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SEMI-DISCRETIZATION OF DELAYED DYNAMICAL SYSTEMS

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ABSTRACT

An efficient numerical method is presented for the stability analysis of linear retarded dynamical systems. The method is based on a special kind of discretization technique with respect to the past effect only. The resulting approximate system is delayed and time-periodic in the same time, but still, it can be transformed analytically into a high dimensional linear discrete system. The method is especially efficient for time varying delayed systems, including the case when the time delay itself varies in time. The method is applied to determine the stability charts of the delayed Mathieu equation with damping.

INTRODUCTION

There are several mechanical models which lead to equations of motions governed by delayed-differential equations (DDEs). In mechanical engineering, for example, the models describing the regenerative effect in machine tool vibrations, the human/machine systems involving the human operator's reflex delay, or the robotics applications like telemanipulation with information delay, can be mentioned (see Stépán, 1989). The corresponding mechanical models are often low degree-of-freedom (DOF) oscillatory systems subjected to the delayed feed-back of the state variables. The stability analysis of these systems is an important and crucial problem.

In the above practical applications, the first mechanical models lead to autonomous DDEs, and the stability analysis of the linearized systems is based on the roots of the characteristic function in the same way, as the Routh-Hurwitz criterion (see Routh, 1877, Hurwitz, 1895) for ordinary differential equations (ODEs). In case of a DDE, the major problem is that the number of the characteristic roots is infinite, and to have asymptotic stability, all these roots have to be situated in the left half of the complex plane.

The advanced mechanical models include parametric excitation, too. In case of a human operator, the reflex delay can vary in time; in case of machining, the cutting speed (see Insperger *et al.*, 2001), or the number of active teeth (Insperger and Stépán, 2000b) can change periodically; while in case of telemanipulation, a time-varying parameter may help to compensate the destabilizing effect of large time delays (see Insperger and Stépán, 2000c). Problems like these require the stability analysis of linear, time-periodic delayed oscillatory systems, described by linear non-autonomous DDEs.

The models in all the applications mentioned above can be viewed as some kinds of generalizations of the delayed Mathieu equation. First, the existing results are summarized in this direction. Then the so-called semi-discretization method is introduced for time-periodic DDEs like the damped and delayed Mathieu equation. Finally, the stability charts of these systems are presented for different parameter combinations, and physical conclusions are derived.

PRELIMINARIES

The underlying problem in all the applications mentioned in the Introduction is the delayed Mathieu equation:

$$\ddot{x}(t) + (d + e\cos t)x(t) = bx(t - 2p) \qquad (1)$$

In case of $b = 0$, the stability chart of the classical Mathieu equation, the so-called Strutt–Ince diagram was published by van der Pol and Strutt (1928). This is presented in Fig. 1 with S denoting the parameter domains where the trivial solution is stable (but not asymptotically stable) in Lyapunov sense (see Nayfeh and Mook, 1979). The stability chart for the case $e = 0$ was published only forty years later by Hsu and Bhatt (1966), the structure of this chart is much simpler, though. In Fig. 2, S denotes those parameter domains, where $x \equiv 0$ is asymptotically stable in Lyapunov sense.

