Rezgéstan – 11. gyakorlat

Több DoF rezgőrendszerek

Emlékeztető:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot q_k} - \frac{\partial T}{\partial q_k} + \frac{\partial \mathcal{D}}{\partial \dot q_k} + \frac{\partial U}{\partial q_k} = Q_k^* \qquad k = 1,2\dots n$$

n DoF

↳ Mátrix együtthatós DE

$$\underline{\underline{M}}\,\underline{\ddot q} + \underline{\underline{K}}\,\underline{\dot q} + \underline{\underline{S}}\,\underline{q} = \underline{Q} \qquad \underline{q} = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$$

általános koordináta vektor

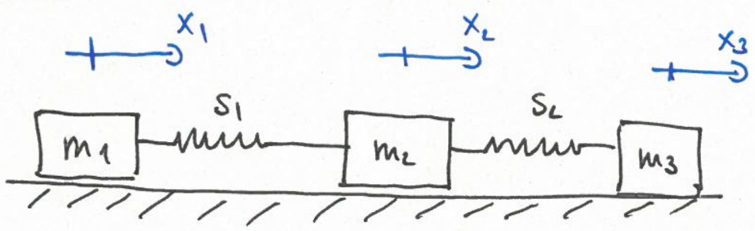
$$\underline{\underline{M}} = [m_{ij}] = \left[\left. \frac{\partial^2 T}{\partial \dot q_i \partial \dot q_j} \right|_{\underline{q}=\underline{0}} \right] = \underline{\underline{M}}^T$$

$$\underline{\underline{K}} = [k_{ij}] = \left[\left. \frac{\partial^2 \mathcal{D}}{\partial \dot q_i \partial \dot q_j} \right|_{\underline{q}=\underline{0}} \right] = \underline{\underline{K}}^T$$

$$\underline{\underline{S}} = [s_{ij}] = \left[\left. \frac{\partial^2 U}{\partial q_i \partial q_j} \right|_{\underline{q}=\underline{0}} \right] = \underline{\underline{S}}^T$$

linearizált alak!
időfüggetlen geom kényszerek!
kis lengések!

1. Feladat



Adatok:

$m_1 = 2\,[kg]$
$m_2 = 4\,[kg]$
$m_3 = 5\,[kg]$
$s_1 = 200\,[N/m]$
$s_2 = 500\,[N/m]$

Feladat: sajátkörfrekvenciák
lengésképek

$n = 3$ DoF $\qquad \underline{q} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Csillapítatlan és gerjesztetlen!

$$\frac{d}{dt}\frac{\partial T}{\partial \dot q_k} - \frac{\partial T}{\partial q_k} + \frac{\partial U}{\partial q_k} = 0$$

A kinetikus Energia:

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 + \frac{1}{2} m_3 \dot{x}_3^2$$

$$\frac{\partial T}{\partial \dot{x}_1} = m_1 \dot{x}_1 \rightarrow \frac{\partial^2 T}{\partial \dot{x}_1^2} = \boxed{m_1 = m_{11}}$$

$$\rightarrow \frac{\partial^2 T}{\partial \dot{x}_1 \partial \dot{x}_2} = \boxed{0 = m_{12} = m_{21}}$$

$$\frac{\partial T}{\partial \dot{x}_2} = m_2 \dot{x}_2 \rightarrow \frac{\partial^2 T}{\partial \dot{x}_2^2} = \boxed{m_2 = m_{22}}$$

$$\frac{\partial T}{\partial \dot{x}_3} = m_3 \dot{x}_3 \rightarrow \frac{\partial^2 T}{\partial \dot{x}_2 \partial \dot{x}_3} = \boxed{0 = m_{23} = m_{32}}$$

$$\rightarrow \frac{\partial^2 T}{\partial \dot{x}_3^2} = \boxed{m_3 = m_{33}}$$

$$\rightarrow \frac{\partial^2 T}{\partial \dot{x}_1 \partial \dot{x}_3} = \boxed{0 = m_{13} = m_{31}}$$

$$\underline{\underline{M}} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 5 \end{bmatrix} [kg]$$

A potenciális Energia

$$U = \frac{1}{2} s_1 (x_1 - x_2)^2 + \frac{1}{2} s_2 (x_2 - x_3)^2$$

$$\frac{\partial U}{\partial x_1} = s_1 (x_1 - x_2) \Rightarrow \frac{\partial^2 U}{\partial x_1^2} = \boxed{s_1 = s_{11}}$$

$$\Rightarrow \frac{\partial^2 U}{\partial x_1 \partial x_2} = \boxed{-s_1 = s_{12} = s_{21}}$$

$$\frac{\partial U}{\partial x_2} = s_2 (x_2 - x_3) - s_1 (x_1 - x_2) \Rightarrow \frac{\partial^2 U}{\partial x_2^2} = \boxed{s_1 + s_2 = s_{22}}$$

$$\Rightarrow \frac{\partial^2 U}{\partial x_2 \partial x_3} = \boxed{-s_2 = s_{23} = s_{32}}$$

$$\frac{\partial U}{\partial x_3} = -s_2 (x_2 - x_3) \Rightarrow \frac{\partial^2 U}{\partial x_3^2} = \boxed{s_2 = s_{33}} \; ; \; \frac{\partial^2 U}{\partial x_3 \partial x_1} = \boxed{0 = s_{13} = s_{31}}$$

