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## COMPARISON OF ANALYTICAL AND NUMERICAL SIMULATIONS FOR VARIABLE SPINDLE SPEED TURNING

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### ABSTRACT

*The turning process with varying spindle speed is investigated. The well-known single degree of freedom turning model is presented and the governing delay-differential equation with time varying delay is analyzed. Three different numerical techniques are used to solve the governing equation: (1) direct Euler simulation with linear interpolation of the delayed term, (2) Taylor expansion of the time delay variation combined with Euler integration and (3) semi-discretization method. The results of the three methods are compared. Stability charts are constructed, and some improvements in the process stability is shown, especially for low spindle speed domains.*

### INTRODUCTION

The history of machine tool chatter dates back nearly 100 years, when Taylor (1907) described machine tool chatter as the “most obscure and delicate of all problems facing the machinist”. After the extensive work of Tlustý et al., (1962), Tobias (1965), Merrit (1965) and Kudinov (1967), the so-called regenerative effect has become the most commonly accepted explanation for machine tool chatter (Stépán, 1989, Moon, 1998). This effect is related to the cutting force variation due to the wavy workpiece surface cut one revolution ago. The corresponding mathematical models are delay-differential equations (DDEs).

For the simplest model of turning, the governing equation of motion is an autonomous DDE with a corresponding infinite dimensional state space. This fact results in an infinite number of characteristic roots, most of which have negative real parts referring to damped components of the vibration signals.

Prevention of chatter is a primary problem for the machinist. The notion that parametric excitation effects may suppress vibrations during the cutting process comes from the famous problem of stabilizing inverted pendulums by

parametric excitation (see, for example, Insperger and Horváth, 2000). The governing equation of motion of the turning process with parametric excitation is a time periodic DDE.

Periodically varying stiffness was suggested by Segalman and Butcher (2000) to suppress chatter in turning. They investigated the resulting DDE with time periodic coefficients by the harmonic balance method, and found some stability improvements.

In the 1970s, several researchers suggested continuous variation of the spindle speed for chatter suppression (see Inamura and Sata, 1974, Takemura et al., 1974, Hosho et al., 1977, Sexton et al., 1977, Sexton and Stone, 1978). The corresponding mathematical model is a DDE with time varying delay. Inamura and Sata (1974) and Sexton et al. (1977) approximated the quasi-periodic solutions of the time periodic DDE by periodic ones and applied the harmonic balance method to derive stability boundaries. They predicted improvements in stability properties by a factor of 10 for properly chosen parameter values. In spite of some reports on successful experiments, the stability investigations of cutting with time varying spindle speeds were not reliable enough to present a breakthrough in this field.

With their novel approach, Jayaram et al. (2000) created stability charts for turning with varying spindle speed. They used quasi-periodic trial solutions for the periodic DDE, and combined the Fourier expansion with an expansion using Bessel function series. They determined stability boundaries by harmonic balance method and obtained slight improvements in stability properties for low spindle speed domains.

The semi-discretization method was used to obtain stability charts by Insperger et al. (2001). They showed that contrary to cutting processes with constant spindle speed, where only secondary Hopf and period doubling bifurcations may arise, for

machining with varying spindle speed period-one bifurcation is also a possible route for the onset of chatter.

Different shapes of the spindle speed modulation were investigated by Insperger and Stépán (2002a), such as sinusoidal modulation and piecewise linearly increasing or decreasing (saw-like) modulations. They found that the shape of the modulation has no significant effect on the stability charts. Spindle speed can also be modulated randomly in order to suppress chatter, as was investigated by Yilmaz et al. (2002).

The allowable parameters for the spindle speed modulation (amplitude, frequency) are bounded by the technological conditions. The realizable values are 10% for modulation amplitude and 0.5–1 Hz for modulation frequency (Young and Schaut, 2001).

In this paper, turning process with sinusoidal spindle speed modulation is investigated using three different methods:

- 1, The governing DDE with time varying delay is solved by Euler integration.
- 2, The governing DDE with time varying delay is transformed to a DDE with constant delay via Taylor expansion and it is solved by Euler integration.
- 3, The governing DDE with time varying delay is investigated by semi-discretization.

