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Act-and-Wait Control of Discrete Systems with Random Delays

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Abstract— **This paper addresses the stabilization of discrete-time linear systems with random delays, which is a common problem in networked control systems. The delays are assumed to be bounded and longer than a sampling time unit. We apply the act-and-wait control concept to stabilize the system: the controller is on for one sampling period (act) and off for some sampling periods (wait). If the waiting period is longer than the maximum delay in the feedback, then the stability can be described by finite number of eigenvalues. Although the closed-loop stability of the stochastic system with the act-and-wait controller is characterized by the Lyapunov exponent of infinite random matrix products, the dimension of these matrices is finite, which results in a significant reduction of computational complexity. The applicability of this method is demonstrated in a simple example, where we compare deterministic stability with the Lyapunov exponent based results.**

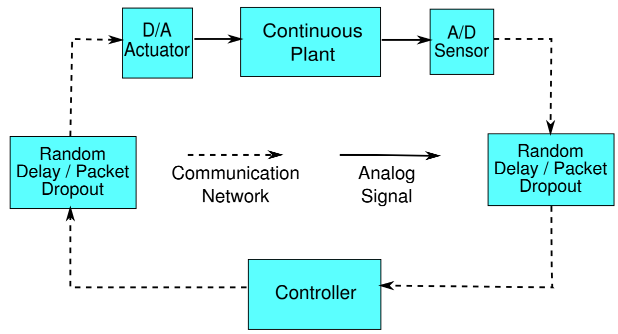

Fig. 1. General networked control systems configuration. The communication network is either a wired or wireless network.

I. INTRODUCTION

Advances in communication technology have made Networked Control Systems (NCSs) a common practice. Sensors, actuators and controllers are connected in a NCS as nodes, instead of point-to-point connections in traditional distributed control systems. Classical control theory considers the transmission of digital signals through an ideal channel, whereas in NCS signals propagate through an unreliable communication network, as shown in Figure 1. NCS's are applicable to diverse fields such as teleoperation [4], mobile sensor networks [17] and collaborative haptics [1]. The main differences between NCSs and standard digital control systems are random delay, packet dropout, and bandwidth limitation [7]. Our focus here is on random delays that comprise network access time and transmission delays. In some approaches, the packet dropout is also modeled as a long transmission delay [22].

Different types of time-delay systems including fixed, time-varying and stochastic time delays are characterized in [19] as a survey. Existing problems in NCS, and the relevant control methods are discussed in [23] and [24]. While deterministic delays have been considered in different fields such as biology [15], population dynamics [13] and machine tool chatter [20], the study of random delays in control systems is fairly recent. There are different approaches for stability analysis of an NCS. Modeling and analysis of real-time systems subject to random time delays in the communication network was discussed in [16]. This thesis presented the time-stamping technique to enable the controller to have an estimate of the delays and take the appropriate action. Based on this formulation, [22] applied a bilinear algorithm (V-K iteration) to design switching and non-switching output feedback controllers, including dynamic controllers.

Periodic control approaches have been shown to have some advantages in stabilizing linear time-invariant (LTI) systems, see, e.g., [18], [2] and [14]. If time delay appears in the feedback loop of a periodic controller, then the governing equation is a time-periodic delay-differential equation (DDE), for which the stability analysis requires numerical techniques, such as the semi-discretization method [11] or the spectral element approach [12].

Recently [8] and [10] introduced a special periodic controller, the so-called act-and-wait controller, to stabilize continuous-time LTI systems with feedback delays. In this method, the control input is switched on (act) and off (wait) periodically in time. It was shown that if the length of the switch-off (waiting) period is selected longer than time delays in a system, then the stabilization of this system is simplified to a finite dimensional pole placement. The discrete-time counterpart of the act-and-wait control concept for sampled systems was introduced in [9].

