Több szabadságfokú lengőrendszerek

Elméleti összefoglaló:

↳ Mozgásegyenlet: $\underline{\underline{M}}\,\ddot{q} + \underline{\underline{K}}\,\dot{q} + \underline{\underline{S}}\,q = \underline{Q}^*$

↳ Felírása Lagrange-egyenlet alapján:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k} + \frac{\partial D}{\partial \dot{q}_k} + \frac{\partial U}{\partial q_k} = \underline{Q}_k^* \qquad k = 1 \ldots n$$

ha n DoF a rendszer

$$q = \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix}$$ általános koordináták

$$\underline{\underline{M}} = \left[\frac{\partial T}{\partial \dot{q}_i \partial \dot{q}_j}\right]\Bigg|_{\underline{0}} = \underline{\underline{M}}^T$$

$$\underline{\underline{S}} = \left[\frac{\partial U}{\partial q_j \partial q_k}\right]\Bigg|_{\underline{0}} = \underline{\underline{S}}^T$$

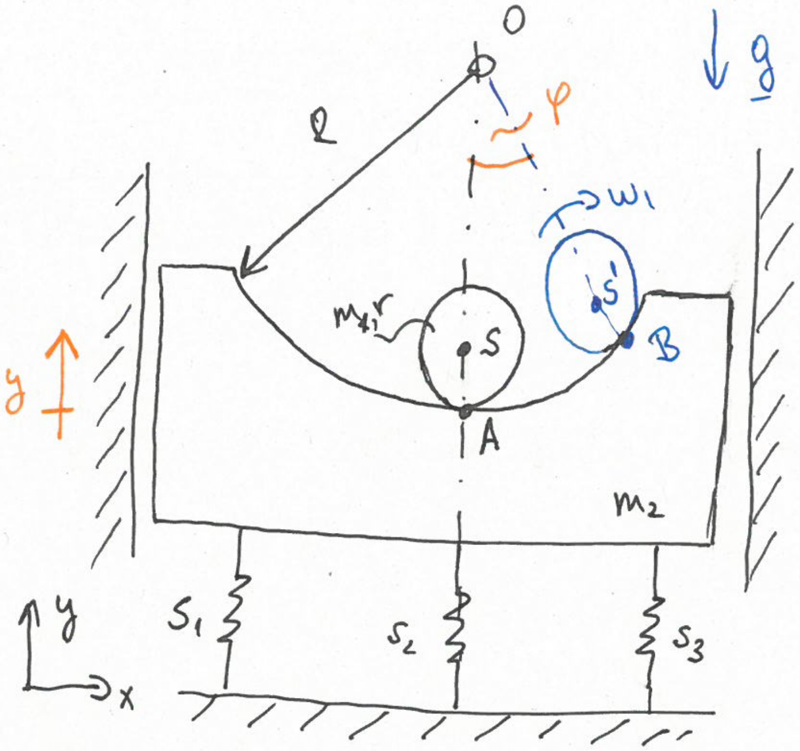
Frekvenciaegyenlet (csak csillapítatlan esetben!)

$$\det(-\alpha^2 \underline{\underline{M}} + \underline{\underline{S}}) = 0$$

↳ n-edfokú polinom α^2-re

$0 \leq \alpha_1 \leq \alpha_2 \ldots \leq \alpha_n$ - sajátkörfrekvenciák!

Feladat



Adatok:

m_1 - golyó tömege

m_2 - asztal tömege

R, r

s_1, s_2, s_3 - rugómerevségek

Feladat:

mozgásegyenletek felírása

1. Hány DoF a rendszer?

↳ az asztal függőlegesen el tud mozdulni $\boxed{q_1 = y}$ (A statikus egyensúlyi helyzettől)

↳ a golyó gördül a vályúban $\boxed{q_2 = \varphi}$

Tehát $n = 2$ DoF $\quad \underline{q} = \begin{bmatrix} y \\ \varphi \end{bmatrix}$

