Chaos in a simply formulated dry-friction oscillator

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Abstract. Chaotic oscillators with Coulomb-type friction were extensively studied in the past. However, these investigations mostly deal with rather complex mechanical models that are closely related to everyday’s engineering applications. In the present paper, we introduce a stick-slip oscillator consisting only of a spring and a block sliding on a rough surface, and the block is periodically forced. A simple friction law is implemented in which we consider sticking and sliding coefficients of friction. We show with the help of numerical simulation that the system can behave chaotically in certain parameter domains. We use common numerical techniques to visualize chaos and also try to estimate the value of the maximal Lyapunov exponent. We also point out the possibility of transient chaos by means of cobweb diagrams.

Keywords: dry-friction, stick-slip, forced oscillator, transient chaos.

1 Introduction

Investigating piecewise-smooth dynamical systems became very prevalent during the last decade. Plenty of studies deal with non-smooth systems subjected to dry-friction and try to unfold non-linear phenomena and chaotic behaviour. However, in most cases the mechanical model is derived from real-world applications such as disc brakes which makes the models rather complex [1,2]. Some other investigations use very simple models with the simplest variant of Coulomb-type friction and carry out a detailed non-smooth bifurcation analysis [3,4]. They investigate bifurcation scenarios that are peculiar to systems with dry-friction.

In this article the very simple frictional oscillator presented in [5–8] will be used and extended with a little more advanced friction model already mentioned by Hinrichs et al. [2]. It will be shown that for certain parameters the system will behave chaotically. Moreover, signs of transient chaos will be highlighted.
The mechanical and mathematical model

The mechanical model consists of a block sliding on a rough surface. It is driven by a harmonic excitation force while supported by a linear spring as depicted in Fig. 1. The equation of motion of the system can be written in the following form:

\[ mz'' + kz = F_0 \cos(\omega_0 (\tau + \tau_0)) - \mu mg f(z'), \]

where \( \tau \) is time and \((\cdot)\)' denotes time derivative, \( z \) is the displacement of the block, \( m \) is the total mass of the moving elements, \( k \) is the spring stiffness, \( F_0 \) is the amplitude of the excitation force, \( \omega_0 \) is the angular excitation frequency, \( \mu \) is the sliding coefficient of friction, \( g \) is the gravitational acceleration and \( f(z') \) can be an arbitrary function of the velocity that models the friction characteristics.

After transforming the system into dimensionless form (See [7]) we obtain the following equations that describe the motion:

\[ \ddot{x} + x = \cos(\Omega (t + t_0)) - S f(\dot{x}) \]

\[ f(\dot{x}) \in \begin{cases} 1 & \text{if } \dot{x} > 0 \\ [-S_1/S, S_1/S] & \text{if } \dot{x} = 0 \\ -1 & \text{if } \dot{x} < 0 \end{cases} \]

where \((\cdot)\) denotes non-dimensional time derivative. We rescaled time by \( t = \tau \sqrt{k/m} \) and displacement by \( x = zk/F_0 \). The non-dimensional constants are \( \Omega = \omega \sqrt{m/k} \), \( S = \mu mg/F_0 \) and \( S_1 = \mu_1 mg/F_0 \). Here the friction function assumes that the friction force at sticking can be greater than at sliding. Therefore \( \mu_1 \) will be the sticking coefficient of friction. This model will be the subject of our further investigations.
3 Numerical analysis

Exact solutions exist that satisfy Eq.(1) between two consecutive stops, i.e. when the velocity is zero in the form:

\[ x^{\pm}(t) = A^{\pm} \cos(t) + B^{\pm} \sin(t) + L \cos(\Omega t) + K \sin(\Omega t) \mp S, \quad (2) \]

where \( A^{\pm} \) and \( B^{\pm} \) are determined by initial conditions. The plus and minus signs indicate different constants for positive and negative velocity respectively. \( K, L \) can be expressed as

\[ K = \frac{\sin(\Omega t_0)}{\Omega^2 - 1}, \quad L = -\frac{\cos(\Omega t_0)}{\Omega^2 - 1}. \]

This enables us to avoid solving the system of ordinary differential equations however, we also tried this latter approach. When using Eq.(2) the only uncertainty will be finding the time instants when velocity changes sign and also when the stuck body starts sliding again. A Matlab program was developed that considers these cases and performs the simulation of the model.

Earlier investigations [7] of one of the present authors showed that the behaviour of the system changes at resonant excitation frequencies, namely, the symmetric solutions become asymmetric in the sense that the absolute values of turnaround displacements during one excitation period become different. Also the resonant bands open up if a friction coefficient for sticking different from sliding is introduced. This led us to choose parameters for resonant solutions. We stuck to the case \( \Omega = 0.5 \) and \( S_1 = 0.4 \) and carried out numerical simulations. First a brute-force bifurcation analysis was done for positive initial conditions where the last ten sticking displacements were registered at the end of each run.