### SINGLE DEGREE OF FREEDOM MODELLING FOR SPINDLE SPEED MODULATED TURNING

The mechanical model of the turning process in case of orthogonal cutting is shown in Figure 1. The mass  $m$  of the tool, the damping coefficient  $c$ , and the spring stiffness  $k$  can be determined via modal analysis of the machine tool that has a well-separated dominant natural frequency. The structure is assumed to be flexible in the  $x$  direction only. This reduces the model to a single degree of freedom. The prescribed feed motion is uniform with a constant speed  $v$  of the tool. The angular speed of the workpiece is denoted by  $\Omega(t)$ . The cutting force  $F$  depends on the current chip thickness that is influenced by the current and delayed position of the tool,  $x(t)$  and  $x(t-\tau(t))$ , respectively. The linear equation of motion reads

$$\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2x(t) = \frac{Kb}{m}(x(t-\tau(t)) - x(t)) \quad (1)$$

where  $\omega_n = \sqrt{k/m}$  is the dominant natural angular frequency of the system,  $\zeta = c/(2m\omega_n)$  is the damping ratio,  $K$  is the  $x$  direction specific cutting energy and  $b$  is the chip width.

If the spindle speed is constant:  $\Omega(t) \equiv \Omega_0$ , and is given in [rpm], then the time delay can be expressed as  $\tau_0 = 60/\Omega_0$ . In the case of time periodic spindle speed modulation  $\Omega(t+T) = \Omega(t)$ , the time delay is also time periodic with the same period:  $\tau(t+T) = \tau(t)$ . In this case, the time delay can only be given in the implicit form

$$\int_{t-\tau(t)}^t \Omega(s) / 60 \, ds = 1. \quad (2)$$

This means that the workpiece makes one revolution in the time interval  $[t-\tau(t), t]$  for any  $t$ .

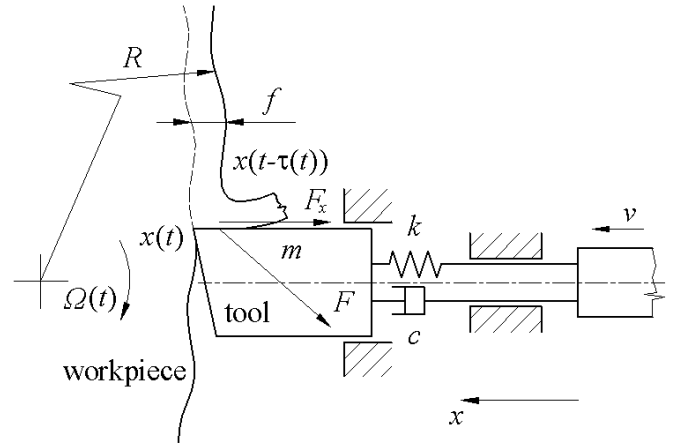


Fig. 1. Mechanical model of turning process

However, if we assume a harmonic spindle speed modulation in the form  $\Omega(t) = \Omega_0(1 + \varepsilon \cos(\omega t))$ , where the modulation amplitude ( $\varepsilon$ ) is small, then the time delay variation can be approximated as

$$\tau(t) = \tau_0(1 - \varepsilon \cos(\omega t)), \quad (3)$$

where  $\tau_0 = 60/\Omega_0$ . In Figure 2, the exact time delay obtained by the numerical solution of equation (2) and the approximated time delay is shown for 10% modulation amplitude ( $\varepsilon = 0.1$ ). The time period of the modulation is  $T = 2\pi/\omega$ . The frequency of the modulation is characterized by the ratio  $T/\tau_0$ .

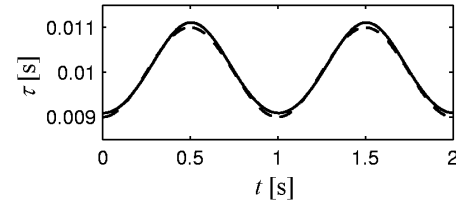


Fig. 2. Exact (continuous) and approximate (dashed) time delay variation for  $\Omega_0=6000$  [rpm],  $\varepsilon=0.1$  and  $T=1$  [s]

Introducing the dimensionless time  $\tilde{t} = t\omega_n$  gives the dimensionless equation of motion

$$\ddot{x}(\tilde{t}) + 2\zeta\dot{x}(\tilde{t}) + x(\tilde{t}) = k_1(x(\tilde{t} - \omega_n\tau(\tilde{t})) - x(\tilde{t})), \quad (4)$$

where  $k_1 = Kb/(m\omega_n^2)$  is proportional to the chip width, or depth of cut. The dimensionless time delay is then

$$\omega_n\tau(\tilde{t}) = \omega_n\tau_0\left(1 - \varepsilon \cos\left(\frac{\omega}{\omega_n}\tilde{t}\right)\right), \quad (5)$$

where the dimensionless time period of the modulation is  $\omega_n T$ .