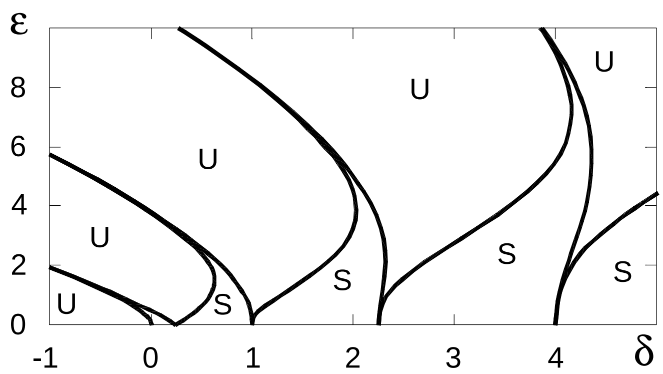

Fig. 1 The Strutt–Ince diagram of Eq. (1) for *b*=0

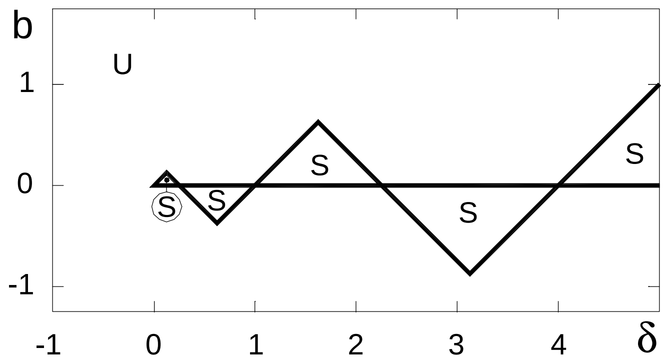

Fig.2 The Hsu–Bhatt diagram of Eq. (1) for *e* = 0

Insperger and Stépán (2000b) used the weight functions of Fargue (1973) to approximate the delayed term in Eq. (1). Although, the resulting finite dimensional systems convergence slowly to the infinite dimensional DDE, the results of the numerical investigations clearly indicated that the stability boundaries remain lines for the $e > 0$ case. Using the infinite determinant method of Hill (1886), Insperger and Stépán (2000a) proved that the stability boundaries in the parameter plane (d, b) remain lines passing along the boundary curves of the Strutt-Ince diagram (see Fig. 3) for the general case of Eq. (1).

SEMI-DISCRETIZATION METHOD

The so-called semi-discretization is a well known technique used, for example, in computational fluid mechanics. The basic idea is, that the corresponding partial differential equation (PDE) is discretized along the spatial coordinates only, while the time coordinates are unchanged. From dynamical systems viewpoint, the PDE has an infinite dimensional state space, which is approximated by the finite dimensional state space of a high dimensional ODE.

The same idea can be used for any DDE, but its implementation is not trivial. The infinite dimensional nature of the DDE is due to the presence of past effects described by functions embedded also in the time domain, above the past interval $[t - t, t]$, where t denotes the time delay, that is length of the delay effect. In this paper, the method is presented for the general second order time periodic DDE

$$\ddot{x}(t) + b_0(t)\dot{x}(t) + c_0(t)x(t) = b_1(t)\dot{x}(t - t) + c_1(t)x(t - t)\,, \quad (2)$$

where the coefficients $b_0(t)$, $c_0(t)$, $b_1(t)$ and $c_1(t)$ are periodic functions of period T. This equation is a generalization of the delayed Mathieu equation in the following aspects: it contains viscous damping, all the coefficients may depend on the time periodically, and the time period T is not necessarily equal to the time delay t which could also be time-periodic. Note, that $t \equiv t_0 = T = 2p$ for the special case of Eq. (1).