In this paper, we consider the stability problem for a discrete LTI system with random communication delays. In [16] and [22], it was assumed that the time delays are less than one sampling time unit. Here we assume that these delays are bounded and larger than a sampling time unit. This work is aimed to understand how the feedback delays in the control signal influence stability of NCS, and to find stability boundaries. In Section II, a discrete time system with random delays in the feedback control is presented. The time

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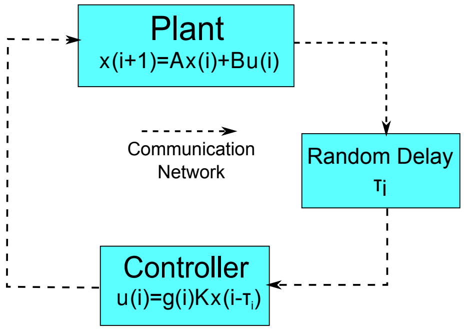

Fig. 2. A schematic diagram of the NCS under study.

stamping method and the act-and-wait control is applied to this system. In Section III, the Lyapunov exponent stability is discussed. It will be shown that stability analysis is simplified due to the structure of the act-and-wait controller. In Section IV, we apply the act-and-wait controller to a simple example with random delays. The Lyapunov exponent based stability results are compared with existing conservative criteria. In Section V, conclusions are drawn.

II. PROBLEM STATEMENT

Consider the discrete-time system

$$\mathbf{x}(i+1) = \mathbf{A}\mathbf{x}(i) + \mathbf{B}\mathbf{u}(i), \tag{1}$$

where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{u} \in \mathbb{R}^m$, $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$ are real matrices and $i = 1, 2, ...$ We assume that this discrete system is the plant of the NCS, as illustrated in Figure 2. Due to the presence of the communication network in the NCS random delays will be introduced in the feedback.

It is assumed that the delays, denoted by r_i's, are random, non-negative integers, i.e. $0 \le r_i \le r_{\max}$, where $r_{\max}$ is the maximum delay.

The control strategy is the act-and-wait control [9], which is a time-periodic state feedback. This control approach applies a signal for a while and then waits to evaluate the response. Based on this response the controller applies next appropriate control signal. The act-and-wait controller applied here is given in the form

$$\mathbf{u}(i) = g(i)\mathbf{K}\mathbf{x}(i - r_i), \tag{2}$$

where $\mathbf{K} \in \mathbb{R}^{m\times n}$ is the gain matrix, r_i's are random delays and $g(i)$ is a periodic function defined as

$$g(i) = \begin{cases} 1 & \text{if} \quad i \bmod P = 0 \\ 0 & \text{otherwise} \end{cases}, i = 1, 2, ... \tag{3}$$

where $P \in \mathbb{N}$ is the period of the act-and-wait control. Here, the controller is acting for a single discrete step, and it is inactive (waiting) for $P-1$ discrete steps. System (1) with the act-and-wait controller (2), gives rise to the following random delayed difference equation

$$\mathbf{x}(i+1) = \mathbf{A}\mathbf{x}(i) + g(i)\mathbf{B}\mathbf{K}\mathbf{x}(i - r_i). \tag{4}$$

In this paper, our goal is to study the stability of this system.

Remark: $P = 1$ (i.e. $g(i) \equiv 1$) corresponds to simple continuous state feedback control with random delays. This special case has been extensively studied (see e.g. [20], [22]).

III. RECASTING THE PROBLEM

As the delay is bounded, we can apply the time-stamping technique [16] by introducing a new variable $\mathbf{z}_i = [\mathbf{x}^T(i),\ \mathbf{x}^T(i-1), ...,\ \mathbf{x}^T(i - r_{\max})]^T \in \mathbb{R}^{(1+r_{\max})n}$. After some manipulations, system (4) can be written as the following *non-autonomous* difference equation

$$\mathbf{z}_{i+1} = \mathbf{G}_i\,\mathbf{z}_i, \tag{5}$$

where

$$\mathbf{G}_i = \begin{pmatrix} \mathbf{A} & \mathbf{0} & \dots & g(i)\mathbf{B}\mathbf{K} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{I} & \mathbf{0} & \dots & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \dots & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \\ \vdots & & & \ddots & & & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \dots & \mathbf{I} & \mathbf{0} \end{pmatrix}, \tag{6}$$

and the matrix $g(i)\mathbf{B}\mathbf{K}$ is located in the $(1 + r_i)^{th}$ column. Then

$$\mathbf{z}_{i+1} = \mathbf{G}_i\mathbf{G}_{i-1}.....\mathbf{G}_1\mathbf{z}_1. \tag{7}$$