Látjuk, hogy csillapítatlan, gerjesztetlen:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k} + \frac{\partial U}{\partial q_k} = Q_k^{*} \qquad k = 1,2$$

alakú lesz a Lagrange egyenlet

A kinetikus energia

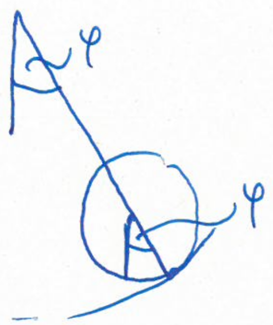
$$T = \frac{1}{2} m_2 \dot{y}_2^2 + \underbrace{\frac{1}{2} m_1 v_S^2 + \frac{1}{2} \Theta_S \omega_1^2}$$

mivel függőlegesen mozog

$\underline{v}_B \neq \underline{0} \Rightarrow$ nincs álló pont!

A korong súlypontja (R-r) sugarú körön mozog O körül!

$$\underline{v}_S = \underline{v}_O + \underline{\omega}_O \times \underline{r}_{OS} = \begin{bmatrix} 0 \\ \dot{y} \\ 0 \end{bmatrix} + \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 0 & \dot{\varphi} \\ (R-r)\sin\varphi & -(R-r)\cos\varphi & 0 \end{vmatrix} = \begin{bmatrix} \dot{\varphi}(R-r)\cos\varphi \\ \dot{y} + (R-r)\sin\varphi \\ 0 \end{bmatrix}$$



Másrészről

$$\underline{v}_S = \underline{v}_B + \underline{\omega}_1 \times \underline{r}_{BS} = \begin{bmatrix} 0 \\ \dot{y} \\ 0 \end{bmatrix} + \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 0 & \omega_1 \\ -r\sin\varphi & r\cos\varphi & 0 \end{vmatrix} = \begin{bmatrix} -\omega_1 r \cos\varphi \\ \dot{y} - \omega_1 r \sin\varphi \\ 0 \end{bmatrix}$$

Összevetve: $\dot{\varphi}(R-r)\cos\varphi = -\omega_1 r \cos\varphi$

$$\boxed{\omega_1 = \dot{\varphi} \frac{(R-r)}{r}}$$

Hasonlóan (relatív kinematikával)

$$\underline{v}_S = \underline{v}_{S\,száll} + \underline{v}_{S\,rel} = \underline{v}_{S\,száll} + \underline{v}_B + \underline{\omega}_1 \times \underline{r}_{BS} = \begin{bmatrix} -\omega_1 r\cos\varphi \\ \dot{y} - \omega_1 r \sin\varphi \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ \dot{y} \end{bmatrix} \qquad \begin{bmatrix} 0 \\ \dot{y} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} -\omega_1 r\cos\varphi \\ -\omega_1 r\sin\varphi \\ 0 \end{bmatrix}$$

A fentiekből $\quad \omega^2 = \dot{\varphi}^2 \dfrac{(R-r)^2}{r^2}$

$$v_S^2 = \underline{\omega_1^2 r^2 \cos^2\varphi} + \dot{y}^2 + \underline{\omega_1^2 r^2 \sin^2\varphi} - 2\dot{y}\,\omega_1 r \sin\varphi$$

$\omega_1^2 r^2$

$$v_S^2 = \dot{y}^2 + \omega_1^2 r^2 - 2\dot{y}\,\omega_1 r \sin\varphi = \underline{\underline{\dot{y}^2 + \dot{\varphi}^2 (R-r)^2 + 2\dot{y}(R-r)\dot{\varphi}\sin\varphi}}$$

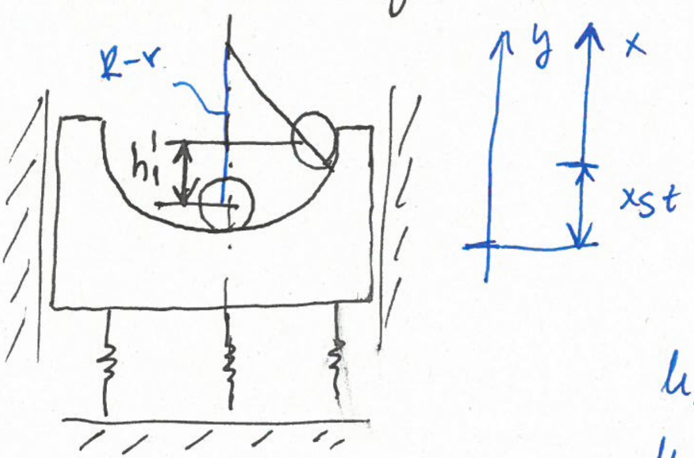
Tehát a kinetikus energia:

$$T = \frac{1}{2} m_1 \left(\dot{y}^2 + \dot{\varphi}^2 (R-r)^2 + 2\dot{y}\dot{\varphi}(R-r)\sin\varphi\right) + \frac{1}{2}\,\frac{1}{2} m_1 \cancel{r^2}\, \dot{\varphi}^2 \frac{(R-r)^2}{\cancel{r^2}} + \frac{1}{2} m_2 \dot{y}^2$$

$$T = \left(\frac{1}{2} m_1 + \frac{1}{2} m_2\right)\dot{y}^2 + \frac{1}{2}\Big(\underbrace{m_1 (R-r)^2 + \frac{1}{2} m_1 (R-r)^2}_{\frac{3}{2} m_1 (R-r)^2}\Big)\dot{\varphi}^2 + m_1 (R-r)\sin\varphi\,\dot{\varphi}\dot{y}$$

$$T = \frac{1}{2}(m_1 + m_2)\dot{y}^2 + \frac{1}{2}\left(\frac{3}{2} m_1 (R-r)^2\right)\dot{\varphi}^2 + m_1 (R-r)\sin\varphi\,\dot{\varphi}\dot{y}$$

- Potenciális energia



$$U = \frac{1}{2} s_1 x^2 + \frac{1}{2} s_2 x^2 + \frac{1}{2} s_3 x^2 + m_1 g h_1 + m_2 g h_2$$

$$h_1 = x + h_1' = x + (R-r)(1-\cos\varphi)$$

$$h_2 = x$$

$$U = \frac{1}{2}(s_1 + s_2 + s_3)x^2 + m_1 g\left(x + (R-r)(1-\cos\varphi)\right) + m_2 g x$$

$$U = \frac{1}{2}(s_1 + s_2 + s_3)x^2 + (m_1 + m_2) g x + m_1 g (R-r)(1-\cos\varphi)$$

konst!

$$x = y - x_{st} \;\rightarrow\; U = \frac{1}{2}(s_1+s_2+s_3)y^2 + \boxed{\frac{1}{2}(s_1+s_2+s_3)x_{st}^2} - \underline{(s_1+s_2+s_3)x_{st}\,y}$$

konst.

$$+ \underline{(m_1+m_2) g y} - \boxed{(m_1+m_2) g x_{st}} + m_1 g (R-r)(1-\cos\varphi)$$

egyensúlyi egy. miatt

Tehát:

$$\mathcal{U} = \frac{1}{2}(s_1+s_2+s_3)y^2 + m_1 g (R-r)(1-\cos\varphi) + \underbrace{\left((m_1+m_2)g-(s_1+s_2+s_3)x_{st}\right)}_{=0!}y$$
$$+ \text{const}$$

$$\tilde{\mathcal{U}} = \frac{1}{2}(s_1+s_2+s_3)y^2 + m_1 g (R-r)(1-\cos\varphi) \quad \text{is jó lenne}$$

A deriváltak

$$\frac{\partial T}{\partial \dot y} = (m_1+m_2)\dot y + m_1 (R-r)\sin\varphi \cdot \dot\varphi$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot y} = (m_1+m_2)\ddot y + m_1 (R-r)\cos\varphi\, \dot\varphi^2 + m_1 (R-r)\sin\varphi \cdot \ddot\varphi$$

$$\frac{\partial T}{\partial y} = 0 \;;\; \frac{\partial \tilde{\mathcal{U}}}{\partial y} = \underline{\underline{(s_1+s_2+s_3)y}}$$

$$\frac{\partial T}{\partial \dot\varphi} = \frac{3}{2} m_1 (R-r)^2\dot\varphi + m_1 (R-r)\sin\varphi\, \dot y$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot\varphi} = \frac{3}{2} m_1 (R-r)^2\ddot\varphi + m_1 (R-r)\cos\varphi\, \dot y\dot\varphi + m_1 (R-r)\sin\varphi\, \ddot y$$

$$\frac{\partial T}{\partial \varphi} = m_1 (R-r)\cos\varphi\, \dot\varphi \dot y$$

$$\frac{\partial \tilde{\mathcal{U}}}{\partial \varphi} = m_1 g (R-r)\sin\varphi$$