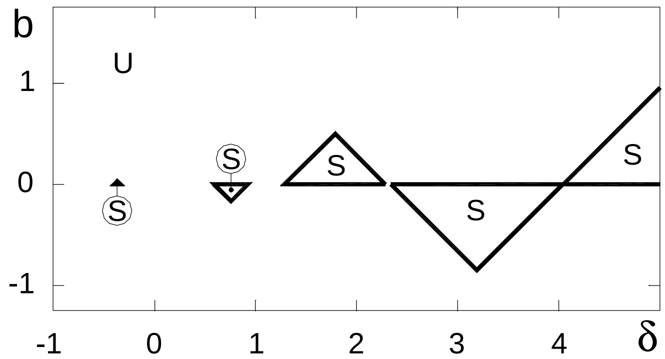

Fig. 3 Stabilty chart of Eq. (1) for *e* = 1

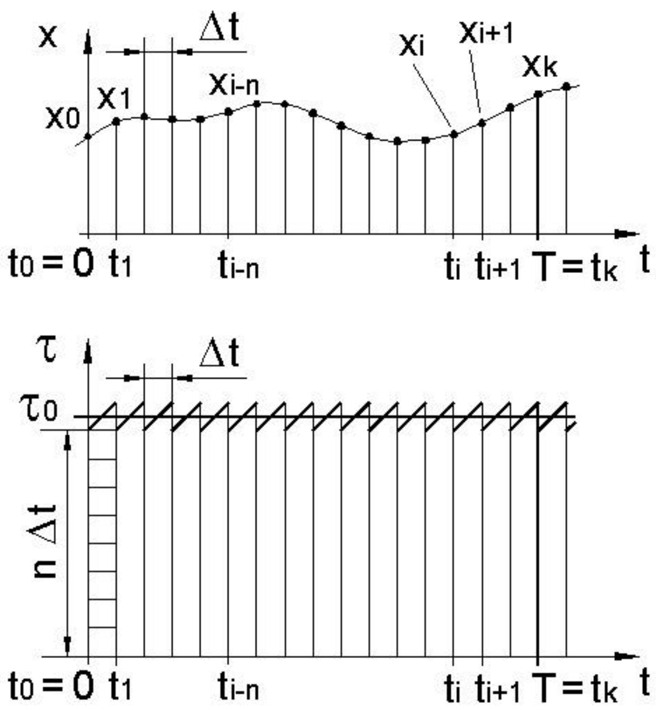

Fig. 4. Semi-discretization

The basic idea of semi-discretization is as follows. Find integers k and n with the help of which the time period T and the constant time delay t_0 can be divided in the following way: the intervals $[t_i, t_{i+1}]$ of length Dt, $i = 0,1,\ldots,k-1$, cover one time period, while the length of the past effect in time is covered by n+1/2 intervals as shown in Fig. 4. Thus, k and n satisfy

$$Dt = t_{i+1} - t_i = \frac{T}{k} = \frac{t_0}{n+1/2}\,. \qquad (3)$$

The larger the integer n is, the better the approximation is. That is why we call n as approximation parameter. For example, in the special case of Eq. (1), when $T = t_0 = 2p$, the integers $k = n$ or $k = n-1$ can be chosen, but they satisfy the above condition for $n \to \infty$ only. In other words, if the principal period T of Eq. (2), is not a multiple of the interval length Dt, that is $T \neq kDt$ exactly, then an approximate principal period $\tilde{T} \approx T$ should be applied, which satisfies $\tilde{T} = kDt$. By decreasing Dt, that is by increasing the approximation parameter n, the errors decrease.

Consider Eq. (2) in the time interval $t \in [t_i, t_{i+1}]$, where $t_i = iDt$, $i = 0,1,2,\ldots$. The point of semi-discretization is that the right hand side of Eq. (2) is approximated with a constant value, while the left hand side is left in the original differential form. In other words, the delayed terms are discretized, the actual time domain terms are not. To achieve this goal, let us approximate the constant time delay t_0 by the piece-wise linear, time-dependent, Dt-periodic varying delay function $t(t)$ as shown by the saw-like function in Fig. 4. In spite of the fact that the resulting system looks more complicated due to the appearance of the time-periodic delay instead of the constant one, the new approximate equation has a finite dimensional representation and it can be handled analytically in closed form. The reason is as follows:

$$t_0 \approx t + (n - \mathrm{int}(t/Dt))Dt, \quad t \in [0,\infty) \qquad (4)$$

$$x(t - t_0) \approx x((i-n)Dt) = x_{i-n}, \quad t \in [t_i, t_{i+1}) \qquad (5)$$

$$\dot{x}(t - t_0) \approx \dot{x}((i-n)Dt) = \dot{x}_{i-n}, \quad t \in [t_i, t_{i+1}) \qquad (6)$$

which means that the delayed state variables become constant for each discretized time interval. If the time dependent coefficients $b_1(t)$ and $c_1(t)$ in Eq. (2) are approximated by constant (say average) values b_{1i} and c_{1i} for $t \in [t_i, t_{i+1}]$ and for each $i = 0,1,2,\ldots$, then the forcing term on the right hand side of Eq. (2) is approximated by a piece-wise constant function

$$b_1(t)\dot{x}(t - t_0) + c_1(t)x(t - t_0) \approx b_{1i}\dot{x}_{i-n} + c_{1i}x_{i-n} = f_{i-n}\,, \qquad (7)$$

If the same approximation is used for the periodic functions $b_0(t) \approx b_{0i}$, $c_0(t) \approx c_{0i}$ for each time interval of discretization, the resulted equation is piece-wise autonomous:

$$\ddot{x}(t) + b_{0i}\dot{x}(t) + c_{0i}x(t) = f_{i-n}\,, \quad t \in [t_i, t_{i+1}], \quad i = 0,1,2,\ldots, \qquad (8)$$

