tau) = 1, \) it is easy to see that \( d_{1} = [1 \ 0 \ \cdots \ 0]^T. \)

By definition, the coefficients of \( d_{\nu} \) can be either approximated by the approximation method described earlier in Eqn. (30), or by using the formula
\[ d_{\nu}^{(k)} = -2(-1)^{\nu-1} J_{\nu-1}(\pi), \]
where \( J_{\nu}(\pi) \) denotes the Bessel-function of the first kind of the \( k^{th} \) order, evaluated at \( \pi. \) By investigating the integral square error of the 7-term expansion for different number of polynomials (see Fig. 6), the sufficient number of polynomials was determined to be 8. The result for applying the Magnus
expansion on the delayed Mathieu equation by using different number of Magnus terms is shown in Fig. 7.

Since the shifted Chebyshev polynomials are defined over the interval $[0,1]$, Eqn. (12) cannot be used for obtaining time series past the first principal period. Instead, for obtaining solutions for the length of 5 principal periods, we use the formula

$$\Phi(t) = \Phi(s) \Phi^k(T),$$

(48)

where $t = kT + s$, $0 \leq s < T$, $k = 0, 1, 2, ...$

Figure 6: INTEGRAL SQUARE ERROR OF THE DELAYED MATHIEU EQUATION IN CASE OF DIFFERENT NUMBER OF CHEBYSHEV POLYNOMIALS USED, FOR 7 MAGNUS TERMS, $a = 1.5$, $b = 0.5$, $c = -0.2$, $\omega = 10$.

Figure 7: INTEGRAL SQUARE ERROR OF THE DELAYED MATHIEU EQUATION FOR THE CASE OF DIFFERENT NUMBER OF MAGNUS TERMS OBTAINED USING CHEBYSHEV POLYNOMIALS, $a = 1.5$, $b = 0.5$, $c = -0.2$, $\omega = 10$.

After directly constructing $\Phi(t)$ with the help of Magnus’ expansion, we can investigate the eigenvalues of the principal matrix $\mathbf{C} = \Phi(2\pi)$ using the Floquet theorem. By definition, if all of the eigenvalues of the principal matrix are within the unit circle as $|\hat{\mu}_i| < 1$, while if $|\hat{\mu}_i| > 1$ for any $i$, the system is unstable. The stability map constructed for the delayed Mathieu equation described in Eqn. (8) can be seen in Fig. 8.

Figure 8: STABILITY CHART FOR THE DELAYED MATHIEU EQUATION, USING 8 CHEBYSHEV POLYNOMIALS FOR 7 MAGNUS TERMS, AT $b = 0.15$. THE STABLE PARAMETER SETS ARE INDICATED BY THE DARK REGIONS.

The stability map matches the maps obtained earlier in [10] as well as the one constructed by the method described in [11].

6.2. Harmonic system with delay

For a second example, we choose a coupled system described as

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} p & q \sin(\omega t) \\ r \cos(\omega t) & p \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ c_{12} \end{bmatrix}, \quad x_1(t-\tau), \ x_2(t-\tau).$$

(49)

Following the method that we used in the previous example, we describe the system by Eqn. (44), using the constant matrices $\mathbf{A}$, and the periodic functions $\hat{f}_1(\tau)$ and $\hat{f}_2(\tau) = 1$. A third period function, $\hat{f}_2(\tau) = \sin(2\pi \tau)$ is required. The terms of the Magnus’ expansion are constructed in the same way, using Eqn. (28) through Eqn. (43), and the coefficients of the Chebyshev-expansion of $\sin(2\pi \tau)$ can be calculated as

$$\mathbf{d}_i^{(k)} = -2 \cdot (-1)^{i-2} J_{i-1}(\pi), \quad k = 1, 2, 3, \ldots$$

(50)
The error calculations for this example are shown in Fig. 9, where 7 Magnus terms and 8 Chebyshev polynomials and 10 collocation points were used.

![Figure 9. INTEGRAL SQUARE ERROR OF THE TIME RESPONSE APPROXIMATION FOR Eqn. (49) FOR THE CASE OF DIFFERENT NUMBER OF MAGNUS TERMS OBTAINED USING CHEBYSHEV POLYNOMIALS. $p = 0.1, \quad q = r = 0.1, \quad c_{12} = 0.1, \quad \omega = 10$.](image)

7. CONCLUSIONS

The approximation method of Magnus’ expansion was extended to systems of delayed differential equations. Magnus’ expansion solves a system of linear time-invariant differential equations by finding the logarithm of the fundamental solution matrix. This technique was proposed to be used on a system of DDEs with periodic coefficients by converting them to a finite number of ODEs using the mapping method called Chebyshev spectral time approximation. After finding the sufficient number of Chebyshev collocation points used for CSCTA, the stability and time response for systems of abstract ODEs were investigated with two methods for obtaining the terms of Magnus’ expansion: (1) the direct integration; and (2) a method using Chebyshev polynomials. Error calculations for both methods showed that the second technique provides significantly more accurate results. This technique allowed us to use the expansion of large system of ODEs. After finding the sufficient number of Chebyshev polynomials, the time-response solution was obtained for two examples: the delayed Mathieu equation and a harmonic system with delays. Also, the stability chart corresponding to the Mathieu equation with a single delay was presented.

It was shown that Magnus’ expansion can effectively be used for DDEs with a high accuracy, both for stability and time response calculations. While in this paper examples for simple systems with a single time delay term were investigated, the method can easily be extended to systems with multiple time delays. A major advantage of using Chebyshev polynomials for constructing the terms of the expansion is that this approach enables accurate solutions to be obtained for any times. On the other hand, by using the direct integration method, the error accumulates due to the numerical integration through time, while using the second method this error is not present as the integration operation matrix provides the exact integral values of the polynomials. The approximations were made with up to 7 Magnus terms. The accuracy of the results can be improved by increasing this number of terms, but an increase in the computational capacity is needed for their application.

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9. REFERENCES