A mozgástörvény

(1) $$(m_1+m_2)\ddot y + m_1 (R-r)\cos\varphi\, \dot\varphi^2 + m_1 (R-r)\sin\varphi\, \ddot\varphi + s_e y = 0$$

(2) $$\frac{3}{2} m_1 (R-r)^2\ddot\varphi + \cancel{m_1 (R-r)\cos\varphi\, \dot y\dot\varphi} + m_1 (R-r)\sin\varphi\, \ddot y - \cancel{m_1 (R-r)\cos\varphi\, \dot\varphi \dot y}$$
$$+ m_1 g (R-r)\sin\varphi = 0$$

- <u>Linearizálás</u>: $\begin{bmatrix} y \\ \varphi \end{bmatrix} = \underline{0}$ e.h. $\sin\varphi \approx \varphi$; $\dot{\varphi}^2 \approx 0$
  $\cos\varphi \approx 1$; $\sin\varphi\,\ddot{\varphi} \approx 0$

$$\left.\begin{aligned} &(1)\quad (m_1+m_2)\ddot{y} + s_e y = 0 \\ &(2)\quad \frac{3}{2} m_1 (R-r)^2 \ddot{\varphi} + m_1 g (R-r)\varphi = 0 \end{aligned}\right\} \text{ lineáris DE}$$

<u>Mátrix DE</u>: $\underline{\underline{M}}\,\ddot{\underline{q}} + \underline{\underline{S}}\,\underline{q} = \underline{0}$

<u>Tömeg mátrix</u>

$$\frac{\partial T}{\partial \dot{y}} = (m_1+m_2)\dot{y} + m_1(R-r)\sin\varphi\,\dot{\varphi}$$

$$\frac{\partial^2 T}{\partial \dot{y}^2} = m_1+m_2 \Big|_{\substack{\varphi=0\\y=0}} = \underline{\underline{m_1+m_2}} = m_{11}$$

$$\frac{\partial^2 T}{\partial \dot{y}\,\partial\dot{\varphi}} = m_1(R-r)\sin\varphi\Big|_{\substack{\varphi=0\\y=0}} = 0 = \underline{\underline{m_{12}}}$$

$$\frac{\partial T}{\partial \dot{\varphi}} = \frac{3}{2}(m_1)(R-r)^2\dot{\varphi} + m_1(R-r)\sin\varphi\,\dot{y}$$

$$\frac{\partial^2 T}{\partial \dot{\varphi}^2} = \frac{3}{2}m_1(R-r)^2 = \underline{\underline{m_{22}}}$$

$$\underline{\underline{M}} = \begin{bmatrix} m_1+m_2 & 0 \\ 0 & \frac{3}{2}m_1(R-r)^2 \end{bmatrix} \quad (S1)$$

<u>Merevségi mátrix</u>:

$$\frac{\partial U}{\partial y} = s_e y$$

$$\frac{\partial^2 U}{\partial y^2} = s_e = s_{11}$$

$$\frac{\partial^2 U}{\partial y\,\partial\varphi} = 0 = s_{12} = s_{21}$$

$$\frac{\partial U}{\partial \varphi} = m_1 g(R-r)\sin\varphi$$

$$\frac{\partial^2 U}{\partial \varphi^2} = m_1 g(R-r)\cos\varphi\Big|_{\substack{\varphi=0\\y=0}} = m_1 g(R-r) = s_{22}$$

$$\underline{\underline{S}} = \begin{bmatrix} s_e & 0 \\ 0 & m_1 g(R-r) \end{bmatrix}$$