The general solution of Eq. (8) (if b_{0i} is small enough) reads

$$x(t) = K_{1i}\exp\big(g(t-t_i)\big)\cos\big(w_i(t-t_i)\big) + K_{2i}\exp\big(g(t-t_i)\big)\sin\big(w_i(t-t_i)\big) + \frac{1}{c_{0i}}f_{i-n}\,, \qquad (9)$$

where the constants K_{1i} and K_{2i} are determined by the initial conditions $x(t_i) = x_i$, $\dot{x}(t_i) = \dot{x}_i$, and w_i and g characterize the natural frequency and damping, respectively, having different values for each time interval. They are determined by the left-hand side of Eq. (8) as follows

$$g = -\frac{b_{0i}}{2}\,, \qquad (10)$$

$$w_i = \frac{1}{2}\sqrt{4c_{0i} - b_{0i}^{\,2}}\,. \qquad (11)$$

Calculation of the solution (9) at time t_{i+1} results the values

$$x_{i+1} = a_{1i}x_i + a_{2i}\dot{x}_i + a_{3i}f_{i-n}\,, \qquad (12)$$

$$\dot{x}_{i+1} = b_{1i}x_i + b_{2i}\dot{x}_i + b_{3i}f_{i-n}\,, \qquad (13)$$

where

$$a_{1i} = \left(\cos(w_i Dt) - \frac{g}{w_i}\sin(w_i Dt)\right)\exp\big(gDt\big), \qquad (14)$$

$$a_{2i} = \frac{1}{w_i}\sin(w_i Dt)\exp\big(gDt\big), \qquad (15)$$

$$a_{3i} = \frac{1}{c_{0i}}(1 - a_{1i})\,, \qquad (16)$$

$$b_{1i} = -\frac{g^2 + w_i^{\,2}}{w_i}\sin(w_i Dt)\exp\big(gDt\big), \qquad (17)$$

$$b_{2i} = \frac{1}{w_i}\big(g\sin(w_i Dt) + w_i\cos(w_i Dt)\big)\exp\big(gDt\big), \qquad (18)$$

$$b_{3i} = -\frac{1}{c_{0i}}\,b_{1i}\,. \qquad (19)$$

From Eq. (7), the shift of the indices gives

$$f_i = b_{1i}\dot{x}_i + c_{1i}x_i\,. \qquad (20)$$

Equations (12), (13) and (20) give the connection between the states of the system at time instants t_i and t_{i+1}. This connection can be presented as a discrete map

$$\mathbf{y}_{i+1} = \mathbf{A}_i \mathbf{y}_i \,, \tag{21}$$

where the state variables are arranged into the n+2 dimensional state vector

$$\mathbf{y}_i = \mathrm{col}(\dot{x}_i \;\; x_i \;\; f_{i-1} \cdots f_{i-n}) \,, \tag{22}$$

and the coefficient matrix has the form

$$\mathbf{A}_i = \begin{pmatrix} b_{2i} & b_{1i} & 0 & 0 & \cdots & 0 & b_{3i} \\ a_{2i} & a_{1i} & 0 & 0 & \cdots & 0 & a_{3i} \\ b_{1i} & c_{1i} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix} . \tag{23}$$

The new variables denoted by f_j in Eq. (22) are the constant excitation forces in the time intervals $t \in [t_j, t_{j+1}]$.

The next step is to determine the transition matrix F over the principal period $T = kDt$ (or over the approximated principal period $\tilde{T} = kDt$). This serves a finite dimensional approximation of the monodromy operator in the infinite dimensional version of the Floquet Theory presented by Hale and Lunel (1993) or Farkas (1994). The transition matrix gives the connection between $\mathbf{y}_0$ and $\mathbf{y}_k$ in the form

$$\mathbf{y}_k = \mathrm{F} \;\mathbf{y}_0 \,. \tag{24}$$

The coupling of the solutions over the intervals $[t_i, t_{i+1}]$, $i = 0, 1, \ldots, k-1$ results

$$\mathbf{y}_k = \mathbf{A}_{k-1}\mathbf{y}_{k-1} = \mathbf{A}_{k-1}\mathbf{A}_{k-2}\mathbf{y}_{k-2} = \ldots = \mathbf{A}_{k-1}\ldots\mathbf{A}_0\mathbf{y}_0 \,, \tag{25}$$

that is, the transition matrix F is given by the simple matrix multiplication

$$\mathrm{F} = \mathbf{A}_{k-1}\,\mathbf{A}_{k-2}\ldots\mathbf{A}_1\mathbf{A}_0 \,. \tag{26}$$

At this point, the stability investigation of Eq. (2) is reduced to the problem whether the eigenvalues of F are in modulus less than 1. Any standard numerical algorithm can be used for this last step.