## GOVERNING DIMENSIONLESS EQUATION

For the sake of simplicity, equation (4) is rewritten in the form

$$\ddot{x}(t) + 2\zeta \dot{x}(t) + x(t) = k_1(x(t - \tau(t)) - x(t)), \quad (6)$$

with

$$\tau(t + T) = \tau(t) = \tau_0(1 - \varepsilon \cos(\omega t)) \quad (7)$$

where – by abuse of notation –  $t$  denotes dimensionless time,  $\tau(t)$  is the dimensionless delay,  $\omega = 2\pi/T$  is the dimensionless modulation frequency and  $T$  is the dimensionless time period.

In the subsequent analysis, equation (6) is investigated with the time delay expression (7), and stability charts are constructed for 10% modulation amplitude ( $\varepsilon = 0.1$ ), and for the modulation periods  $T/\tau_0 = 2$  and  $T/\tau_0 = 5$ .

The stability chart of equation (6) with  $\tau(t) \equiv \tau_0$  (that is with  $\varepsilon = 0$ ) is shown in Figure 3. The axes are dimensionless spindle speed  $\Omega_0/(60f_n)$  and the “dimensionless depth of cut”  $k_1$ . Here,  $f_n = \omega_n/2\pi$  is the natural frequency of the machine tool and  $\Omega_0 = 60/\tau_0$ . Below the boundary curves, the cutting operation is stable, above them, chatter arises.

Our purpose here is to investigate how the stability lobes in Figure 3 change when spindle speed modulation is applied.

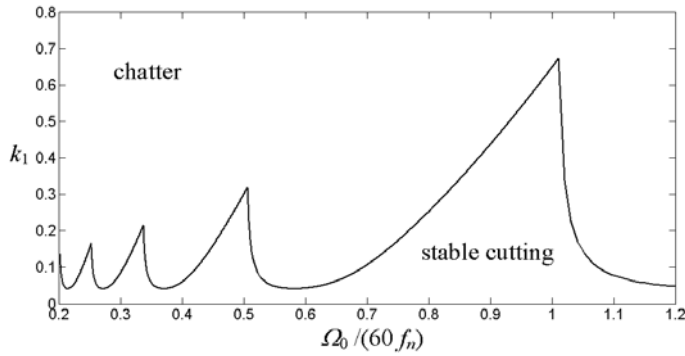


Fig. 3. Example stability chart of turning process with constant spindle speed

## STABILITY ANALYSIS VIA DIRECT EULER INTEGRATION

The difficulty with the discretization of DDEs is that the time delay  $\tau$  is not always a multiple of the length  $\Delta t$  of the discretization interval, therefore the delayed term can not always be expressed as a delayed discrete value of the state variable. Here, the following linear interpolation will be used: if  $m\Delta t < \tau < (m+1)\Delta t$ , where  $m$  is integer, then the delayed term  $x(t - \tau)$  is approximated as a linear combination of the “neighboring” discrete values  $x_{i-m} = x(t_i - m\Delta t)$  and  $x_{i-m-1} = x(t_i - (m+1)\Delta t)$ .

Consider the following discrete approximation of the state variable and its derivatives in equation (6):

$$x(t_i) = x_i, \quad (8)$$

$$\dot{x}(t_i) \approx \frac{x_{i+1} - x_i}{\Delta t}, \quad (9)$$

$$\ddot{x}(t) \approx \frac{x_{i+2} - 2x_{i+1}x_i}{\Delta t^2}, \quad (10)$$

$$x(t_i - \tau_i) \approx \frac{(m_i + 1)\Delta t - \tau_i}{\Delta t} x_{i-m_i} + \frac{\tau_i - m_i\Delta t}{\Delta t} x_{i-m_i-1}, \quad (11)$$

where  $\Delta t = t_{i+1} - t_i$ ,  $\tau_i = \tau(t_i)$ , and  $i$  and  $m_i$  are integers. Integer  $m_i$  is defined as  $m_i = \text{int}(\tau(t_i)/\Delta t)$ , where  $\text{int}$  is the function that rounds positive numbers towards zero (e.g.  $\text{int}(4.32) = 4$ ).

Substitution of approximations (8)-(11) into equation (6) yields the recursive formula

$$x_{i+2} = (2 - 2\zeta \Delta t)x_{i+1} + (2\zeta \Delta t - 1 - \Delta t^2(1 + k_1))x_i + \Delta t^2 k_1 \frac{(m_i + 1)\Delta t - \tau_i}{\Delta t} x_{i-m_i} + \Delta t^2 k_1 \frac{\tau_i - m_i\Delta t}{\Delta t} x_{i-m_i-1}. \quad (12)$$

Numerical integration gives a series of the state variable  $x_i$ . For computations, the time-step was chosen as  $\Delta t = \tau_0/1000$  and the values of  $x_i$  was computed over the interval  $[0, 10T]$ . For the case  $T/\tau_0 = 2$ , this means that  $x_i$  is computed for  $n = 20000$  steps. Consequently, for the case  $T/\tau_0 = 5$ , the number of computation steps is  $n = 50000$ .