Further simplification can be achieved by grouping the matrices $\mathbf{G}_i$'s in one period. For simplicity we assume i=NP, and therefore

$$\begin{aligned} \mathbf{z}_{i+1} = \mathbf{z}_{NP+1} = \left(\mathbf{G}_{NP}\,\mathbf{G}_{NP-1}\dots\mathbf{G}_{(N-1)P+1}\right)\dots \\ \left(\mathbf{G}_{2P}\,\mathbf{G}_{2P-1}\dots\mathbf{G}_{P+1}\right)\left(\mathbf{G}_{P}\,\mathbf{G}_{P-1}\dots\mathbf{G}_{1}\right)\mathbf{z}_1 \\ = \mathbf{H}_N\dots\mathbf{H}_2\,\mathbf{H}_1\,\mathbf{z}_1 \end{aligned} \tag{8}$$

where

$$\mathbf{H}_j = \mathbf{G}_{jP}\,\mathbf{G}_{jP-1}\dots\mathbf{G}_{(j-1)P+1}. \tag{9}$$

During the "waiting" steps when $(i \bmod P) \neq 0$, matrix $g(i)\mathbf{B}\mathbf{K}$ vanishes, for example, $\mathbf{G}_1 = \mathbf{G}_2 = \dots = \mathbf{G}_{P-1}$. Therefore, $\mathbf{H}_j$ is simplified even more as

$$\mathbf{H}_j = \mathbf{G}_{jP}\left(\mathbf{G}_{jP-1}\right)^{P-1}. \tag{10}$$

Let us consider the period to be longer than the maximum delay, i.e.

$$P > r_{\max}. \tag{11}$$

In this case, $\mathbf{H}_j$ will exhibit the simple structure

$$\mathbf{H}_j = \begin{pmatrix} \mathbf{M}_j & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{A}^{P-1} & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{A}^{P-r_{max}} & \mathbf{0} & \dots & \mathbf{0} \end{pmatrix}, \tag{12}$$

with

$$\mathbf{M}_j = \mathbf{A}^P + \mathbf{B}\mathbf{K}\mathbf{A}^{P-1-r_{(j-1)P}}. \tag{13}$$

Equation (8) after $i = NP$ iterations can be written as

$$\mathbf{z}_{NP+1} = \begin{pmatrix} \mathbf{M}_N\mathbf{M}_{N-1}\ldots\mathbf{M}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \mathbf{A}^{P-1}\mathbf{M}_{N-1}\mathbf{M}_{N-2}\ldots\mathbf{M}_1 & \mathbf{0} & \cdots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{A}^{P-r_{\max}}\mathbf{M}_{N-r_{\max}}\ldots\mathbf{M}_1 & \mathbf{0} & \cdots & \mathbf{0} \end{pmatrix}\mathbf{z}_1. \tag{14}$$

Note that the coefficient matrix has $r_{\max}n$ zero eigenvalues, and all the nonzero eigenvalues are determined by the $n \times n$ block $\mathbf{M}_N\mathbf{M}_{N-1}\ldots\mathbf{M}_1$.

IV. STABILITY BASED ON LYAPUNOV EXPONENT

Equation (14) describes a random process because the matrices $\mathbf{M}_j$ involve the random delays r_j's. The Lyapunov exponent for the problem is defined as

$$\lambda = \lim_{N\to\infty}\frac{1}{N}\ln(\| \mathbf{z}_{NP+1} \|). \tag{15}$$

The stochastic growth/instability ($\lambda > 0$) or decay/stability ($\lambda < 0$) of a random process is described by the above Lyapunov exponent. The limit in (15) gives rise to an infinite matrix product. Random matrix products have a wide variety of applications in physics [3] and in mathematics [6]. The computation of Lyapunov exponent for matrix products is in general a complex task [21].