We found period-doubling bifurcation and signs of chaotic behaviour if the ratio \( S/S_1 \) is small enough (See Fig. 2). In this particular case period-adding occurs at \( S/S_1 = 0.17175 \) (Point I in Fig. 2) followed by a chaotic band at \( S/S_1 = 0.139 \) that suddenly vanishes at \( S/S_1 = 0.109 \) (Point II). Furthermore periodic solution exists between \( S/S_1 = 0.089 - 0.109 \) (Point III) and a more extended chaotic motion can be seen below \( S/S_1 = 0.089 \) (Point IV). In addition to the results in [7] we found that the solution’s symmetry also vanishes at \( S/S_1 = 0.835 \) by the separation of sticking and sliding coefficients of friction (Point V). For the characterization of symmetry of the solutions we introduced the following formula:

\[ \mathbb{N} = \sum_{i=1}^{n} x(\sigma_i) \]

assuming \( \sigma_i \) to be time instants in one excitation period when \( \dot{x}(\sigma_i) = 0 \) \( (\sigma_i \in [0; \frac{2\pi}{\Omega}]), \quad i = 1, ..., n \). The symmetry-changing can be seen in panel two of Fig. 2 where \( \mathbb{N} = 0 \) means that the solution is symmetric. Solutions
Fig. 2. Monte Carlo bifurcation diagram for $S/S_1 = 0 - 1$ depicting the sticking displacement and asymmetry of solutions. Parameters were $S_1 = 0.4$ and $\Omega = 0.5$.

Fig. 3. (a) Asymmetric ($S/S_1 = 0.75$) and (b) symmetric ($S/S_1 = 1$) solutions for the parameters $S_1 = 0.4$ and $\Omega = 0.5$. 
become asymmetric in a way that one of the two sticking occurrences vanishes during an excitation period by a *crossing-sliding* bifurcation according to [3]. An asymmetric and a symmetric solution can be seen in Fig. 3 before and after the symmetry breaking. The grey filled area in the figure is the region of sticking. If the solution’s velocity becomes zero inside this area then it will be stuck until the excitation overcomes the sticking friction force and the spring force.

**Fig. 4.** Bifurcation diagram and Lyapunov exponents for $S/S_1 = 0 - 0.25$ with parameters $S_1 = 0.4$ and $\Omega = 0.5$. (Solid lines - direct numerical simulation of nearby trajectories, red circles - method for non-smooth systems [9]).

Fig. 4 is the enlargement of Fig. 2 for the parameters $S/S_1 = 0 - 0.25$ also showing the maximal Lyapunov exponent. Black solid lines indicate results computed from two nearby trajectories’ separation while red circles refer to a method published by Stefanski and Kapitaniak [9] that is developed for piecewise-smooth systems. This latter method relies on the synchronisation of two similar systems by a coupling parameter $q$. In our case the coupled system of first order equations are as follows:

$$
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 + \cos(\Omega (t + t_0)) - Sf(x_1) \\
\dot{x}_3 &= x_4 + q(x_1 - x_3) \\
\dot{x}_4 &= -x_3 + \cos(\Omega (t + t_0)) - Sf(x_3) + q(x_2 - x_4)
\end{align*}
$$
In order to obtain the maximal Lyapunov exponent we have to carry out a brute-force bifurcation analysis taking $q$ as bifurcation parameter. When the two coupled systems are fully synchronised, i.e. the solutions evolve identically then the coupling parameter will become the maximal Lyapunov exponent. An example for the process [9] can be seen in Fig. 5. The method

\begin{equation}
A = \frac{1}{n} \sum_{i=1}^{n} \frac{1}{\Delta t} \frac{|d_{i+1}|}{|d_i|},
\end{equation}

During direct numerical simulation Eq.(3) is evaluated for the nearby trajectories where $d_{i+1}$ is the distance between the trajectories at the $i+1$-st time sample $\Delta t$ time after distance $d_i$ was sampled. One certain issue with this calculation can be when one of the initial trajectories suddenly sticks while the other one still slides. In this case the rate of separation of the trajectories may not be exponential. Another issue arises when both trajectories stick at the same displacement hence the quotient when computing the Lyapunov exponent will be either zero (when both solutions just stuck) or infinity when the earlier distance was zero. These cases have to be neglected in the calculation.

**Fig. 5.** Monte Carlo diagram for the coupling parameter $q$ as bifurcation parameter. Synchronisation occurs at $q = 0.0073$ thus $\Lambda = 0.0073$. ($S/S_1 = 0.125, \Omega = 0.5$)
In order to illustrate chaos cobweb diagrams are often used. This was one of the earliest methods that we deployed searching for chaos. At that early stage we integrated the first order system of ODE’s to obtain numerical solution. We discovered that at certain parameters the solution seems to escape from a repellor into an attractor. This scenario is depicted in Fig. 6. The parameters were \( S/S_1 = 0.125, S_1 = 0.4 \) and \( \Omega = 0.5 \). If we look

\[
\begin{align*}
S/S_1 &= 0.125, \\
S_1 &= 0.4, \\
\Omega &= 0.5.
\end{align*}
\]

at Fig. 2, we see that for these values there is purely chaotic behaviour to expect. However not far away from this point in Fig. 4 a negative spike can be discovered in the Lyapunov exponent plot \((S/S_1 = 0.116)\). If we run simulation with the analytical solutions and plot the displacement against time, we see a slight change in behaviour (See Fig. 7). This qualitative change is hardly visible in any cobweb diagram since the magnitude of displacements are almost the same, however it may be a sign of transient chaos that should be investigated in detail.

4 Conclusions and future work

In this work we carried out a numerical bifurcation analysis of a simply formulated dry-friction oscillator. We showed that by using different coefficients of friction for sliding and sticking a qualitative change occurs in the behaviour of the system. Namely, symmetric solutions become asymmetric. Moreover, by further decreasing the sliding coefficient of friction chaotic motion arises that is followed by a periodic band and end up in chaos. This chaotic behaviour was proved by calculating the maximal Lyapunov exponent in two
different ways that show acceptable coincidence. Finally, we highlighted the possibility of transient chaos that may arise inside a very narrow band of the bifurcation parameter.

Further investigation will focus on the chaotic behaviour of the system if friction coefficients are closer to each other since this case is closer to reality. Closer attention should be paid to unfold the transient chaotic behaviour.

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References