An initial condition for the recursive computation is the following:

$$x_j = 0 \quad \text{for} \quad j = -m_1, -m_1 + 1, \dots, 1 \quad \text{and} \quad x_2 = 1. \quad (13)$$

The first step ( $i=1$ ) of recursion gives the value of  $x_{i+2} = x_3$ , the second step gives  $x_4$ , etc.

The process is said to be stable if the amplitude of the signal is decreasing. This implies the stability condition:

$$\sum_{i=3}^{n/2+2} |x_i| < \sum_{i=n/2+3}^{n+2} |x_i|, \quad (14)$$

where  $n$  is the number of computation steps.

Theoretically, this stability condition is not exact, since the series  $x_i$  is determined just for the single initial condition (13) from the  $m_1+3$  possible initial conditions of equation (12) (or, furthermore, from the infinite number of possible initial conditions of equation (6)). However, as it will be shown later, condition (14) gives satisfactory results.

## STABILITY ANALYSIS VIA TAYLOR EXPANSION AND EULER INTEGRATION

In this approach, the delayed term in equation (6) is approximated by the Taylor series expansion around  $\tau_0$ :

$$\begin{aligned} x(t - \tau(t)) &= x(t - \tau_0 + \tau_0 \varepsilon \cos(\omega t)) \\ &= x(t - \tau_0) + \dot{x}(t - \tau_0) \tau_0 \varepsilon \cos(\omega t) \\ &\quad + \frac{1}{2} \ddot{x}(t - \tau_0) (\tau_0 \varepsilon \cos(\omega t))^2 \\ &\quad + \frac{1}{6} \dddot{x}(t - \tau_0) (\tau_0 \varepsilon \cos(\omega t))^3 + \dots \end{aligned} \quad (15)$$

The mathematical justification of this approximation is quite poor, since the Taylor expansion does not uniformly converge on the set of closed intervals above the past.

In the next three subsections, first, second, and third order expansion will be investigated.

### First order expansion

If the first order expansion

$$x(t - \tau(t)) \approx x(t - \tau_0) + \dot{x}(t - \tau_0)\tau_0\epsilon \cos(\omega t) \quad (16)$$

is considered, then equation (6) can be approximated as

$$\begin{aligned} \ddot{x}(t) + 2\zeta \dot{x}(t) + (1 + k_1)x(t) \\ = k_1 x(t - \tau_0) + k_1 \dot{x}(t - \tau_0)\tau_0\epsilon \cos(\omega t). \end{aligned} \quad (17)$$

This is a DDE with a constant time delay and with a periodic coefficient in the right-hand-side.

Stability of equation (17) can be determined by Euler integration. The delayed term is approximated as

$$x(t_i - \tau_0) = x_{i-m}, \quad (18)$$

where  $\Delta t = t_{i+1} - t_i$  and the integer  $m$  is chosen so that  $\tau_0 = m\Delta t$  is satisfied. The first derivative of the delayed term is approximated as follows:

$$\dot{x}(t_i - \tau_0) \approx \frac{x_{i-m+1} - x_{i-m}}{\Delta t}. \quad (19)$$

For the approximations of the actual time domain terms and derivatives, expressions (8)-(10) can be used. As a result of the discretization, the following recursive formula can be given:

$$\begin{aligned} x_{i+2} = (2 - 2\zeta \Delta t)x_{i+1} + (2\zeta \Delta t - 1 - \Delta t^2(1 + k_1))x_i \\ + k_1(\Delta t^2 - \Delta t C_i)x_{i-m} + k_1 \Delta t C_i x_{i-m+1}, \end{aligned} \quad (20)$$

where  $C_i = \tau\epsilon \cos(\omega t_i)$ . The initial condition for the recursive computation is chosen as

$$x_j = 0 \quad \text{for} \quad j = -m_1 + 1, -m_1 + 2, \dots, 1 \quad \text{and} \quad x_2 = 1. \quad (21)$$

Stability properties can now be determined in the manner described by equation (14).

### Second order expansion

If the second order expansion

$$\begin{aligned} x(t - \tau(t)) \approx x(t - \tau_0) + \dot{x}(t - \tau_0)\tau_0\epsilon \cos(\omega t) \\ + \frac{1}{2}\ddot{x}(t - \tau_0)(\tau_0\epsilon \cos(\omega t))^2 \end{aligned} \quad (22)$$

is considered, then equation (6) can be approximated as

$$\begin{aligned} \ddot{x}(t) + 2\zeta \dot{x}(t) + (1 + k_1)x(t) \\ = k_1 x(t - \tau_0) + k_1 \dot{x}(t - \tau_0)\tau_0\epsilon \cos(\omega t) \\ + \frac{1}{2}k_1 \ddot{x}(t - \tau_0)(\tau_0\epsilon \cos(\omega t))^2. \end{aligned} \quad (23)$$

In this DDE, the second derivative of both the actual time domain term  $x(t)$  and the delayed term  $x(t - \tau)$  appear. This class of equations is referred to as neutral functional differential equations (NFDEs) (see Kolmanovskii and Nosov, 1986).