As it was noted above, the coefficient matrix in (14) has $r_{\max}n$ zero eigenvalues, and the nonzero eigenvalues are comprised of the eigenvalues of

$$\mathbf{L} = \mathbf{M}_N\mathbf{M}_{N-1}\ldots\mathbf{M}_1. \tag{16}$$

This means that the Lyapunov exponent is simply the logarithm of the dominant eigenvalue of matrix $\mathbf{L}$.

Note that, using the act-and-wait control system, we reduced the $n(r_{\max}+1)$-dimensional matrix in (15) to an n-dimensional one in (16), reducing the computational burden to a great extent. Still, computing the above matrix product for large N values has some practical limitations. Namely, the elements of $\mathbf{L}$ might grow exponentially and naive computation will lead to overflow. In order to avoid this problem, we follow the re-normalization procedure described in [5].

V. DOUBLE INTEGRATOR WITH RANDOM DELAYS AND ACT-AND-WAIT CONTROL

In this example, we will apply the act-and-wait controller to a simple dynamic system (the double integrator) with random delay, and explore the stability chart in the parameter space of the control gains. We also compare the Lyapunov exponent results with a naive result based on fixed delays. This comparison reveals the significance of considering the random nature of the delays.

Consider the double integrator system

$$\dot{\mathbf{x}}(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\mathbf{x}(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix}u(t). \tag{17}$$

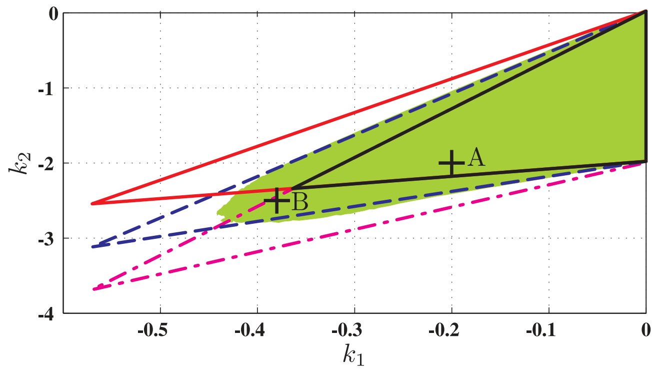

Fig. 3. Stability region for the stochastic system obtained by the Lyapunov exponent analysis (green); the stability boundaries for the deterministic system with fixed delays $r = 4$ (solid red), $r = 5$ (dashed blue) and $r = 6$ (dash-dotted magenta); and their intersection as a conservative estimate (solid black).

The sampled system of (17) with the sampling period being 1 results in the following discrete system

$$\mathbf{x}(i+1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\mathbf{x}(i) + \begin{pmatrix} 1 \\ 1 \end{pmatrix}u(i), \tag{18}$$

$$u(i) = g(i)\mathbf{K}\mathbf{x}(i - r_i), \tag{19}$$

where $\mathbf{K} = [\ k_1 \quad k_2\ ]$ is the gain matrix ($k_1$is the proportional gain and k_2 is the differential gain). In (19), $g(i)$ is the *P*-periodic switching function defined in (3) and $r_i \in \{4, 5, 6\}$ is a uniformly distributed random delay.