STABILITY CHARTS FOR THE DAMPED DELAYED MATHIEU EQUATION

As an example, the stability chart of the delayed damped Mathieu equation

$$\ddot{x}(t) + b_0\dot{x}(t) + (\,d + e\cos t)x(t) = c_1 x(t - 2p) \tag{27}$$

is determined for various parameters. The accuracy of the method can be checked in the undamped case $b_0 = 0$ the exact charts of which are known analytically (see Insperger and Stépán (2000a), and Fig. 3 for $e = 1$). Similar stability charts are shown in Fig. 5 determined with $n = 20$ approximation number for different e values.

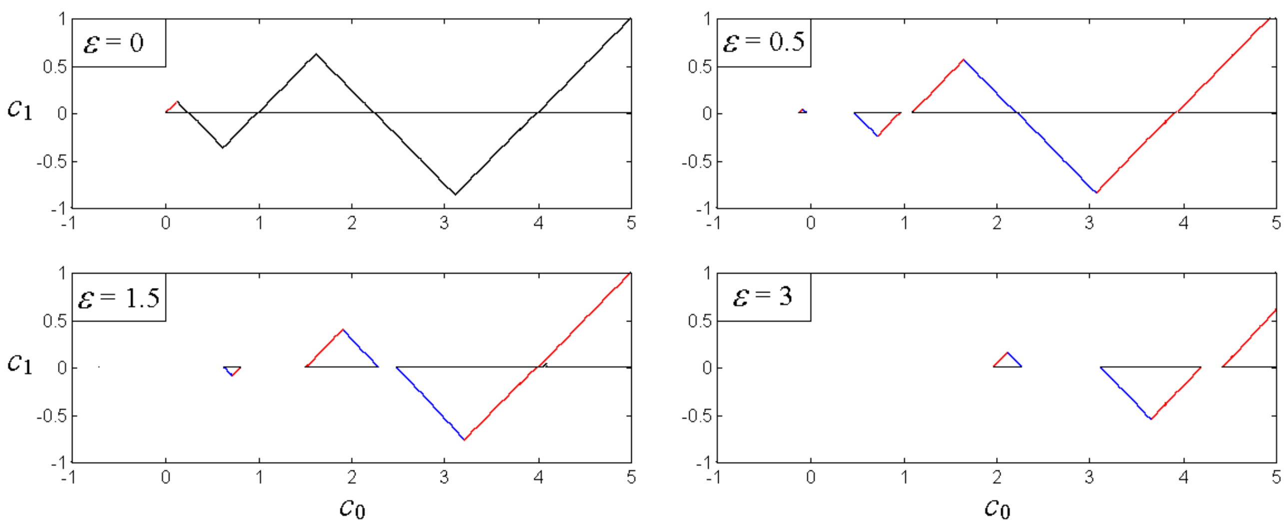

Fig. 5 Stability charts of the delayed Mathieu equation without damping

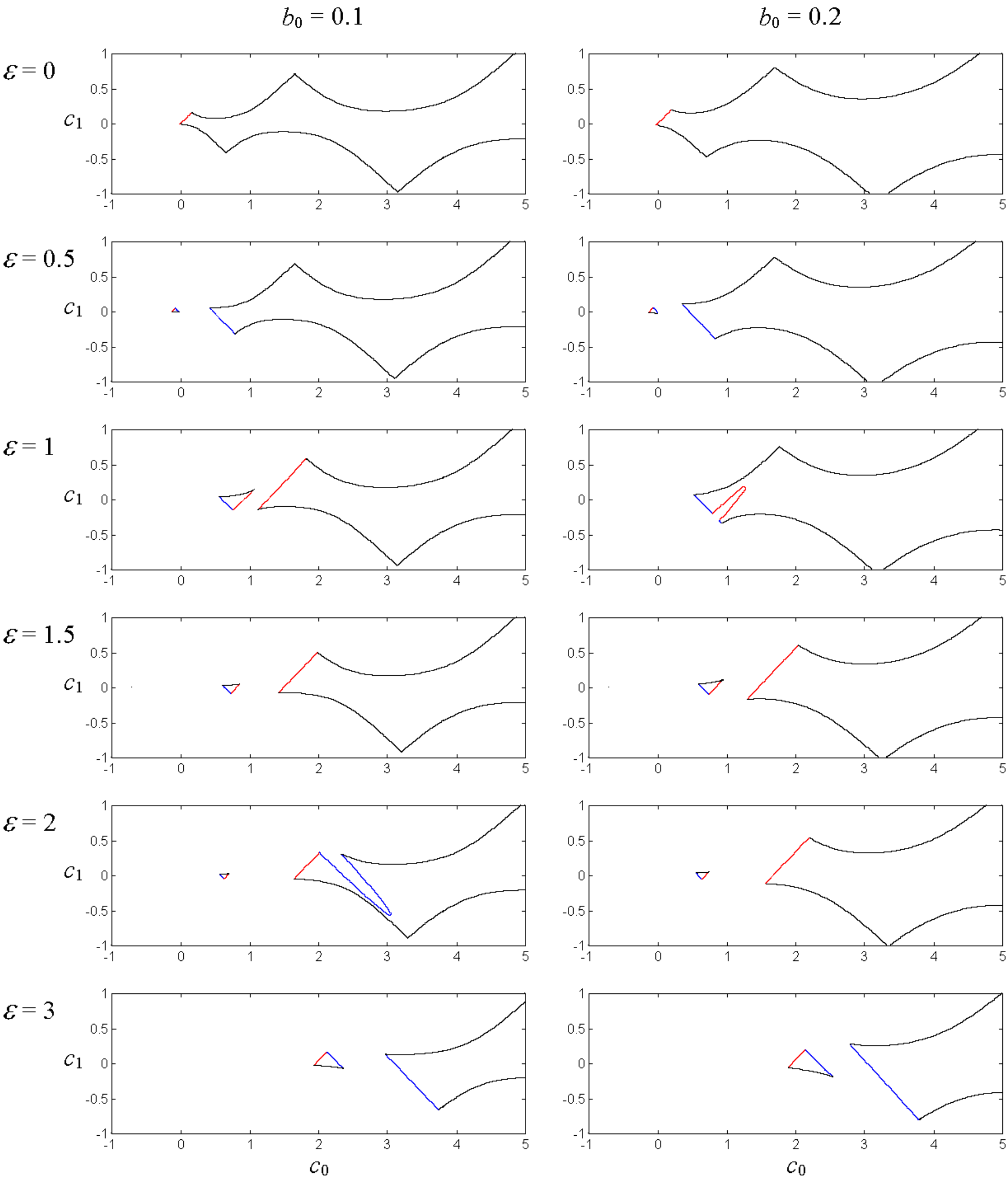

Fig. 6. Stability charts of the damped delayed Mathieu equation (27)

The charts in Fig. 5 reconstruct the exact linear stability boundaries well. Still, a slight deviation can be identified in the case $e = 1.5, c_0 \approx 4, c_1 \approx 0$ that is due to the fact that the time period $\tilde{T}$ in the approximating discrete system is not exactly equal to the time period $T = 2p$ in Eq. (27).

In Fig.6, two different viscous damping values are added to the delayed Mathieu equation. The $e = 0$ case shows that the increasing damping improves the stability properties and unifies the disjoint domains of stability of the undamped case. Still, the parametric excitation makes these unified domains disjoint again, and also shifts them as the amplitude e of parametric excitation increases.

In Figures 5 and 6, the black stability boundary curves refer to Hopf bifurcation, the blue ones for flip bifurcation or period doubling, the red ones for bifurcations topologically equivalent to the saddle-node bifurcations.

The convergence of the method can easily be seen by applying the theorem, that the solutions of a differential equation depend continuously on the system parameters. If the systems described by Eq. (2) and Eq. (6) are close to each other, then their solutions are also close to each other. By converging the approximation parameter n to infinity, the solutions of the two system are also converging to each other.

CONCLUSION

The stability charts of the damped delayed Mathieu equation give a clear picture about the complicated and physically almost unpredictable stability properties of these systems. The deep understanding of these linear systems is important in several applications in the design of machining technology, in robotics telemanipulation or in human-machine systems.

The reliable and efficient calculation of these stability charts is presented with the help of the so-called semi-discretization method applied for DDEs. The main point of semi-discretization is that the constant time delay is approximated with a piecewise linear periodic function as shown in Fig. 4. Although, the resulted approximating system is also a DDE with time periodic delay, and it seems to be more complicated than the original DDE with constant time delay, it can still be treated as a finite dimensional discrete system. In other words, the infinite dimensional characteristic of the parametrically excited delayed system disappears when the time delay is piecewise linear with slope 1.

For the approximation parameter $n = 1$, the resulted approximating system corresponds to the so-called zero order holder (ZOH) appearing in the models of computer controlled machines (see Stépán, 2000).

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