Substitution of expressions (8)-(10), (18), (19) and the approximation

$$\ddot{x}(t_i - \tau_0) \approx \frac{x_{i-m+2} - 2x_{i-m+1} + x_{i-m}}{\Delta t^2} \quad (24)$$

into equation (6) results in the recursive formula

$$\begin{aligned} x_{i+2} = (2 - 2\zeta \Delta t)x_{i+1} + (2\zeta \Delta t - 1 - \Delta t^2(1 + k_1))x_i \\ + k_1(\Delta t^2 - \Delta t C_i + \frac{1}{2}C_i^2)x_{i-m} \\ + k_1(\Delta t C_i - C_i^2)x_{i-m+1} + \frac{1}{2}k_1 C_i^2 x_{i-m+2}, \end{aligned} \quad (25)$$

where, again,  $C_i = \tau\epsilon \cos(\omega t_i)$ . For computations, the initial condition is given as (21), again.

### Third order expansion

If the third order expansion

$$\begin{aligned} x(t - \tau(t)) \approx x(t - \tau_0) + \dot{x}(t - \tau_0)\tau_0\epsilon \cos(\omega t) \\ + \frac{1}{2}\ddot{x}(t - \tau_0)(\tau_0\epsilon \cos(\omega t))^2 \\ + \frac{1}{6}\ddot{\ddot{x}}(t - \tau_0)(\tau_0\epsilon \cos(\omega t))^3 \end{aligned} \quad (26)$$

is considered, then equation (6) can be approximated as

$$\begin{aligned} \ddot{x}(t) + 2\zeta \dot{x}(t) + (1 + k_1)x(t) \\ = k_1 x(t - \tau_0) + k_1 \dot{x}(t - \tau_0)\tau_0\epsilon \cos(\omega t) \\ + \frac{1}{2}k_1 \ddot{x}(t - \tau_0)(\tau_0\epsilon \cos(\omega t))^2 \\ + \frac{1}{6}k_1 \ddot{\ddot{x}}(t - \tau_0)(\tau_0\epsilon \cos(\omega t))^3. \end{aligned} \quad (27)$$

In this DDE, the third derivative of the delayed term  $x(t - \tau)$  appears, while only the second derivatives of the actual time domain term  $x(t)$  occur. This type of equations, where the highest derivative is delayed, is called an advanced functional differential equation (AFDE). The reason for the phrase “advanced” is demonstrated by the following example. Consider the simple AFDE

$$\dot{x}(t) = \ddot{x}(t - \tau). \quad (28)$$

By a  $\tau$ -shift transformation, and by using the new variable  $z = \dot{x}$ , equation (28) can be written in the form

$$\dot{z}(t) = z(t + \tau). \quad (29)$$

Here, the rate of change of state is determined by the future values of state, i.e., an advanced state determines the present state. Therefore, AFDEs are always unstable and have minor physical relevance at the moment. Consequently, equation (27) is unstable for any parameters, which confirms the weakness of the Taylor expansion.

In spite of all the problems with AFDEs, Euler integration can be used to solve the AFDE (27). Expressions (8)-(10), (18), (19), (24) and the approximation

$$\ddot{x}(t_i - \tau_0) = \frac{x_{i-m+3} - 3x_{i-m+2} + 3x_{i-m+1} - x_{i-m}}{\Delta t^3} \quad (30)$$

can be used for the numerical integration. After substitution into equation (6), the following recursive formula is obtained:

$$\begin{aligned} x_{i+2} = & (2 - 2\zeta \Delta t)x_{i+1} + (2\zeta \Delta t - 1 - \Delta t^2(1 + k_1))x_i \\ & + k_1 \left( \Delta t^2 - \Delta t C_i + \frac{1}{2} C_i^2 - \frac{1}{6\Delta t} C_i^3 \right) x_{i-m} \\ & + k_1 \left( \Delta t C_i - C_i^2 + \frac{1}{2\Delta t} C_i^3 \right) x_{i-m+1} \\ & + k_1 \left( \frac{1}{2} C_i^2 - \frac{1}{2\Delta t} C_i^3 \right) x_{i-m+2} + k_1 \frac{1}{6\Delta t} C_i^3 x_{i-m+3}, \end{aligned} \quad (31)$$

where, again,  $C_i = \tau \varepsilon \cos(\omega t_i)$  and the initial condition is the same as (21).