Tehát a merevségi mátrix

$$\underline{\underline{S}} = \begin{bmatrix} s_1 & -s_1 & 0 \\ -s_1 & s_1+s_2 & -s_2 \\ 0 & -s_2 & s_2 \end{bmatrix} = \begin{bmatrix} 200 & -200 & 0 \\ -200 & 700 & -500 \\ 0 & -500 & 500 \end{bmatrix} \left[\frac{N}{m}\right]$$

↳ A mozgásegyenlet: $\underline{\underline{M}}\,\ddot{\underline{q}} + \underline{\underline{S}}\,\underline{q} = \underline{0}$ alakú

$\underline{q}(t) = \underline{A}\,e^{\lambda t}$ alakú a megoldás $\boxed{\lambda = \pm i\alpha}$

↳ Frekvenciaegyenlet:

$$\det(-\alpha^2 \underline{\underline{M}} + \underline{\underline{S}}) = 0$$

$$\begin{vmatrix} -2\alpha^2+200 & -200 & 0 \\ -200 & -4\alpha^2+700 & -500 \\ 0 & -500 & -5\alpha^2+500 \end{vmatrix} = 0$$

Figyelni kell:
- (α^2)-re polinom!
- alternáló előjelek!

$$-40\alpha^6 + 15000\alpha^4 - 1100000\,\alpha^2 = 0$$

a szépen megoldhatóval túl vagyok a nagyságrendi különbségek!

$\left(\frac{\alpha}{10}\right)$ az új változó

$$-40\cdot\cancel{10^6}\left(\frac{\alpha}{10}\right)^6 + 15\,0\cancel{00}\cdot\cancel{10^4}\left(\frac{\alpha}{10}\right)^4 - 110\cancel{0000}\cdot\cancel{10^2}\left(\frac{\alpha}{10}\right)^2 = 0 \quad \Big/ : 10^6$$

$$-40\left(\frac{\alpha}{10}\right)^6 + 150\cdot\left(\frac{\alpha}{10}\right)^4 - 110\left(\frac{\alpha}{10}\right)^2 = 0$$

$$\alpha^2\left(-40\alpha^4 + 15000\,\alpha^2 - 1100000\right) = 0$$

↳ $\alpha_1^2 = 0 \rightarrow \alpha_1 = 0\left[\frac{rad}{s}\right]$ (merevtest szerű mozgás)

↳ $\alpha_2^2 = 100 \rightarrow \alpha_2 = 10\left[\frac{rad}{s}\right]$

↳ $\alpha_3^2 = 275 \rightarrow \alpha_3 = 16{,}58\left[\frac{rad}{s}\right]$

Lengésképek :

- $\alpha_1 = 0 \left[\frac{rad}{s}\right]$ $\qquad \underbrace{(-\overset{=0}{\alpha_1^2} \underline{\underline{M}} + \underline{\underline{S}})\, \underline{A}_1 = \underline{0}}_{\underline{\underline{S}}\, \underline{A}_1 = \underline{0}}$

Tetszőlegesen választhatjuk

$\boxed{A_{11} = 1}$

DE nálunk $\boxed{A_{i1} = 1}$

$$\begin{bmatrix} s_1 & -s_1 & 0 \\ -s_1 & s_1+s_2 & -s_2 \\ 0 & -s_2 & s_2 \end{bmatrix} \begin{bmatrix} A_{11} \\ A_{12} \\ A_{13} \end{bmatrix} = \underline{0}$$

- $s_1 A_{11} - s_1 A_{12} = 0 \quad \Rightarrow \quad \boxed{A_{12} = A_{11} = 1}$
- $-s_1 A_{11} + (s_1+s_2) A_{12} - s_2 A_{13} = 0 \quad \Rightarrow \quad \boxed{A_{13} = 1}$

$$\underline{A}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

- $\alpha_2 = 10 \left[\frac{rad}{s}\right]$ $\qquad (-\alpha_2^2 \underline{\underline{M}} + \underline{\underline{S}})\, \underline{A}_2 = \underline{0}$

$$\begin{bmatrix} -\alpha_2^2 m_1 + s_1 & -s_1 & 0 \\ -s_1 & -\alpha_2^2 m_2 + (s_1+s_2) & -s_2 \\ 0 & -s_2 & -\alpha^2 m_3 + s_2 \end{bmatrix} \begin{bmatrix} A_{21} \\ A_{22} \\ A_{23} \end{bmatrix} = \underline{0}$$