We apply the act-and-wait control with period $P = 7 > r_{\max}$, to satisfy condition (11). The system is stable if the Lyapunov exponent λ is negative for the chosen values of k_1 and k_2. The Lyapunov exponents are calculated for $NP = 10^5$. The corresponding stability diagram in the plane (k_1, k_2) is shown in Figure 3. The stable domains are indicated by green color. For comparison, the stability boundaries are also presented for the fixed delay values $r = 4$, $r = 5$ and $r = 6$. In this case, matrices $\mathbf{M}_j$ are identical (since r is fixed), and the system is stable if the eigenvalues of $\mathbf{M}_j$ are all inside the unit circle. The stability boundaries for the cases $r = 4$, $r = 5$ and $r = 6$ are triangles indicated by solid red, dashed blue and dash-dotted magenta lines in Figure 3. The intersection of these three triangles is the smaller triangle (black thick line), which can be considered as a conservative estimate for the stability region. As can be seen, this region is considerably smaller than the true (stochastic) stability region.

To further elaborate this point, Figure 4 presents the time response of (18)-(19) with the gain matrix $\mathbf{K} = [-0.2 \quad -2.0]$ (corresponding to point A in Figure 3), which is stable by both the "deterministic" method and based on the Lyapunov exponent. The control signal is also presented in the figure. Figure 5 shows the same plots for the gain matrix $\mathbf{K} = [-0.38 \quad -2.5]$ (corresponding to point B in Figure 3). This case was predicted to be stable by the analysis of the Lyapunov exponents, while it was predicted to be unstable by the conservative "deterministic" approach.

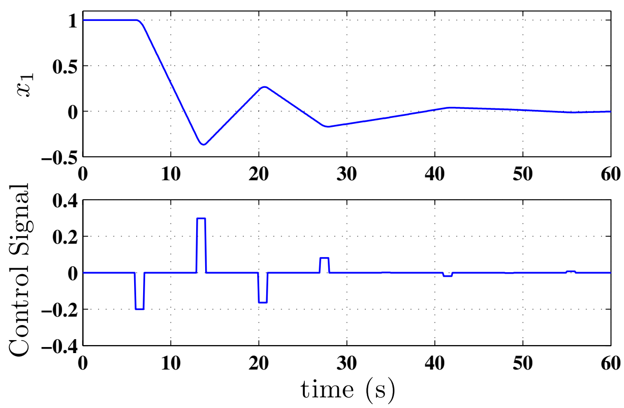

Fig. 4. Time response of (18)-(19) with the gain matrix $\mathbf{K} = [-0.2, \ -2.0]$ that is stable based on both the Lyapunov exponent approach and the deterministic stability.

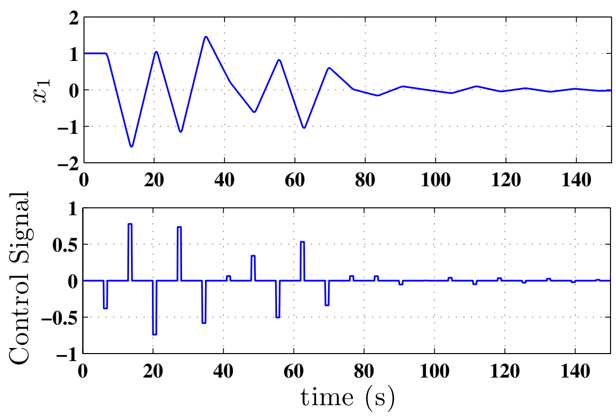

Fig. 5. Time response of (18)-(19) with the gain matrix $\mathbf{K} = [-0.38, \ -2.5]$ that is stable only based on the Lyapunov exponent approach.

VI. DISCUSSION

In this paper, a discrete dynamical system with random feedback delays was considered. We applied the time-periodic act-and-wait controller to the system. The stability analysis of the system resulted in an infinite random matrix product, which we investigated by the Lyapunov exponent. It was shown that if the length of the waiting period was chosen longer than the maximum time delay, then the dimension of the multiplied matrices was reduced, resulting in a significant reduction of computational complexity. The proposed method was applied to the simple example of the double integrator. The stability region derived based on the Lyapunov exponent was compared with conservative deterministic estimations.

Although the concept of not acting for a while during a control process may seem unnatural, it is still a natural control logic for systems with feedback delays. This is how, for example, one would adjust the shower temperature considering the delay between the controller (tap) and the sensed output (water temperature at skin).

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