As it will be shown later by stability charts, the discrete map (31) is not necessary unstable as it is expected from the advanced nature of equation (27). This means that the Euler integration is not an accurate method for the numerical solution of AFDEs.

## STABILITY ANALYSIS VIA SEMI-DISCRETIZATION METHOD

Semi-discretization is a technique to construct solutions for delayed equations (see Insperger and Stépán, 2002b).

Consider the time interval series  $[t_i, t_{i+1}]$ ,  $i \in \mathbf{Z}$  of interval length  $\Delta t = t_{i+1} - t_i$ , and with starting time  $t_0 = 0$ . Let  $\Delta t$  be chosen such that  $T = N\Delta t$ , where  $N$  is an integer. Also, define the “current” time delay by  $\tau_i = \frac{1}{\Delta t} \int_{t_i}^{t_{i+1}} \tau(s) ds$ , and the integer  $m_i = \text{int}(\tau_i / \Delta t)$ , where  $\text{int}$  has the same meaning as stated previously. The delayed term is approximated as

$$x(t - \tau(t)) \approx x(t - \tau_i) \approx \alpha_i x(t_{i-m_i}) + \beta_i x(t_{i-m_i-1}), \quad (32)$$

where

$$\alpha_i = \frac{(m_i + 1)\Delta t - \tau_i}{\Delta t} \quad \text{and} \quad \beta_i = \frac{\tau_i - m_i \Delta t}{\Delta t}$$

are a kind of weighting coefficients. Finally, let the integer  $m_{\max}$  be defined as  $m_{\max} = 1 + \max\{m_i, i = 0, 1, \dots, k\}$ .

Equation (6) can then be written in the form

$$\ddot{x}(t) + 2\zeta \dot{x}(t) + (1 + k_1)x(t) = k_1 (\alpha_i x(t_{m_i}) + \beta_i x(t_{m_i+1})) \quad (33)$$

For the initial conditions  $x(t_i) = x_i$  and  $\dot{x}(t_i) = \dot{x}_i$ , equation (33) can be solved as an ordinary differential equation. The solution and its derivative at time instant  $t_{i+1}$  read

$$x_{i+1} = x(t_{i+1}) = a_{00}x_i + a_{01}\dot{x}_i + b_{0\alpha}x_{i-m_i} + b_{0\beta}x_{i-m_i-1}, \quad (34)$$

$$\dot{x}_{i+1} = \dot{x}(t_{i+1}) = a_{10}x_i + a_{11}\dot{x}_i + b_{1\alpha}x_{i-m_i} + b_{1\beta}x_{i-m_i-1}, \quad (35)$$

where

$$\begin{aligned} a_{00} &= \kappa_{10} \exp(\lambda_1 \Delta t) + \kappa_{20} \exp(\lambda_2 \Delta t), \\ a_{01} &= \kappa_{11} \exp(\lambda_1 \Delta t) + \kappa_{21} \exp(\lambda_2 \Delta t), \\ a_{10} &= \kappa_{10} \lambda_1 \exp(\lambda_1 \Delta t) + \kappa_{20} \lambda_2 \exp(\lambda_2 \Delta t), \\ a_{11} &= \kappa_{11} \lambda_1 \exp(\lambda_1 \Delta t) + \kappa_{21} \lambda_2 \exp(\lambda_2 \Delta t), \\ b_{0\alpha} &= \alpha_i (\sigma_1 \exp(\lambda_1 \Delta t) + \sigma_2 \exp(\lambda_2 \Delta t) + k_1 / (1 + k_1)), \\ b_{0\beta} &= \beta_i (\sigma_1 \exp(\lambda_1 \Delta t) + \sigma_2 \exp(\lambda_2 \Delta t)), \\ b_{1\alpha} &= \alpha_i (\sigma_1 \lambda_1 \exp(\lambda_1 \Delta t) + \sigma_2 \lambda_2 \exp(\lambda_2 \Delta t)), \\ b_{1\beta} &= \beta_i (\sigma_1 \lambda_1 \exp(\lambda_1 \Delta t) + \sigma_2 \lambda_2 \exp(\lambda_2 \Delta t)), \end{aligned}$$

and

$$\begin{aligned} \lambda_{1,2} &= \frac{-2\zeta \pm \sqrt{4\zeta^2 - 4(1 + k_1)}}{2}, \\ \kappa_{10} &= \frac{\lambda_2}{\lambda_2 - \lambda_1}, \quad \kappa_{11} = \frac{-1}{\lambda_2 - \lambda_1}, \quad \sigma_1 = \frac{-\lambda_2}{\lambda_2 - \lambda_1} \frac{k_1}{1 + k_i}, \\ \kappa_{20} &= \frac{-\lambda_1}{\lambda_2 - \lambda_1}, \quad \kappa_{21} = \frac{1}{\lambda_2 - \lambda_1}, \quad \sigma_2 = \frac{\lambda_1}{\lambda_2 - \lambda_1} \frac{k_1}{1 + k_i}. \end{aligned}$$