$\boxed{A_{21} = 1}$

$$A_{22} = \frac{-\alpha_2^2 m_1 + s_1}{s_1} A_{21} = \underline{0} \qquad \text{(1. sorból)}$$

$$A_{23} = \frac{-s_1 A_{21} + (s_1+s_2-\alpha_2^2 m_2) A_{22}}{s_2} = -\frac{s_1}{s_2} A_{21} = \underline{\underline{0{,}4}}$$

$$\underline{A}_2 = \begin{bmatrix} 1 \\ 0 \\ -0{,}4 \end{bmatrix}$$

- $\alpha_3 = 16{,}58 \left[\frac{rad}{s}\right]$ $\qquad (-\alpha_3^2 \underline{\underline{M}} + \underline{\underline{S}})\, \underline{A}_3 = \underline{0}$ $\qquad \boxed{A_{31} = 1}$

↳ hasonlóan az előző esethez

$$A_{32} = \frac{-\alpha_3^2 m_1 + s_1}{s_1} A_{31} = -1{,}75$$

$$A_{33} = \frac{-s_1 A_{31} + (s_1+s_2-\alpha_3^2 m_3) A_{32}}{s_2} = 1$$

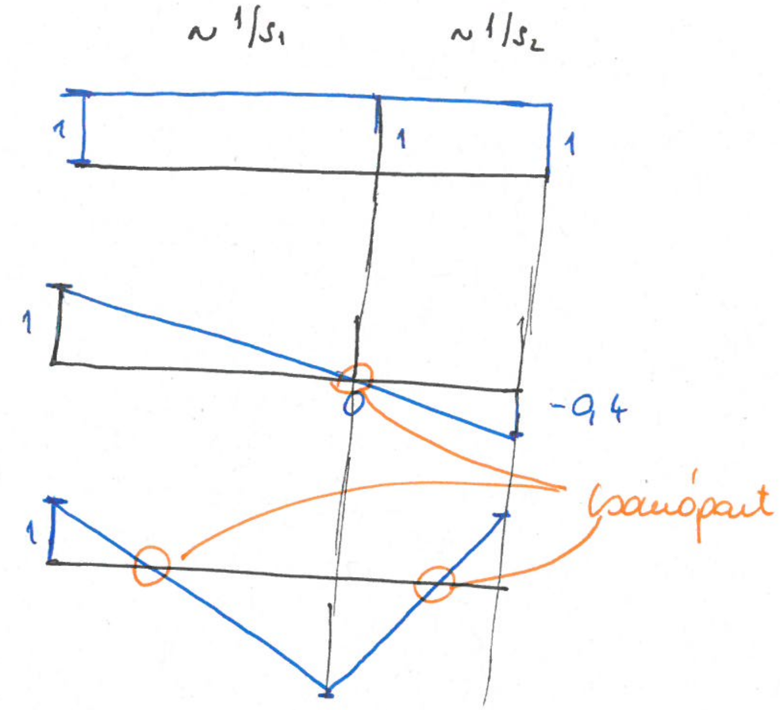
$$\underline{A}_3 = \begin{bmatrix} 1 \\ -1{,}75 \\ 1 \end{bmatrix}$$

Lengésképek ábrázolása

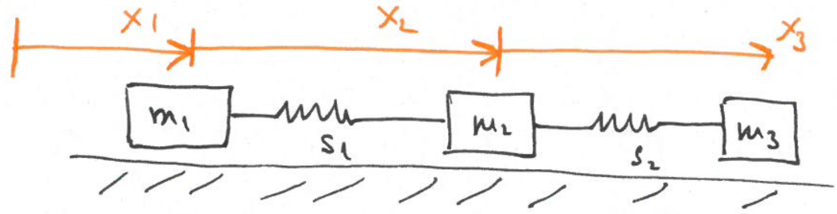
- $\alpha_1 = 0 \left[\frac{rad}{s}\right] \quad \underline{A}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

- $\alpha_2 = 10 \left[\frac{rad}{s}\right] \quad \underline{A}_2 = \begin{bmatrix} 1 \\ 0 \\ -0,4 \end{bmatrix}$

- $\alpha_3 = 16,58 \left[\frac{rad}{s}\right] \quad \underline{A}_3 = \begin{bmatrix} 1 \\ -1,75 \\ 1 \end{bmatrix}$



Másik opció: relatív koordinátákkal



$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 (\dot{x}_1 + \dot{x}_2)^2 + \frac{1}{2} m_3 (\dot{x}_1 + \dot{x}_2 + \dot{x}_3)^2$$

$$U = \frac{1}{2} s_1 x_2^2 + \frac{1}{2} s_2 x_3^2$$

↳ a deriválásokat követően:

$$\underline{\underline{M}} = \begin{bmatrix} m_1 + m_2 + m_3 & m_2 + m_3 & m_3 \\ m_2 + m_3 & m_2 + m_3 & m_3 \\ m_3 & m_3 & m_3 \end{bmatrix} \qquad \underline{\underline{S}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & s_1 & 0 \\ 0 & 0 & s_2 \end{bmatrix}$$

↳ A mátrixok mások lesznek!

↳ De a sajátkörfrekvenciák, lengésképek ugyanazok!