Equations (34) and (35) define the discrete map

$$\mathbf{y}_{i+1} = \mathbf{A}_i \mathbf{y}_i, \quad (36)$$

where the  $m_{\max}+2$  dimensional state vector is

$$\mathbf{y}_i = \text{col}(\dot{x}_i \ x_i \ x_{i-1} \ \dots \ x_{i-m_{\max}}),$$

and the coefficient matrix has the form

$$\mathbf{A}_i = \begin{pmatrix} a_{11} & a_{10} & 0 & \dots & 0 & b_{0\alpha} & b_{0\beta} & 0 & \dots \\ a_{01} & a_{00} & 0 & \dots & 0 & b_{1\alpha} & b_{1\beta} & 0 & \dots \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \dots \\ 0 & 0 & 0 & \ddots & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \dots & \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}.$$

In the interval  $[t_i, t_{i+1}]$ , the numbers  $b_{0\alpha}$  and  $b_{1\alpha}$  are in the  $(m_i+2)^{\text{th}}$  column of matrix  $\mathbf{A}_i$ , while the numbers  $b_{0\beta}$ ,  $b_{1\beta}$  are in the  $(m_i+3)^{\text{th}}$  column. In other discretization intervals, these elements are shifted to other columns of the matrix corresponding to the actual value of  $m_i$ .

The principal period  $T$  of the spindle speed modulation is a  $N$ -multiple of the interval length  $\Delta t$ . Thus, a transition matrix can be given by coupling the solutions for each interval:

$$\Phi = \mathbf{A}_{N-1} \mathbf{A}_{N-2} \dots \mathbf{A}_1 \mathbf{A}_0. \quad (37)$$

The criterion of asymptotic stability is that all eigenvalues  $\mu$  of the transition matrix  $\Phi$  are in modulus less than one (see Lakshmikantham and Trigiante, 1988).

For computations, a discretization step  $\Delta t = \tau_0 / 20$  was used that resulted in  $m_{\max}=23$  and a  $25 \times 25$  element transition matrix.

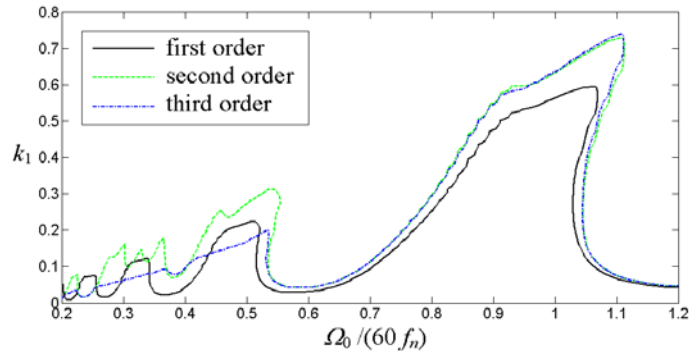
### STABILITY CHARTS

Stability charts were determined using MATLAB. Turning with constant spindle speed was investigated by setting  $\varepsilon=0$  in the MATLAB code in order to check the accuracy of the methods. Then,  $\varepsilon=0.1$  was used to obtain stability charts for varying spindle speed cases.

First, the accuracy of the Taylor approximation of various order was investigated. The recursive formulas (20), (25) and (31) were computed, and stability was determined using equation (14). The results are shown in Figure 4.

First, second and third order Taylor approximations were applied. As it was mentioned earlier, the third order case results in the AFDE (27), that is always unstable for any parameters. However the discrete solution (31) of AFDE (27) does not provide this total instability as it can be seen in Figure 4. However, for low spindle speeds, the third order approximation leads to a decrease of the stable domains. Consequently, the second order case is more reliable than the first or the third order ones, therefore, the second order case is used to compare the results of the Taylor expansion method to the direct Euler integration and to the semi-discretization.

A turning process with 10% spindle speed modulation amplitude was investigated. Figure 5 and 6 presents stability charts for  $T/\tau_0=2$  and 5, respectively. This means that the

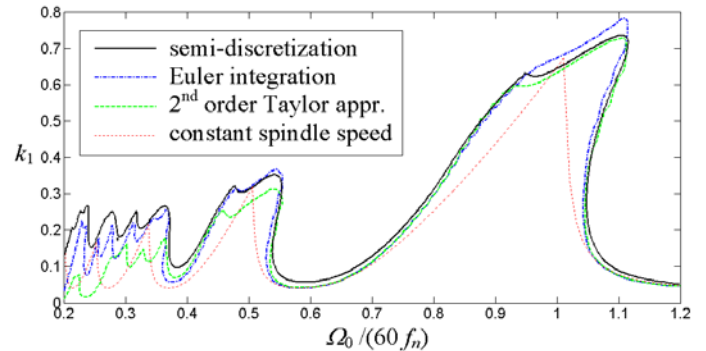


**Fig. 4.** Stability chart of varying spindle speed turning with 10% modulation amplitude and  $T/\tau_0=2$  determined by the Taylor approximation of various order

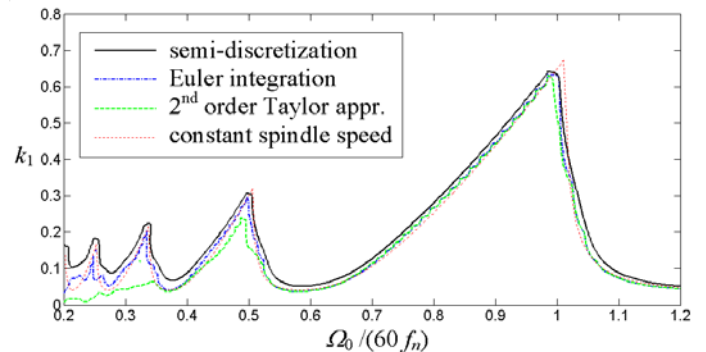
spindle speed completes one period of oscillation once during 2 or 5 revolutions of the workpiece. The figures show that the boundary curve obtained by the semi-discretization method is similar to the ones obtained by direct Euler integration. The second order Taylor approximation shows larger deviations, especially in the low spindle speed domain.

From a computational viewpoint, the semi-discretization method is more efficient than the other two methods. The computation time for semi discretization was 5-10 minutes, while the Euler integrations required more than 10 hours. One explanation for the difference in computation times is that the semi-discretization method needs to compute solutions over the interval  $[0, T]$ , while simulations were completed over the interval  $[0, 10T]$  to obtain a sufficiently long data series for stability determination. Also, the discretization step for Euler integration was  $\Delta t = \tau_0 / 1000$ , while for semi-discretization, it was  $\Delta t = \tau_0 / 20$ . This can be explained by another advantage of the semi-discretization method, namely, that it discretizes only the delayed term, and an almost exact solution is given in each discretization interval by solving the ordinary differential equation (33).

In Figures 5 and 6, the lobes of the traditional turning process are denoted by dotted lines so that the effect of spindle speed modulation can be observed. For the case,  $T/\tau_0=2$ , new stable parameters can be achieved for high spindle speed as well. For low spindle speed domains, the relative improvements in stability properties are more essential. For the case,  $T/\tau_0=5$ , slight increase of the boundary curves is obtained for low



**Fig. 5.** Stability charts of varying spindle speed turning with 10% modulation amplitude and  $T/\tau_0=2$



**Fig. 6.** Stability charts of varying spindle speed turning with 10% modulation amplitude and  $T/\tau_0=5$

spindle speeds, while no improvements can be observed for high spindle speeds. If larger values of the ratio  $T/\tau_0$  are used, then the system can be considered quasi-autonomous and the charts converge to those of the conventional turning process, as it can be seen in Figure 6.

## CONCLUSIONS

Three numerical methods were used to predict changes in stability for turning with varying spindle speed: (1) direct Euler integration with a linear interpolation of the delayed term, (2) Taylor expansion of the delayed term combined with Euler integration, and (3) semi-discretization method.

The direct Euler integration is a commonly used and widely accepted numerical method, however, for accurate approximation, the computation time is enormous relative to the computation time of semi-discretization.

A Taylor expansion of the varying time delay was also investigated. This type of approximation, however, must be used cautiously, since the higher order expansion of the delay variation term results higher derivatives of delayed terms and an AFDE is obtained as approximation. This contradiction can be explained if the weakness of the Taylor expansion is understood: it does not converge uniformly to the weight function of past states above the time in the past.

The semi-discretization method was found the most reliable and computationally efficient approximation technique for the analysis of equation (6).

Stability charts were constructed for 10% modulation amplitude with  $T/\tau_0=2$  and 5. It was shown, that the improvements of stability properties are better for low modulation period, while for high modulation period, the system can be considered quasi-autonomous and the charts converge to the ones of the conventional turning process.

It was shown that spindle speed variation is an effective way of chatter suppression for lower spindle speeds and little improvement arises for higher spindle speeds.

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